Czechoslovak Mathematical Journal

H. Karami; S. M. Sheikholeslami; Abdollah Khodkar

Lower bounds on signed edge total domination numbers in graphs

Czechoslovak Mathematical Journal, Vol. 58 (2008), No. 3, 595–603

Persistent URL: `http://dml.cz/dmlcz/140408`

Czechoslovak Mathematical Journal, 58 (133) (2008), 595–603

LOWER BOUNDS ON SIGNED EDGE TOTAL DOMINATION NUMBERS IN GRAPHS

H. Karami, S. M. Sheikholeslami, Tabriz,
and Abdollah Khodkar, Carrollton

(Received April 10, 2006)

Abstract. The open neighborhood $N_G(e)$ of an edge e in a graph G is the set consisting of all edges having a common end-vertex with e. Let f be a function on $E(G)$, the edge set of G, into the set $\{-1,1\}$. If $\sum\limits_{x\in N_G(e)} f(x)\geqslant 1$ for each $e\in E(G)$, then f is called a signed edge total dominating function of G. The minimum of the values $\sum\limits_{e\in E(G)} f(e)$, taken over all signed edge total dominating function f of G, is called the signed edge total domination number of G and is denoted by $\gamma'_{st}(G)$. Obviously, $\gamma'_{st}(G)$ is defined only for graphs G which have no connected components isomorphic to K_2. In this paper we present some lower bounds for $\gamma'_{st}(G)$. In particular, we prove that $\gamma'_{st}(T)\geqslant 2-m/3$ for every tree T of size $m\geqslant 2$. We also classify all trees T with $\gamma'_{st}(T)=2-m/3$.

Keywords: signed edge domination, signed edge total dominating function, signed edge total domination number

MSC 2010: 05C69, 05C05

1. Introduction

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. We use [2] for terminology and notation which are not defined here and consider simple connected graphs only. Two edges e_1,e_2 of G are called *adjacent* if they are distinct and have a common end-vertex. The *open neighborhood* $N_G(e)$ of an edge $e\in E(G)$ is the set of all edges adjacent to e. Its *closed neighborhood* is $N_G[e]=N_G(e)\cup\{e\}$. For a function $f\colon E(G)\longrightarrow\{-1,1\}$ and a subset S of $E(G)$ we define $f(S)=\sum\limits_{e\in S} f(e)$. The *edge-neighborhood* $E_G(v)$ of a vertex $v\in V(G)$ is the set of all edges at the

Research supported by a Faculty Research Grant, University of West Georgia.

vertex v. For each vertex $v \in V(G)$ we also define $f(v) = \sum_{e \in E_G(v)} f(e)$. A function $f\colon E(G) \longrightarrow \{-1, 1\}$ is called a *signed edge total dominating function* (SETDF) of G, if $f(N_G(e)) \geqslant 1$ for each edge $e \in E(G)$. It is clear that there exists an SETDF only for graphs G which have no connected components isomorphic to K_2. Throughout this paper we assume G is a simple connected graph of order $n \geqslant 3$. The minimum of the values $f(E(G))$, taken over all signed edge total dominating functions f of G, is called the *signed edge total domination number* of G. The signed edge total domination number was introduced by B. Zelinka in [5] and denoted by $\gamma'_{st}(G)$. The signed edge total dominating function f of G with $f(E(G)) = \gamma'_{st}(G)$ is called the $\gamma'_{st}(G)$-*function*.

Similarly, a function $f\colon E(G) \longrightarrow \{-1, 1\}$ is called a *signed edge dominating function* (SEDF) of G, if $f(N_G[e]) \geqslant 1$ for each edge $e \in E(G)$. The minimum of the values $f(E(G))$, taken over all signed edge dominating functions f of G, is called the *signed edge domination number* of G. The signed edge domination number was introduced by B. Xu in [3] and denoted by $\gamma'_s(G)$.

Here are some well-known results on $\gamma'_s(G)$ and $\gamma'_{st}(G)$.

Theorem A [1], [4]. *For every tree T of order $n \geqslant 2$, $\gamma'_s(T) \geqslant 1$.*

Theorem B [5]. *Let G be a graph with m edges and with no K_2-components. Then $\gamma'_{st}(G) \equiv m \pmod 2$.*

Theorem C [5]. *Let P_m be a path of length $m \geqslant 2$. Then $\gamma'_{st}(P_m) = m$.*

Theorem D [5]. *Let C_m be a cycle of length $m \geqslant 3$. Then $\gamma'_{st}(C_m) = m$.*

Theorem E [5]. *Let T be a star with $m \geqslant 2$ edges. If m is odd, then $\gamma'_{st}(T) = 3$. If m is even, then $\gamma'_{st}(T) = 2$.*

The following terminology and notation are useful to prove our results. A graph G with an SETDF f of G, denoted by (G, f), is called a *signed total graph*. For simplicity, given a signed total graph (G, f), an edge e is said to be a $+1$ edge of (G, f) if $f(e) = 1$. Similarly, an edge e is said to be a -1 edge of (G, f) if $f(e) = -1$. We write $E^+(G, f) = \{e \in E(G);\ f(e) = 1\}$ and $E^-(G, f) = \{e \in E(G);\ f(e) = -1\}$.

For any signed total graph (G, f), the two spanning subgraphs $G^+(f)$ and $G^-(f)$ of G are defined as $V(G^+(f)) = V(G^-(f)) = V(G)$ and $E(G^+(f)) = E^+(G, f)$ and $E(G^-(f)) = E^-(G, f)$. For every vertex $v \in V(G)$ we have $f(v) = \deg_{G^+(f)}(v) - \deg_{G^-(f)}(v)$.

2. A lower bound for SETDN of trees

In this section we study the signed edge total domination number of trees. We first prove that for every tree T of size $m \geqslant 2$, $\gamma'_{st}(T) \geqslant 2 - m/3$. Then we characterize all trees T for which $\gamma'_{st}(T) = 2 - m/3$.

Theorem 1. *For every tree T of size $m \geqslant 2$, $\gamma'_{st}(T) \geqslant 2 - m/3$.*

Proof. The proof is by induction on m. The statement holds for all trees of size $m = 2, 3, 4$. Assume T is an arbitrary tree of size $m \geqslant 5$ and that the statement holds for all trees with smaller sizes. Let f be a γ'_{st}-function of T. We consider two cases.

Case 1. There is a non-pendant edge $e = uv \in E$ for which $f(e) = -1$.

Let T_1 and T_2 be the connected components of $T - e$ with $u \in T_1$ and $v \in T_2$. Obviously, the sizes of T_1 and T_2 are greater than 1 and $\gamma'_{st}(T) = f(E(T_1)) - 1 + f(E(T_2))$. For $i = 1, 2$, the function f, restricted to T_i, is an SETDF of T_i, hence, $\gamma'_{st}(T_i) \leqslant f(E(T_i))$. By the inductive hypothesis, $\gamma'_{st}(T_i) \geqslant 2 - m_i/3$, where m_i is the size of T_i. Thus

$$(1) \qquad \gamma'_{ts}(T) \geqslant -1 + (2 - m_1/3) + (2 - m_2/3) = 3 - (m-1)/3 > 2 - m/3.$$

Case 2. The only edges e for which $f(e) = -1$ are pendant edges.

By assumption we have $f(v) \geqslant 0$ for each $v \in V(T)$ with $\deg(v) \geqslant 2$. Let $Z = \{v \in V(T);\ \deg(v) \geqslant 2 \text{ and } f(v) = 0\}$. First, let $Z = \emptyset$. Then f is an SEDF of T. Since $m \geqslant 5$, by Theorem A we have

$$(2) \qquad \gamma'_{st}(T) = f(E(T)) \geqslant \gamma'_{s}(T) \geqslant 1 > 2 - m/3.$$

Let $Z \neq \emptyset$. It is easy to see that Z is an independent set in T. Let $Z = \{u_i;\ 1 \leqslant i \leqslant k\}$. Obviously, there is no $+1$ pendant edge at u_i for each i. Let $N'(u_i) = \{u \in N(u_i);\ \deg(u) \geqslant 2\}$. Let first $|N'(u_i)| \geqslant 2$ for some i. Without loss of generality we may assume $|N'(u_1)| \geqslant 2$ and $v_1, v_2 \in N'(u_1)$. Let T_1 and T_2 be the connected components of $T - u_1v_1$ for which $v_1 \in V(T_1)$. Let T'_1 be obtained from T_1 by adding a new pendant edge v_1w_1 and let T'_2 be obtained from T_2 by deleting one of the -1 pendant edges at u_1. Now define $g_1\colon E(T'_1) \longrightarrow \{-1, +1\}$ by

$$g(v_1w_1) = +1 \text{ and } g(e) = f(e) \text{ if } e \in E(T_1).$$

Obviously, g is an SETDF of T'_1 and $f|_{T'_2}$ is an SETDF of T'_2. By the inductive hypothesis, $\gamma'_{st}(T'_i) \geqslant 2 - m_i/3$, where m_i is the size of T'_i and $m_1 + m_2 = m - 1$.

Thus

$$\begin{aligned}(3)\qquad \gamma'_{st}(T) = f(E(T)) &= g(E(T'_1)) + f|_{T'_2}(E(T'_2)) - 1\\ &\geqslant -1 + (2 - m_1/3) + (2 - m_2/3)\\ &> 2 - m/3.\end{aligned}$$

Now let $|N'(u_i)| = 1$ and $N'(u_i) = \{v_i\}$ for $1 \leqslant i \leqslant k$. It is clear that $f(v_i) \geqslant 3$ for each i. Let T' be obtained from T by deleting all leaves and the vertices of Z. Then since $|N'(u_i)| = 1$ for each i, T' is a tree. Let $w \in \{v_1, v_2, \ldots, v_k\}$. Hence, $f(w) \geqslant 3$ and $\deg(w) \geqslant 3$. We consider three subcases.

Subcase 2.1. $\deg_{T'}(w) \geqslant 1$, $e = ww_1 \in E(T')$ and $f(w_1) = 1$ in T.

By the construction of T' we have $deg_T(w_1) \geqslant 2$. Since $f(w_1) = 1$ and each edge at w_1 in T' is a $+1$ edge, there exists a pendant edge e' in T at w_1. Let T_1 and T_2 be the connected components of $T - e$ containing w, w_1, respectively. Let T'_1 be obtained from T_1 by adding a new pendant edge ww' at w and $T'_2 = T_2 - e'$. It is easy to see that the sizes of T'_1 and T'_2 are greater than 1. Define $g_1\colon E(T'_1) \longrightarrow \{-1, +1\}$ by

$$g(ww') = 1 \text{ and } g(e) = f(e) \text{ if } e \in E(T_1).$$

Obviously, g and $f|_{T'_2}$ are SETDFs of T'_1 and T'_2, respectively. By the inductive hypothesis, $\gamma'_{st}(T'_i) \geqslant 2 - m_i/3$ where m_i is the size of T'_i and $m_1 + m_2 = m - 1$. Thus

$$(4)\qquad \gamma'_{st}(T) = f(E(T)) = g(E(T'_1)) + f|_{T'_2}(E(T'_2)) - 1 > 2 - m/3.$$

Subcase 2.2. $\deg_{T'}(w) \geqslant 1$, $e = ww_1 \in E(T')$ and $f(w_1) \geqslant 2$ in T.

Let T_1 and T_2 be the connected components of $T - e$. Let T'_1 and T'_2 be obtained from T_1 and T_2 by adding new pendant edges ww' and $w_1w'_1$, respectively. Define $g_1\colon E(T'_1) \longrightarrow \{-1, +1\}$ by

$$g(ww') = 1 \text{ and } g(e) = f(e) \text{ if } e \in E(T_1),$$

and $g_2\colon E(T'_2) \longrightarrow \{-1, +1\}$ by

$$g(w_1w'_1) = 1 \text{ and } g(e) = f(e) \text{ if } e \in E(T_2).$$

Obviously, g_i is an SETDF of T'_i for $i = 1, 2$. Let $m_i = |E(T'_i)|$. Then we have $m_1 + m_2 = m + 1$. By the inductive hypothesis,

$$(5)\qquad \gamma'_{st}(T) = f(E(T)) = g_1(E(T'_1)) + g_2(E(T'_2)) - 1 > 2 - m/3.$$

S u b c a s e 2.3. $\deg_{T'}(w) = 0$.

This implies that $wu_i \in E(T)$ for each $1 \leqslant i \leqslant k$. If there exist two pendant edges at w, say e', e'', such that $f(e') = -1$ and $f(e'') = 1$, then using the inductive hypothesis on $T - \{e', e''\}$ we have

$$\gamma'_{st}(T) \geqslant 2 - (m-2)/3 > 2 - m/3. \tag{6}$$

Finally, let f assign -1 to all pendant edges at w and let r be the number of pendant edges at w. By assumption $k - r = f(w) \geqslant 3$. Furthermore, since $f(u_i) = 0$, there exists a pendant edge u_iv_i for each i. Therefore, $m \geqslant 2k + r$ and hence, $r \leqslant m/3 - 2$. On the other hand, we have $\gamma'_{st}(T) = -r$. Therefore, $\gamma'_{st}(T) \geqslant 2 - m/3$. This completes the proof. □

Now we characterize all trees that attain this bound. We use the notation of Theorem 1.

Theorem 2. *Let $T = (V, E)$ be a tree of size $m \geqslant 2$. Then $\gamma'_{st}(T) = 2 - m/3$ if and only if $V = \{w, u_i, v_i, w_j;\ 1 \leqslant i \leqslant k,\ k \geqslant 3$ and $1 \leqslant j \leqslant k-3\}$, and $E(T) = \{ww_j, wu_i, u_iv_i;\ 1 \leqslant i \leqslant k$ and $1 \leqslant j \leqslant k-3\}$.*

P r o o f. Let $\gamma'_{st}(T) = 2 - m/3$. Obviously, $m \equiv 0 \pmod 3$. By Theorems C, D and E we must have $m \geqslant 6$. Let f be a γ'_{st}-function of T. By (1), f must assign 1 to all non-pendant edges of T. Obviously, $f(v) \geqslant 0$ for each $v \in V(T)$ with $\deg(v) \geqslant 2$. By (2), we have $Z \neq \emptyset$. Let $Z = \{u_i;\ 1 \leqslant i \leqslant k\}$. Obviously, there is no $+1$ pendant edge at u_i for each i and Z is an independent set of T. By (3), $|N'(u_i)| = 1$ for each i. Since $f(u_i) = 0$, there exists precisely one pendant edge at u_i, hence $\deg(u_i) = 2$ for each i. By (4) and (5), the subtree T' of T is of order one. Let $w \in T'$. Then $w \in \cap_{i=1}^{k} N'(u_i)$. By (6), f assigns -1 to all pendant edges at w. Let r be the number of pendant edges at w. Then we have $2 - (2k+r)/3 = f(E(T)) = -r$, which implies $r = k - 3$ and $k \geqslant 3$.

Conversely, let G be a graph with the structure described in the theorem. By Theorem 1 we have $\gamma'_{st}(G) \geqslant 2 - (3k-3)/3$. Define $g\colon E(T) \longrightarrow \{-1, +1\}$ by

$$g(wu_i) = 1,\ g(u_iv_i) = -1\ (1 \leqslant i \leqslant k) \text{ and } g(ww_j) = -1\ (1 \leqslant j \leqslant k-3).$$

Obviously, g is an SETDF of T and $g(E(T)) = 2 - (3k-3)/3$. This completes the proof. □

3. Lower bounds

In this section we find some lower bounds for signed edge total domination numbers of simple connected graphs. Let G be a simple connected graph of order n and size $m \geqslant 2$. For every edge $e = uv \in V(G)$, the degree of e, $d(e)$, is defined by $d(e) = \deg(u) + \deg(v) - 2$. First we present a lower bound in terms of n, m, δ and Δ.

Theorem 3. *For every simple connected graph of order $n \geqslant 3$, size m and $\delta \geqslant 2$,*

$$\gamma'_{st}(G) \geqslant \left\lceil \frac{m - (\Delta - \delta)(\Delta - 1)(n - \delta)}{2(\Delta - 1)} \right\rceil.$$

Proof. Let f be a γ'_{st}-function of G. We have

$$\begin{aligned} (7) \qquad 2\gamma'_{st}(G) = 2f(E(T)) &= 2(|E^+(G,f)| - |E^-(G,f)|) \\ &= \sum_{u \in V(G^+(f))} \deg_{G+(f)}(u) - \sum_{u \in V(G^-(f))} \deg_{G^-(f)}(u) \\ &= \sum_{u \in V(G)} f(u). \end{aligned}$$

For $uv \in E(G)$ we have $f(u) + f(v) - 2f(uv) \geqslant 1$. Therefore

$$\begin{aligned} (8) \qquad m + 2\gamma'_{st}(G) &\leqslant \sum_{uv \in E(G)} (f(u) + f(v) - 2f(uv)) + 2 \sum_{uv \in E(G)} f(uv) \\ &= \sum_{uv \in E(G)} (f(u) + f(v)) \\ &= \sum_{u \in V(G)} f(u) \deg_G(u). \end{aligned}$$

Let $B_1 = \{u \in V(G);\ f(u) \geqslant 1\}$, $B_2 = \{u \in V(G);\ f(u) \leqslant -1\}$ and $B_3 = \{u \in V(G);\ f(u) = 0\}$. Obviously, for each $u \in B_2$ we have $N_G(u) \subseteq B_1 \cup B_3$. Hence,

$$(9) \qquad \delta \leqslant |N_G(u)| \leqslant |B_1| + |B_3| = n - |B_2|.$$

Thus by (7) and (8) we have

$$\begin{aligned}
m + 2\gamma'_{st}(G) &\leqslant \sum_{u\in V(G)} f(u)\deg_G(u) \\
&= \sum_{u\in B_1} f(u)\deg_G(u) + \sum_{u\in B_2} f(u)\deg_G(u) \\
&\leqslant \Delta \sum_{u\in B_1} f(u) + \delta \sum_{u\in B_2} f(u) \\
&= \Delta \sum_{u\in V(G)} f(u) + (\delta - \Delta) \sum_{u\in B_2} f(u) \\
&= 2\Delta\gamma'_{st}(G) + (\delta - \Delta) \sum_{u\in B_2} f(u).
\end{aligned}$$

Hence,

$$2(\Delta - 1)\gamma'_{st}(G) \geqslant m + (\Delta - \delta) \sum_{u\in B_2} f(u). \tag{10}$$

Now for each $u \in B_2$ there exists $v \in N_G(u)$ such that $f(uv) = -1$. So we have $f(u)+f(v) \geqslant 1+2f(uv) = -1$. Since $f(v) \leqslant \Delta-2$, it follows that $f(u) \geqslant -(\Delta-1)$. Using (9) and (10) we have $2(\Delta-1)\gamma'_{st}(G) \geqslant m - (\Delta-\delta)(n-\delta)(\Delta-1)$. Now the result follows. □

The following result is an immediate consequence of Theorem 3.

Corollary 4. *For every simple k-regular graph G with $k \geqslant 2$, $\gamma'_{st}(G) \geqslant \lceil \frac{1}{2}m \times (k-1)\rceil$.*

Theorem 5. *For every simple connected graph G with $2 \leqslant \delta \leqslant \Delta \leqslant 4$, $\gamma'_{st}(G) \geqslant 0$.*

Proof. Let f be a γ'_{st}-function of G. Since $2 \leqslant \delta \leqslant \Delta \leqslant 4$, we have $|N_G(e) \cap E^+(G,f)| \geqslant 2$ and $|N_G(e) \cap E^-(G,f)| \leqslant 2$. Now it is clear that

$$\begin{aligned}
2|E^-(G,f)| &\leqslant \sum_{e\in E^-(G,f)} |N_G(e) \cap E^+(G,f)| \\
&= \sum_{e\in E^+(G,f)} |N_G(e) \cap E^-(G,f)| \\
&\leqslant 2|E^+(G,f)|.
\end{aligned}$$

Thus $|E^-(G,f)| \leqslant |E^+(G,f)|$ and hence, $\gamma'_{st}(G) = |E^+(G,f)| - |E^-(G,f)| \geqslant 0$. □

Theorem 6. *For every simple connected graph G of order $n \geqslant 3$ and size m,*

$$\gamma'_{st}(G) \geqslant m\Big(\frac{2m}{n(\Delta-1)} - \frac{\varepsilon_o}{2m(\Delta-1)} - 1\Big)$$

where ε_o is the number of edges of odd degree. Furthermore, this bound is sharp.

Proof. Let A be the set of edges of even degree. It is easy to see that if $uv \in A$, then $|N_G(e) \cap E^+(G,f)| \geqslant \frac{1}{2}(\deg(u)+\deg(v))$ and if $e \in E(G) \setminus A$, then $|N_G(e) \cap E^+(G,f)| \geqslant \frac{1}{2}(\deg(u)+\deg(v)-1)$. Thus

$$\begin{aligned}
\sum_{uv\in E(G)} |N(e)\cap E^+(G,f)| &\geqslant \frac{1}{2}\sum_{uv\in E(G)} (\deg(u)+\deg(v)) - \frac{1}{2}\varepsilon_o \\
&= \frac{1}{2}\sum_{u\in V(G)} \deg(u)^2 - \frac{1}{2}\varepsilon_o \\
&\geqslant \frac{1}{2n}\Big(\sum_{u\in V(G)} \deg(u)\Big)^2 - \frac{1}{2}\varepsilon_o \\
&= \frac{2m^2}{n} - \frac{1}{2}\varepsilon_o.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
2(\Delta-1)|E^+(G,f)| &\geqslant \sum_{e\in E^+(G,f)} |N_G(e)| \\
&= \sum_{e\in E^+(G,f)} (|N_G(e)\cap E^+(G,f)| + |N_G(e)\cap E^-(G,f)|) \\
&= \sum_{e\in E^+(G,f)} |N_G(e)\cap E^+(G,f)| + \sum_{e\in E^+(G,f)} |N_G(e)\cap E^-(G,f)| \\
&= \sum_{e\in E^+(G,f)} |N_G(e)\cap E^+(G,f)| + \sum_{e\in E^-(G,f)} |N_G(e)\cap E^+(G,f)| \\
&= \sum_{e\in E(G)} |N_G(e)\cap E^+(G,f)|.
\end{aligned}$$

Therefore $|E^+(G,f)| \geqslant \dfrac{m^2}{n(\Delta-1)} - \dfrac{\varepsilon_o}{4(\Delta-1)}$. This implies that

$$\gamma'_{st}(G) = 2|E^+(G,f)| - m \geqslant m\Big(\frac{2m}{n(\Delta-1)} - \frac{\varepsilon_o}{2m(\Delta-1)} - 1\Big).$$

Theorem D shows that this bound is sharp and the proof is complete. □

References

[1] *H. Karami, A. Khodkar and S. M. Sheikholeslami*: Signed edge domination numbers in trees. Ars Combinatoria. To appear.
[2] *D. B. West*: Introduction to Graph Theory. Prentice-Hall, Inc, 2000.
[3] *B. Xu*: On signed edge domination numbers of graphs. Discrete Mathematics *239* (2001), 179–189.
[4] *B. Xu*: On lower bounds of signed edge domination numbers in graphs. J. East China Jiaotong Univ. *1* (2004), 110–114. (In Chinese.)
[5] *B. Zelinka*: On signed edge domination numbers of trees. Math. Bohem. *127* (2002), 49–55.

Authors' addresses: H. Karami, Department of Mathematics, Azarbaijan University of Tarbiat Moallem, Tabriz, I.R. Iran; Abdollah Khodkar, Department of Mathematics, University of West Georgia, Carrollton, GA 30118, USA, akhodkar@westga.edu S. M. Sheikholeslami, Department of Mathematics, Azarbaijan University of Tarbiat Moallem, Tabriz, I. R. Iran, s.m.sheikholeslami@azaruniv.edu.