Ren Ganglian; Zhang Wenpeng

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ON THE MEAN VALUE OF A SUM ANALOGOUS TO CHARACTER SUMS OVER SHORT INTERVALS

Ren Ganglian and Zhang Wenpeng, Shaanxi

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Abstract. The main purpose of this paper is to study the mean value properties of a sum analogous to character sums over short intervals by using the mean value theorems for the Dirichlet L-functions, and to give some interesting asymptotic formulae.

Keywords: character sums, short intervals, even power mean, asymptotic formula

MSC 2010: 11L40

1. Introduction

Let $q \geqslant 3$ be an integer, and χ a Dirichlet character modulo q. Over the past several decades, many authors investigated various arithmetical properties of the character sums

$$\sum_{a=N+1}^{N+H} \chi(a).$$

Pólya [1] and Vinogradov [2] studied the character sums when the modulus q is equal to a prime p and obtained the following inequality

$$\biggl|\sum_{a=1}^{x} \chi(a)\biggr| \leqslant c\sqrt{p}\ln p,$$

where c is a constant. Actually, one can establish the above inequality with the constant $c = 1$. If χ is a primitive character modulo q, A. V. Sokolovskiĭ [3] proved the existence of N with

$$\biggl|\sum_{N\leqslant n\leqslant N+[q/2]} \chi(n)\biggr| > \sqrt{1-\frac{8\ln q}{q}}\cdot\frac{1}{2\sqrt{2}}\cdot\sqrt{q},$$

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where $[y]$ denotes the greatest integer which less than or equal to y. For general non-principal character, D. A. Burgess [4] obtained the mean value estimate of character sums

$$\sum_{n=1}^{k}\left|\sum_{m=1}^{h}\chi(n+m)\right|^2 < k\cdot h,$$

where h is any positive integer. Xu Zhefeng and Zhang Wenpeng [5] gave the $2k$-th power mean of the character sums over the interval $\left[1, \frac{1}{4}q\right)$

$$\sum_{\chi(-1)=1}{}^{*}\left|\sum_{a<q/4}\chi(a)\right|^{2k} = \frac{J(q)q^k}{16}\left(\frac{\pi}{8}\right)^{2k-2}\prod_{p\mid q}\left(1-\frac{1}{p^2}\right)^{2k-1}$$
$$\prod_{p\nmid 2q}\left(1-\frac{1-C_{2k-2}^{k-1}}{p^2}\right)+O\left(q^{k+\varepsilon}\right).$$

In this paper, we study the even power mean of analogous character sums over the short intervals

$$\sum_{\chi \bmod q}{}^{*}\left|\sum_{a<q/2}(-1)^a\chi(a)\right|^{2k} \quad \text{and} \quad \sum_{\chi \bmod q}{}^{*}\left|\sum_{a<q/4}(-1)^a\chi(a)\right|^{2k}.$$

First we transform the sum to L-functions. Then using the mean value theorems for Dirichlet L-functions, we study the mean value properties of the sums over short intervals, and obtain a sharper asymptotic formula for them. That is, we shall prove the following:

Theorem 1. *Let $q>8$ be an odd integer. Then we have the following asymptotic formulae:*

$$\sum_{\substack{\chi \bmod q\\ \chi(-1)=1}}{}^{*}\left|\sum_{a<q/2}(-1)^a\chi(a)\right|^{2k}$$
$$=\frac{J(q)q^k}{4}\left(\frac{\pi}{4}\right)^{2k-2}\prod_{p\mid q}\left(1-\frac{1}{p^2}\right)^{2k-1}\prod_{p\nmid 2q}\left(1-\frac{1-C_{2k-2}^{k-1}}{p^2}\right)+O\left(q^{k+\varepsilon}\right),$$

and

$$\sum_{\substack{\chi \bmod q\\ \chi(-1)=1}}{}^{*}\left|\sum_{a<\frac{q}{4}}(-1)^a\chi(a)\right|^{2k}$$
$$=\frac{J(q)q^k}{2^{k+1}}\left(\frac{\pi}{4}\right)^{2k-2}\prod_{p\mid q}\left(1-\frac{1}{p^2}\right)^{2k-1}\prod_{p\nmid 2q}\left(1-\frac{1-C_{2k-2}^{k-1}}{p^2}\right)+O\left(q^{k+\varepsilon}\right),$$

where $\sum\limits_{\chi(-1)=1}^{*}$ *denotes the summation over all primitive characters modulo* q *such that* $\chi(-1) = 1$*,* ε *is any fixed positive number,* $J(q)$ *denotes the number of all primitive characters modulo* q*,* $\prod\limits_{p|q}$ *denotes the product over all prime divisors* p *of* q *and* $C_m^n = m!/n!(m-n)!$.

Theorem 2. *Let* $q > 8$ *be an odd integer. Then we have the two asymptotic formulae:*

$$\sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}}^{*} \left| \sum_{a<q/2} (-1)^a \chi(a) \right|^4 = \frac{3J(q)q^2}{16} \prod_{p|q} \frac{(p^2-1)^3}{p^4(p^2+1)} + O\left(q^{2+\varepsilon}\right),$$

and

$$\begin{aligned}\sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}}^{*} \left| \sum_{a<q/4} (-1)^a \chi(a) \right|^4 &= \frac{15J(q)q^2}{64} \prod_{p|q} \frac{(p^2-1)^3}{p^4(p^2+1)} \\ &- \frac{3\sqrt{2}J(q)q^2}{8} \prod_{\substack{p\nmid q \\ p\equiv 1,7 (\bmod 8)}} \left(1+\frac{2}{p^2-1}\right) \prod_{\substack{p\nmid q \\ p\equiv 3,5 (\bmod 8)}} \frac{p^4+1}{(p^2+1)^2} \prod_{p|q}\left(1-\frac{1}{p^2}\right)^2 \\ &+ \frac{J(q)q^2}{4} \prod_{\substack{p\nmid q \\ p\equiv 1,7 (\bmod 8)}} \left(1+\frac{2}{p^2-1}\right) \prod_{\substack{p\nmid q \\ p\equiv 3,5 (\bmod 8)}} \frac{p^2-1}{p^2+1} \prod_{p|q}\left(1-\frac{1}{p^2}\right)^2 \\ &+ \frac{J(q)q^2}{8} \prod_{\substack{p\nmid q \\ p\equiv 1,7 (\bmod 8)}} \left(1+\frac{2}{p^2-1}\right) \prod_{\substack{p\nmid q \\ p\equiv 3,5 (\bmod 8)}} \left(\frac{p^2-1}{p^2+1}\right)^3 \prod_{p|q}\left(1-\frac{1}{p^2}\right)^2 + O(q^{2+\varepsilon}).\end{aligned}$$

Taking $k = 2$ in Theorem 1 and noting that

$$\begin{aligned}\prod_{p|q}\left(1-\frac{1}{p^2}\right)^3 \prod_{p\nmid 2q}\left(1+\frac{1}{p^2}\right) &= \prod_{p}\left(1+\frac{1}{p^2}\right) \prod_{p|q}\left(1-\frac{1}{p^2}\right)^3 \prod_{p|2q} \frac{1}{1+1/p^2} \\ &= \frac{4}{5}\frac{\zeta(2)}{\zeta(4)} \prod_{p|q} \frac{(p^2-1)^3}{p^4(p^2+1)},\end{aligned}$$

and

$$\sum_{\chi \bmod q}^{*} = \sum_{\chi(-1)=1}^{*} + \sum_{\chi(-1)=-1}^{*}$$

we may immediately get

Corollary 1. *Let $q > 8$ be an odd integer. Then we have the following asymptotic formulas*

$$\sum_{\chi \bmod q}^{*} \left| \sum_{a<q/2} (-1)^a \chi(a) \right|^4 = \frac{3}{8} J(q) q^2 \prod_{p|q} \frac{(p^2-1)^3}{p^4(p^2+1)} + O\left(q^{2+\varepsilon}\right)$$

and

$$\begin{aligned}\sum_{\chi \bmod q}^{*} \left| \sum_{a<\frac{q}{4}} (-1)^a \chi(a) \right|^4 &= \frac{21 J(q) q^2}{64} \prod_{p|q} \frac{(p^2-1)^3}{p^4(p^2+1)} \\ &- \frac{3\sqrt{2} J(q) q^2}{8} \prod_{\substack{p\nmid q \\ p\equiv 1,7 (\bmod 8)}} \left(1 + \frac{2}{p^2-1}\right) \prod_{\substack{p\nmid q \\ p\equiv 3,5 (\bmod 8)}} \frac{p^4+1}{(p^2+1)^2} \prod_{p|q} \left(1 - \frac{1}{p^2}\right)^2 \\ &+ \frac{J(q) q^2}{4} \prod_{\substack{p\nmid q \\ p\equiv 1,7 (\bmod 8)}} \left(1 + \frac{2}{p^2-1}\right) \prod_{\substack{p\nmid q \\ p\equiv 3,5 (\bmod 8)}} \frac{p^2-1}{p^2+1} \prod_{p|q} \left(1 - \frac{1}{p^2}\right)^2 \\ &+ \frac{J(q) q^2}{8} \prod_{\substack{p\nmid q \\ p\equiv 1,7 (\bmod 8)}} \left(1 + \frac{2}{p^2-1}\right) \prod_{\substack{p\nmid q \\ p\equiv 3,5 (\bmod 8)}} \left(\frac{p^2-1}{p^2+1}\right)^3 \prod_{p|q} \left(1 - \frac{1}{p^2}\right)^2 + O(q^{2+\varepsilon}).\end{aligned}$$

2. Some lemmas

To prove the theorem, we need the following lemmas.

Lemma 1. *Let χ be a primitive Dirichlet character modulo q with $q \geqslant 3$, then for any real number $\lambda \in [0,1]$ with $\lambda \neq r/q$, we have*

$$\sum_{0<n\leqslant \lambda q} \chi(n) = \begin{cases} \dfrac{\tau(\chi)}{\pi} \displaystyle\sum_{n=1}^{\infty} \frac{\bar{\chi}(n)\sin(2\pi n\lambda)}{n}, & \text{if } \chi(-1) = 1; \\ \dfrac{\tau(\chi)}{\mathrm{i}\pi} \left(L(1,\bar{\chi}) - \displaystyle\sum_{n=1}^{\infty} \bar{\chi}(n) \frac{\cos 2\pi n\lambda}{n} \right), & \text{if } \chi(-1) = -1, \end{cases}$$

where $\tau(\chi) = \sum_{a=1}^{q} \chi(a) e(a/q)$ is the Gauss sum, $e(y) = \mathrm{e}^{2\pi \mathrm{i} y}$, and $L(s,\chi)$ denotes the Dirichlet L-function corresponding to χ.

P r o o f. See Section 3.1 of [6]. □

Lemma 2. *Let $q > 8$ be an odd integer and χ a primitive Dirichlet character modulo q. Then we have*

$$(1)\quad \sum_{a=1}^{[q/2]} \chi(a) = 0, \qquad \text{if } \chi(-1) = 1,$$

$$(2)\quad \sum_{a=1}^{[q/2]} \chi(a) = \frac{(2-\bar{\chi}(2))}{\mathrm{i}\pi}\tau(\chi)L(1,\bar{\chi}), \qquad \text{if } \chi(-1) = -1,$$

$$(3)\quad \sum_{a=1}^{[q/4]} \chi(a) = \frac{\tau(\chi)}{\pi}L(1,\bar{\chi}\chi_4), \qquad \text{if } \chi(-1) = 1,$$

$$(4)\quad \sum_{a=1}^{[q/4]} \chi(a) = \frac{2+\bar{\chi}(2)-\bar{\chi}(4)}{2\pi\mathrm{i}}\tau(\chi)L(1,\bar{\chi}), \qquad \text{if } \chi(-1) = -1,$$

$$(5)\quad \sum_{a=1}^{[q/8]} \chi(a) = \frac{\tau(\chi)}{\pi}\Big[\frac{\bar{\chi}(2)}{2}L(1,\bar{\chi}\chi_4) + \frac{1}{\sqrt{2}}L(1,\bar{\chi}\chi_{8_1})\Big], \qquad \text{if } \chi(-1) = 1,$$

and

$$(6)\quad \sum_{a=1}^{[\frac{q}{8}]} \chi(a) = \frac{\tau(\chi)}{\mathrm{i}\pi}\Big[\frac{4+\bar{\chi}(4)-\bar{\chi}(8)}{4}L(1,\bar{\chi}) - \frac{1}{\sqrt{2}}L(1,\bar{\chi}\chi_{8_2})\Big], \quad \text{if } \chi(-1) = -1,$$

where χ_4 is the primitive Dirichlet character modulo 4, and χ_{8_1} and χ_{8_2} are the two primitive Dirichlet characters modulo 8 with $\chi_{8_1}(3) = 1$ and $\chi_{8_2}(3) = -1$ respectively.

Proof. We only prove formula (5), the others can be obtained in the same way. Since q is an odd number, it is easily seen that $r/q \neq \frac{1}{8}$, and from Lemma 1 we can have

$$\begin{aligned}
\sum_{a=1}^{[q/8]} \chi(a) &= \frac{\tau(\chi)}{\pi}\sum_{n=1}^{\infty}\frac{\bar{\chi}(n)\sin(\frac{1}{4}\pi n)}{n} \\
&= \frac{\tau(\chi)}{\pi}\Bigg(\sum_{n=1}^{\infty}\frac{\bar{\chi}(2n)\sin(\frac{1}{2}\pi n)}{2n} + \sum_{n=1}^{\infty}\frac{\bar{\chi}(2n-1)\sin(\frac{1}{4}(2n-1)\pi)}{2n-1}\Bigg) \\
&= \frac{\tau(\chi)}{\pi}\Bigg(\frac{\bar{\chi}(2)}{2}\sum_{n=1}^{\infty}\frac{\bar{\chi}\chi_4(n)}{n} + \sum_{n=1}^{\infty}\frac{\bar{\chi}(n)\chi_2(n)(-1)^{[(n-1)/4]}}{\sqrt{2}n}\Bigg) \\
&= \frac{\tau(\chi)}{\pi}\Big(\frac{\bar{\chi}(2)}{2}L(1,\bar{\chi}\chi_4) + \frac{1}{\sqrt{2}}L(1,\bar{\chi}\chi_{8_1})\Big).
\end{aligned}$$

This proves formula (5). Similarly we can deduce the other formulas. □

Lemma 3. *Let q and r be integers with $q \geqslant 2$ and $(r,q)=1$, and χ a Dirichlet character modulo q. Then we have the identities*

$$\sum_{\chi \bmod q}^{*} \chi(r) = \sum_{d\mid(q,r-1)} \mu\Big(\frac{q}{d}\Big)\varphi(d)$$

and

$$J(q) = \sum_{d\mid q} \mu(d)\varphi\Big(\frac{q}{d}\Big),$$

where $\sum_{\chi \bmod q}^{*}$ *denotes the summation over all primitive characters modulo q, and $J(q)$ denotes the number of all primitive characters modulo q.*

Proof. See Lemma 4 in the reference [7]. □

Lemma 4. *Let q be any integer with $q>2$, let $\tau_k(n)$ denote the k-th divisor function (i.e., the number of solutions of the equation $n_1n_2\dots n_k=n$ in positive integers $n_1,n_2,\dots,n_k$) and let $\tau(n)$ be the divisor function. Then we have the identity*

$$\sum_{\substack{n=1\\(n,q)=1}}^{\infty} \frac{\tau_k^2(n)}{n^2} = \zeta^{2k-1}(2)\prod_{p\mid q}\Big(1-\frac{1}{p^2}\Big)^{2k-1}\prod_{p\nmid q}\Big(1-\frac{1-C_{2k-2}^{k-1}}{p^2}\Big),$$

where $\zeta(s)$ is the Riemann zeta function.

Proof. See Lemma 3 in the reference [8]. □

Lemma 5. *Let $q>8$ be an odd integer, χ be a Dirichlet character modulo q, χ_4 be the primitive character modulo 4, χ_{8_1} and χ_{8_2} be the two primitive characters modulo 8. Then we have the identities*

$$\sum_{\chi(-1)=1}^{*} |L(1,\bar\chi\chi_4)|^{2k}$$
$$= \frac{J(q)}{2}\Big(\frac{\pi^2}{8}\Big)^{2k-1}\prod_{p\mid q}\Big(1-\frac{1}{p^2}\Big)^{2k-1}\prod_{p\nmid 2q}\Big(1-\frac{1-C_{2k-2}^{k-1}}{p^2}\Big)+O(q^{\varepsilon}),$$

$$\sum_{\chi(-1)=1}^{*} |L(1,\bar{\chi}\chi_{8_1})|^{2k}$$
$$= J(q)\Big(\frac{\pi^2}{8}\Big)^{2k-1} \prod_{p|q}\Big(1-\frac{1}{p^2}\Big)^{2k-1} \prod_{p\nmid 2q}\Big(1-\frac{1-C_{2k-2}^{k-1}}{p^2}\Big) + O(q^{\varepsilon}),$$
$$\sum_{\chi(-1)=-1}^{*} |L(1,\bar{\chi})|^4 = \frac{5\pi^4 J(q)}{144}\prod_{p|q}\frac{(p^2-1)^3}{p^4(p^2+1)} + O(q^{\varepsilon}),$$
$$\sum_{\chi(-1)=-1}^{*} |L(1,\bar{\chi}\chi_{8_2})|^4 = \frac{3\pi^4 J(q)}{256}\prod_{p|q}\frac{(p^2-1)^3}{p^4(p^2+1)} + O(q^{\varepsilon}),$$
$$\sum_{\chi(-1)=-1}^{*} \chi(2^m)|L(1,\bar{\chi})|^4 = \frac{(3m+5)\pi^4 J(q)}{72\cdot 2^{m+1}}\prod_{p|q}\frac{(p^2-1)^3}{p^4(p^2+1)} + O(q^{\varepsilon}),$$

and

$$\sum_{\chi(-1)=-1}^{*} \bar{\chi}(2^m)|L(1,\bar{\chi})|^4 = \frac{(3m+5)\pi^4 J(q)}{72\cdot 2^{m+1}}\prod_{p|q}\frac{(p^2-1)^3}{p^4(p^2+1)} + O(q^{\varepsilon}),$$

where $\sum\limits_{\chi \bmod q}^{*}$ *denotes the summation over all primitive characters modulo* q, *and* $J(q)$ *denotes the number of all primitive characters modulo* q.

P r o o f. Using the method from the proof of Lemma 5 in the reference [5], we can deduce the results. □

Lemma 6. *Let* $q > 8$ *be any odd integer, and* χ *the Dirichlet character modulo* q, $m \geqslant 0$. *Then we have the following asymptotic formulae:*

$$\sum_{\chi(-1)=-1}^{*} \chi(2^m)L^2(1,\bar{\chi})L(1,\chi)L(1,\chi\chi_{8_2})$$
$$= \frac{J(q)}{2^{m+1}}\sideset{}{'}\sum_{n=1}^{\infty} n^{-2}\tau(2^m n)\sum_{t|n}\chi_{8_2}(t) + O(q^{\varepsilon}),$$
$$\sum_{\chi(-1)=-1}^{*} \chi(2^m)L^2(1,\chi)L(1,\bar{\chi})L(1,\bar{\chi}\chi_{8_2})$$
$$= \frac{J(q)}{2^{m+1}}\sideset{}{'}\sum_{n=1}^{\infty} n^{-2}\tau(n)\sum_{t|2^m n}\chi_{8_2}(t) + O(q^{\varepsilon}),$$
$$\sum_{\chi(-1)=-1}^{*} \chi(2^m)L^2(1,\bar{\chi})L^2(1,\chi\chi_{8_2}) = \frac{J(q)}{2^{m+1}}\sideset{}{'}\sum_{n=1}^{\infty}\frac{\tau(2^m n)\chi_{8_2}(n)\tau(n)}{n^2} + O(q^{\varepsilon}),$$

$$\sum_{\chi(-1)=-1}^{*} \chi(2^m)|L(1,\bar\chi)|^2|L(1,\bar\chi\chi_{8_2})|^2$$
$$= \frac{J(q)}{2^{m+1}} \sum_{n=1}^{\infty}{}' n^{-2} \sum_{t|2^m n} \chi_{8_2}(t) \sum_{t|n} \chi_{8_2}(t) + O(q^{\varepsilon}),$$
$$\sum_{\chi(-1)=-1}^{*} \chi(2^m)L^2(1,\chi)L^2(1,\bar\chi\chi_{8_2})$$
$$= \frac{J(q)}{2^{m+1}} \sum_{n=1}^{\infty}{}' \frac{\tau(n)\chi_{8_2}(2^m n)\tau(2^m n)}{n^2} + O(q^{\varepsilon}),$$
$$\sum_{\chi(-1)=-1}^{*} \bar\chi(2^m)L^2(1,\bar\chi)L(1,\chi)L(1,\chi\chi_{8_2})$$
$$= \frac{J(q)}{2^{m+1}} \sum_{n=1}^{\infty}{}' n^{-2}\tau(n) \sum_{t|2^m n} \chi_{8_2}(t) + O(q^{\varepsilon}),$$
$$\sum_{\chi(-1)=-1}^{*} \bar\chi(2^m)L^2(1,\chi)L(1,\bar\chi)L(1,\bar\chi\chi_{8_2})$$
$$= \frac{J(q)}{2^{m+1}} \sum_{n=1}^{\infty}{}' n^{-2}\tau(2^m n) \sum_{t|n} \chi_{8_2}(t) + O(q^{\varepsilon}),$$
$$\sum_{\chi(-1)=-1}^{*} \bar\chi(2^m)L(1,\chi)L(1,\chi\chi_{8_2})L^2(1,\bar\chi\chi_{8_2})$$
$$= \frac{J(q)}{2^{m+1}} \sum_{n=1}^{\infty}{}' n^{-2}\chi_{8_2}(n)\tau(n) \sum_{t|2^m n} \chi_{8_2}(t) + O(q^{\varepsilon}),$$
$$\sum_{\chi(-1)=-1}^{*} \bar\chi(2^m)L^2(1,\bar\chi)L^2(1,\chi\chi_{8_2}) = \frac{J(q)}{2^{m+1}} \sum_{n=1}^{\infty}{}' \frac{\tau(n)\chi_{8_2}(2^m n)\tau(2^m n)}{n^2} + O(q^{\varepsilon}),$$
$$\sum_{\chi(-1)=-1}^{*} \bar\chi(2^m)|L(1,\bar\chi)|^2|L(1,\bar\chi\chi_{8_2})|^2$$
$$= \frac{J(q)}{2^{m+1}} \sum_{n=1}^{\infty}{}' n^{-2} \sum_{t|2^m n} \chi_{8_2}(t) \sum_{t|n} \chi_{8_2}(t) + O(q^{\varepsilon}),$$
$$\sum_{\chi(-1)=-1}^{*} \bar\chi(2^m)L^2(1,\chi)L^2(1,\bar\chi\chi_{8_2}) = \frac{J(q)}{2^{m+1}} \sum_{n=1}^{\infty}{}' \frac{\tau(2^m n)\chi_{8_2}(n)\tau(n)}{n^2} + O(q^{\varepsilon}),$$
$$\sum_{\chi(-1)=-1}^{*} \chi(2^m)L(1,\bar\chi)L(1,\bar\chi\chi_{8_2})L^2(1,\chi\chi_{8_2})$$
$$= \frac{J(q)}{2^{m+1}} \sum_{n=1}^{\infty}{}' n^{-2}\chi_{8_2}(n)\tau(n) \sum_{t|2^m n} \chi_{8_2}(t) + O(q^{\varepsilon}),$$

$$\sum_{\chi(-1)=-1}^{*} \chi(2^m)L(1,\chi)L(1,\chi\chi_{8_2})L^2(1,\bar\chi\chi_{8_2})$$
$$= \frac{J(q)}{2^{m+1}} \sum_{n=1}^{\infty}{}' n^{-2}\chi_{8_2}(2^m n)\tau(2^m n)\sum_{t|n}\chi_{8_2}(t) + O(q^{\varepsilon}),$$
$$\sum_{\chi(-1)=-1}^{*} \bar\chi(2^m)L(1,\bar\chi)L(1,\bar\chi\chi_{8_2})L^2(1,\chi\chi_{8_2})$$
$$= \frac{J(q)}{2^{m+1}} \sum_{n=1}^{\infty}{}' n^{-2}\chi_{8_2}(2^m n)\tau(2^m n)\sum_{t|n}\chi_{8_2}(t) + O(q^{\varepsilon}),$$

where $\sum_{n=1}^{\infty}{}'$ *denotes the summation over all positive integers* n *with* $(n,q)=1$.

P r o o f. We only prove the first formula, the others can be obtained by the same method.

For convenience, we put

$$A(y,\bar\chi) = \sum_{2^m N<n\leqslant y} \bar\chi(n)\tau(n), \qquad \text{and} \qquad B(y,\chi) = \sum_{N<n\leqslant y}\chi(n)\sum_{t|n}\chi_{8_2}(t),$$

where N is a parameter with $q\leqslant N<q^3$. Then from Abel's identity we have

$$L^2(1,\bar\chi) = \sum_{n=1}^{\infty}\frac{\bar\chi(n)\tau(n)}{n} = \sum_{1\leqslant n\leqslant 2^m N}\frac{\bar\chi(n)\tau(n)}{n} + \int_{2^m N}^{\infty}\frac{A(y,\bar\chi)}{y^2}\,\mathrm{d}y$$

and

$$\begin{aligned} L(1,\chi)L(1,\chi\chi_{8_2}) &= \sum_{n=1}^{\infty} n^{-1}\chi(n)\sum_{t|n}\chi_{8_2}(t) \\ &= \sum_{1\leqslant n\leqslant N} n^{-1}\chi(n)\sum_{t|n}\chi_{8_2}(t) + \int_{N}^{\infty}\frac{B(y,\chi)}{y^2}\,\mathrm{d}y. \end{aligned}$$

Hence, we can have

$$\begin{aligned} &\sum_{\chi(-1)=-1}^{*} \chi(2^m)L^2(1,\bar\chi)L(1,\chi)L(1,\chi\chi_{8_2}) \\ &= \sum_{\chi(-1)=-1}^{*} \chi(2^m)\biggl(\sum_{1\leqslant n_1\leqslant 2^m N}\frac{\bar\chi(n_1)\tau(n_1)}{n_1} + \int_{2^m N}^{\infty}\frac{A(y,\bar\chi)}{y^2}\,\mathrm{d}y\biggr) \\ &\quad\times\biggl(\sum_{1\leqslant n_2\leqslant N} n_2^{-1}\chi(n_2)\sum_{t|n_2}\chi_{8_2}(t) + \int_{N}^{\infty}\frac{B(y,\chi)}{y^2}\,\mathrm{d}y\biggr) \end{aligned}$$

$$
\begin{aligned}
&= \sideset{}{^*}\sum_{\chi(-1)=-1} \chi(2^m)\Big(\sum_{1\leqslant n_1\leqslant 2^m N}\frac{\bar\chi(n_1)\tau(n_1)}{n_1}\Big)\Big(\sum_{1\leqslant n_2\leqslant N} n_2^{-1}\chi(n_2)\sum_{t|n_2}\chi_{8_2}(t)\Big)\\
&\quad+\sideset{}{^*}\sum_{\chi(-1)=-1} \chi(2^m)\Big(\sum_{1\leqslant n_1\leqslant 2^m N}\frac{\bar\chi(n_1)\tau(n_1)}{n_1}\Big)\Big(\int_N^\infty \frac{B(y,\chi)}{y^2}\,\mathrm{d}y\Big)\\
&\quad+\sideset{}{^*}\sum_{\chi(-1)=-1} \chi(2^m)\Big(\sum_{1\leqslant n_2\leqslant N} n_2^{-1}\chi(n_2)\sum_{t|n_2}\chi_{8_2}(t)\Big)\Big(\int_{2^m N}^\infty \frac{A(y,\bar\chi)}{y^2}\,\mathrm{d}y\Big)\\
&\quad+\sideset{}{^*}\sum_{\chi(-1)=-1} \chi(2^m)\Big(\int_{2^m N}^\infty \frac{A(y,\bar\chi)}{y^2}\,\mathrm{d}y\Big)\Big(\int_N^\infty \frac{B(y,\chi)}{y^2}\,\mathrm{d}y\Big)
\end{aligned}
$$

$$
= M_1 + M_2 + M_3 + M_4. \tag{7}
$$

Now we shall calculate each term in the expression (7).

(i) From Lemma 3 we have

$$
\begin{aligned}
M_1 &= \sideset{}{^*}\sum_{\chi(-1)=-1} \chi(2^m)\Big(\sum_{1\leqslant n_1\leqslant 2^m N}\frac{\bar\chi(n_1)\tau(n_1)}{n_1}\Big)\Big(\sum_{1\leqslant n_2\leqslant N} n_2^{-1}\chi(n_2)\sum_{t|n_2}\chi_{8_2}(t)\Big)\\
&= \frac{1}{2}\sideset{}{'}\sum_{1\leqslant n_1\leqslant 2^m N}\ \sideset{}{'}\sum_{1\leqslant n_2\leqslant N}(n_1n_2)^{-1}\tau(n_1)\sum_{t|n_2}\chi_{8_2}(t)\sum_{d|(q,2^m\bar{n}_1 n_2-1)}\mu\Big(\frac{q}{d}\Big)\varphi(d)\\
&\quad-\frac{1}{2}\sideset{}{'}\sum_{1\leqslant n_1\leqslant 2^m N}\ \sideset{}{'}\sum_{1\leqslant n_2\leqslant N}(n_1n_2)^{-1}\tau(n_1)\sum_{t|n_2}\chi_{8_2}(t)\sum_{d|(q,2^m\bar{n}_1 n_2+1)}\mu\Big(\frac{q}{d}\Big)\varphi(d)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{2}\sum_{d|q}\mu\Big(\frac{q}{d}\Big)\varphi(d)\sideset{}{'}\sum_{\substack{1\leqslant n_1\leqslant 2^m N\\ 2^m n_2\equiv n_1 (\operatorname{mod} d)}}\ \sideset{}{'}\sum_{1\leqslant n_2\leqslant N}(n_1n_2)^{-1}\tau(n_1)\sum_{t|n_2}\chi_{8_2}(t)\\
&\quad-\frac{1}{2}\sum_{d|q}\mu\Big(\frac{q}{d}\Big)\varphi(d)\sideset{}{'}\sum_{\substack{1\leqslant n_1\leqslant 2^m N\\ 2^m n_2\equiv -n_1 (\operatorname{mod} d)}}\ \sideset{}{'}\sum_{1\leqslant n_2\leqslant N}(n_1n_2)^{-1}\tau(n_1)\sum_{t|n_2}\chi_{8_2}(t),
\end{aligned} \tag{8}
$$

where $\sideset{}{'}\sum_{1\leqslant n\leqslant N}$ denotes the summation over n from 1 to N such that $(n,q)=1$.

For simplicity, we split the sum over n_1 or n_2 into four cases: i) $2^m d\leqslant n_1\leqslant 2^m N$ and $d\leqslant n_2\leqslant N$; ii) $2^m d\leqslant n_1\leqslant 2^m N$ and $1\leqslant n_2\leqslant d-1$; iii) $1\leqslant n_1\leqslant 2^m d-1$ and $d\leqslant n_2\leqslant N$; iv) $1\leqslant n_1\leqslant 2^m d-1$ and $1\leqslant n_2\leqslant d-1$. So we have

$$
\begin{aligned}
&\sum_{d|q}\mu\Big(\frac{q}{d}\Big)\varphi(d)\sideset{}{'}\sum_{\substack{2^m d\leqslant n_1\leqslant 2^m N\\ 2^m n_2\equiv n_1 (\operatorname{mod} d)}}\ \sideset{}{'}\sum_{d\leqslant n_2\leqslant N}(n_1n_2)^{-1}\tau(n_1)\sum_{t|n_2}\chi_{8_2}(t)\\
&\ll \sum_{d|q}\varphi(d)\sum_{2^m\leqslant r_1\leqslant \frac{2^m N}{d}}\ \sum_{1\leqslant r_2\leqslant \frac{N}{d}}\ \sideset{}{'}\sum_{\substack{l_1=1\\ l_2\equiv l_1 (\operatorname{mod} d)}}^{d-1}\sideset{}{'}\sum_{l_2=1}^{d-1}\frac{\tau(r_1d+l_1)\tau(r_2d+l_2)}{(r_1d+l_1)(r_2d+l_2)}
\end{aligned}
$$

$$\ll \sum_{d\mid q}\varphi(d)\sum_{2^m\leqslant r_1\leqslant \frac{2^mN}{d}}\ \sum_{1\leqslant r_2\leqslant \frac{N}{d}}\ {\sum_{l_1=1}^{d-1}}' \frac{[(r_1d+l_1)(r_2d+l_1)]^{\varepsilon}}{(r_1d+l_1)(r_2d+l_1)}$$

$$\ll \sum_{d\mid q}\frac{\varphi(d)}{d}\sum_{2^m\leqslant r_1\leqslant \frac{2^mN}{d}}\ \sum_{1\leqslant r_2\leqslant \frac{N}{d}} \frac{[(r_1d+1)(r_2d+1)]^{\varepsilon}}{r_1r_2} \ll q^{\varepsilon},$$

$$\sum_{d\mid q}\mu\Big(\frac{q}{d}\Big)\varphi(d) \sideset{}{'}\sum_{\substack{2^md\leqslant n_1\leqslant 2^mN\\ 2^mn_2\equiv n_1 (\mathrm{mod}\, d)}}\ \sideset{}{'}\sum_{1\leqslant n_2\leqslant d-1}(n_1n_2)^{-1}\tau(n_1)\sum_{t\mid n_2}\chi_{8_2}(t)$$

$$\ll \sum_{d\mid q}\varphi(d)\sum_{2^m\leqslant r_1\leqslant \frac{2^mN}{d}}\ \sum_{1\leqslant n_2\leqslant d-1}(r_1n_2d)^{\varepsilon-1} \ll q^{\varepsilon},$$

and

$$\sum_{d\mid q}\mu\left(\frac{q}{d}\right)\varphi(d) \sideset{}{'}\sum_{\substack{1\leqslant n_1\leqslant 2^md-1\\ n_2\equiv n_1 (\mathrm{mod}\, d)}}\ \sideset{}{'}\sum_{d\leqslant n_2\leqslant N}(n_1n_2)^{-1}\tau(n_1)\sum_{t\mid n_2}\chi_{8_2}(t)$$

$$\ll \sum_{d\mid q}\varphi(d)\sum_{1\leqslant n_1\leqslant 2^md-1}\ \sum_{1\leqslant r_2\leqslant \frac{N}{d}}(n_1r_2d)^{\varepsilon-1} \ll q^{\varepsilon},$$

where we have used the estimate $\tau(n) \ll n^{\varepsilon}$.

For the case $1 \leqslant n_1 \leqslant 2^md-1$, $1 \leqslant n_2 \leqslant d-1$, the solution of the congruence $2^mn_2 \equiv n_1 (\mathrm{mod}\, d)$ is $2^mn_2 = n_1$. Hence,

$$\sum_{d\mid q}\mu\Big(\frac{q}{d}\Big)\varphi(d) \sideset{}{'}\sum_{\substack{1\leqslant n_1\leqslant 2^md-1\\ 2^mn_2\equiv n_1 (\mathrm{mod}\, d)}}\ \sideset{}{'}\sum_{1\leqslant n_2\leqslant d-1}(n_1n_2)^{-1}\tau(n_1)\sum_{t\mid n_2}\chi_{8_2}(t)$$

$$= \sum_{d\mid q}\mu\Big(\frac{q}{d}\Big)\varphi(d) \sideset{}{'}\sum_{1\leqslant n_2\leqslant d-1} 2^{-m}n_2^{-2}\tau(2^mn_2)\sum_{t\mid n_2}\chi_{8_2}(t)$$

$$= \sum_{d\mid q}\mu\Big(\frac{q}{d}\Big)\varphi(d) \sum_{\substack{n_2=1\\ (n_2,q)=1}}^{\infty} 2^{-m}n_2^{-2}\tau(2^mn_2)\sum_{t\mid n_2}\chi_{8_2}(t) + O(q^{\varepsilon}).$$

So we have

$$\frac{1}{2}\sum_{d\mid q}\mu\Big(\frac{q}{d}\Big)\varphi(d) \sideset{}{'}\sum_{\substack{1\leqslant n_1\leqslant 2^mN\\ 2^mn_2\equiv n_1 (\mathrm{mod}\, d)}}\ \sideset{}{'}\sum_{1\leqslant n_2\leqslant N}(n_1n_2)^{-1}\tau(n_1)\sum_{t\mid n_2}\chi_{8_2}(t)$$
$$= \frac{J(q)}{2^{m+1}}\sum_{\substack{n=1\\ (n,q)=1}}^{\infty} n^{-2}\tau(2^mn)\sum_{t\mid n}\chi_{8_2}(t) + O(q^{\varepsilon}). \tag{9}$$

Similarly, we can also get the estimate

$$\frac{1}{2}\sum_{d|q}\mu\Big(\frac{q}{d}\Big)\varphi(d)\sideset{}{'}\sum_{\substack{1\leqslant n_1\leqslant 2^m N\\ 2^m n_2\equiv -n_1 (\operatorname{mod} d)}}\sideset{}{'}\sum_{1\leqslant n_2\leqslant N}(n_1n_2)^{-1}\tau(n_1)\sum_{t|n_2}\chi_{8_2}(t)\ll q^{\varepsilon}. \tag{10}$$

Then from (8), (9) and (10), we have

$$M_1=\frac{J(q)}{2^{m+1}}\sum_{\substack{n=1\\(n,q)=1}}^{\infty} n^{-2}\tau(2^m n)\sum_{t|n}\chi_{8_2}(t)+O(q^{\varepsilon}). \tag{11}$$

(ii) From Lemma 2 of [9], we have the estimate

$$\sum_{\chi\neq\chi_0}|A(y,\bar{\chi})|^2\ll y^{1+\varepsilon}\varphi^2(q) \qquad \text{and} \qquad \sum_{\chi\neq\chi_0}|B(y,\chi)|^2\ll y^{1+\varepsilon}\varphi^2(q),$$

where χ_0 denotes the principal character modulo q. Then from the Cauchy inequality we can easily get

$$\sum_{\chi(-1)=-1}|A(y,\bar{\chi})|\ll\sum_{\chi\neq\chi_0}|A(y,\bar{\chi})|\ll y^{1/2+\varepsilon}q^{3/2}$$

and

$$\sum_{\chi(-1)=-1}|B(y,\chi)|\ll\sum_{\chi\neq\chi_0}|B(y,\chi)|\ll y^{1/2+\varepsilon}q^{3/2}.$$

Using these estimates we have

$$\begin{aligned} M_2&=\sideset{}{^*}\sum_{\chi(-1)=-1}\Bigg(\sum_{1\leqslant n_1\leqslant 2^m N}\frac{\bar{\chi}(n_1)\tau(n_1)}{n_1}\Bigg)\Bigg(\int_N^{\infty}\frac{B(y,\chi)}{y^2}\,\mathrm{d}y\Bigg)\\ &\ll\sum_{1\leqslant n_1\leqslant 2^m N}n_1^{\varepsilon-1}\int_N^{\infty}\frac{1}{y^2}\Bigg(\sum_{\chi(-1)=-1}|B(y,\chi)|\Bigg)\,\mathrm{d}y\\ &\ll N^{\varepsilon}\int_N^{\infty}\frac{q^{3/2}y^{1/2+\varepsilon_1}}{y^2}\,\mathrm{d}y\ll\frac{q^{3/2}}{N^{\frac{1}{2}-\varepsilon}}. \end{aligned} \tag{12}$$

(iii) Similar by to (ii), we can also get

$$M_3\ll\frac{q^{3/2}}{N^{\frac{1}{2}-\varepsilon}}. \tag{13}$$

(iv) By the same argument as in (ii), and noting the absolute convergence of the integrals, we can write

$$
\begin{aligned}
(14)\quad M_4 &= \sideset{}{^*}\sum_{\chi(-1)=-1} \left(\int_{2^m N}^{\infty} \frac{A(y,\bar{\chi})}{y^2}\,\mathrm{d}y\right)\left(\int_{N}^{\infty} \frac{B(y,\chi)}{y^2}\,\mathrm{d}y\right) \\
&\leqslant \int_{2^m N}^{\infty}\int_{N}^{\infty} \frac{1}{y^2 z^2} \sideset{}{^*}\sum_{\chi(-1)=-1} |A(y,\bar{\chi})||B(z,\chi)|\,\mathrm{d}y\,\mathrm{d}z \\
&\ll \int_{2^m N}^{\infty} \frac{1}{y^2}\int_{N}^{\infty} \frac{1}{z^2}\Big(\sum_{\chi\neq\chi_0} |A(y,\bar{\chi})|^2\Big)^{\frac12}\Big(\sum_{\chi\neq\chi_0} |B(z,\chi)|^2\Big)^{\frac12}\,\mathrm{d}y\,\mathrm{d}z \\
&\ll \int_{2^m N}^{\infty} \frac{1}{y^2}\Big(\sum_{\chi\neq\chi_0} |A(y,\bar{\chi})|^2\Big)^{\frac12}\,\mathrm{d}y \times \int_{N}^{\infty} \frac{1}{z^2}\Big(\sum_{\chi\neq\chi_0} |B(z,\chi)|^2\Big)^{\frac12}\,\mathrm{d}z \\
&\ll \int_{2^m N}^{\infty} \frac{\varphi(q)}{y^{3/2-\varepsilon}}\,\mathrm{d}y \times \int_{N}^{\infty} \frac{\varphi(q)}{z^{3/2-\varepsilon}} dz \ll \frac{\varphi^2(q)}{N^{1-\varepsilon}}.
\end{aligned}
$$

Now taking $N = q^4$ and $\varepsilon < \frac12$, combining (7) and (11)–(14) we obtain the asymptotic formula

$$
\sideset{}{^*}\sum_{\chi(-1)=-1} \chi(2^m)L^2(1,\bar{\chi})L(1,\chi)L(1,\chi\chi_{8_2}) = \frac{J(q)}{2^{m+1}} \sideset{}{'}\sum_{n=1}^{\infty} n^{-2}\tau(2^m n)\sum_{t\mid n}\chi_{8_2}(t) + O(q^{\varepsilon}).
$$

This completes the proof of the first formula of Lemma 6. □

Lemma 7. *Let $q > 8$ be any odd integer, and χ the Dirichlet character modulo $q, m \geqslant 0$. Then we have the following asymptotic formulae:*

$$
\begin{aligned}
&\sideset{}{'}\sum_{n=1}^{\infty} n^{-2}\tau(2^m n)\sum_{t\mid n}\chi_{8_2}(t) \\
&= \frac{(3m+4)\pi^4}{144} \prod_{\substack{p\nmid q\\ p\equiv 1,7(\bmod 8)}} \Big(1+\frac{2}{p^2-1}\Big) \prod_{\substack{p\nmid q\\ p\equiv 3,5(\bmod 8)}} \frac{p^4+1}{(p^2+1)^2} \prod_{p\mid q}\Big(1-\frac{1}{p^2}\Big)^2, \\
&\sideset{}{'}\sum_{n=1}^{\infty} \frac{\tau(n)\sum\limits_{t\mid 2^m n}\chi_{8_2}(t)}{n^2} \\
&= \frac{\pi^4}{36} \prod_{\substack{p\nmid q\\ p\equiv 1,7(\bmod 8)}} \Big(1+\frac{2}{p^2-1}\Big) \prod_{\substack{p\nmid q\\ p\equiv 3,5(\bmod 8)}} \frac{p^4+1}{(p^2+1)^2} \prod_{p\mid q}\Big(1-\frac{1}{p^2}\Big)^2,
\end{aligned}
$$

$$\sum_{n=1}^{\infty}{}' n^{-2}\chi_{8_2}(n)\tau(n)\sum_{t\mid 2^m n}\chi_{8_2}(t)$$
$$= \frac{\pi^4}{64}\prod_{\substack{p\nmid q\\ p\equiv 1,7 (\mathrm{mod}\, 8)}}\Big(1+\frac{2}{p^2-1}\Big)\prod_{\substack{p\nmid q\\ p\equiv 3,5 (\mathrm{mod}\, 8)}}\frac{p^4+1}{(p^2+1)^2}\prod_{p\mid q}\Big(1-\frac{1}{p^2}\Big)^2,$$

$$\sum_{n=1}^{\infty}{}' \frac{\tau(2^m n)\chi_{8_2}(n)\tau(n)}{n^2}$$
$$= \frac{(m+1)\pi^4}{64}\prod_{\substack{p\nmid q\\ p\equiv 1,7 (\mathrm{mod}\, 8)}}\Big(1+\frac{2}{p^2-1}\Big)\prod_{\substack{p\nmid q\\ p\equiv 3,5 (\mathrm{mod}\, 8)}}\Big(\frac{p^2-1}{p^2+1}\Big)^3\prod_{p\mid q}\Big(1-\frac{1}{p^2}\Big)^2,$$

$$\sum_{n=1}^{\infty}{}' n^{-2}\sum_{t\mid 2^m n}\chi_{8_2}(t)\sum_{t\mid n}\chi_{8_2}(t)$$
$$= \frac{\pi^4}{48}\prod_{\substack{p\nmid q\\ p\equiv 1,7 (\mathrm{mod}\, 8)}}\Big(1+\frac{2}{p^2-1}\Big)\prod_{\substack{p\nmid q\\ p\equiv 3,5 (\mathrm{mod}\, 8)}}\frac{p^2-1}{p^2+1}\prod_{p\mid q}\Big(1-\frac{1}{p^2}\Big)^2,$$

$$\sum_{n=1}^{\infty}{}' n^{-2}\chi_{8_2}(n)\tau(2^m n)\sum_{t\mid n}\chi_{8_2}(t)$$
$$= \frac{(m+1)\pi^4}{64}\prod_{\substack{p\nmid q\\ p\equiv 1,7 (\mathrm{mod}\, 8)}}\Big(1+\frac{2}{p^2-1}\Big)\prod_{\substack{p\nmid q\\ p\equiv 3,5 (\mathrm{mod}\, 8)}}\frac{p^4+1}{(p^2+1)^2}\prod_{p\mid q}\Big(1-\frac{1}{p^2}\Big)^2.$$

Proof. Noting that $\tau(n)$ is a multiplicative function, we can write

$$(15)\quad \sum_{n=1}^{\infty}{}' n^{-2}\tau(2^m n)\sum_{t\mid n}\chi_{8_2}(t)$$
$$= \tau(2^m)\sum_{\substack{n=1\\ 2\nmid n}}^{\infty}{}' n^{-2}\tau(n)\sum_{t\mid n}\chi_{8_2}(t) + \sum_{\substack{n=1\\ 2\mid n}}^{\infty}{}' n^{-2}\tau(2^m n)\sum_{t\mid n}\chi_{8_2}(t)$$
$$= (m+1)\sum_{\substack{n=1\\ 2\nmid n}}^{\infty}{}' n^{-2}\tau(n)\sum_{t\mid n}\chi_{8_2}(t) + \sum_{\substack{r=1\\ 2\nmid r}}^{\infty}{}'\sum_{j=1}^{\infty}(r\cdot 2^j)^{-2}\tau(2^{m+j}r)\sum_{t\mid 2^j r}\chi_{8_2}(t)$$
$$= \Big(m+1+\sum_{j=1}^{\infty}\frac{m+j+1}{4^j}\Big)\sum_{\substack{n=1\\ 2\nmid n}}^{\infty}{}' n^{-2}\tau(n)\sum_{t\mid n}\chi_{8_2}(t).$$

By using the Euler product formula we can write

$$(16)\quad {\sum_{\substack{n=1\\ 2\nmid n}}^{\infty}}' n^{-2}\tau(n)\sum_{t|n}\chi_{8_2}(t)$$

$$= \prod_{\substack{p\nmid q\\ p\neq 2}}\left(1+\frac{1}{p^2}\tau(p)\sum_{t|p}\chi_{8_2}(t)+\frac{1}{p^4}\tau(p^2)\sum_{t|p^2}\chi_{8_2}(t)+\dots\right)$$

$$= \prod_{\substack{p\nmid q\\ p\equiv 1,7 (\mathrm{mod}\, 8)}}\left(1+\frac{2^2}{p^2}+\frac{3^2}{p^4}+\dots\right)\prod_{\substack{p\nmid q\\ p\equiv 3,5 (\mathrm{mod}\, 8)}}\left(1+\frac{3}{p^4}+\frac{5}{p^8}+\frac{7}{p^{12}}+\dots\right)$$

$$= \prod_{\substack{p\nmid q\\ p\equiv 1,7 (\mathrm{mod}\, 8)}}\left(1-\frac{1}{p^2}\right)^{-2}\left(1+\frac{2}{p^2-1}\right)\prod_{\substack{p\nmid q\\ p\equiv 3,5 (\mathrm{mod}\, 8)}}\left(1-\frac{1}{p^4}\right)^{-2}\left(1+\frac{1}{p^4}\right)$$

$$= \frac{9}{16}\zeta^2(2)\prod_{\substack{p\nmid q\\ p\equiv 1,7 (\mathrm{mod}\, 8)}}\left(1+\frac{2}{p^2-1}\right)\prod_{\substack{p\nmid q\\ p\equiv 3,5 (\mathrm{mod}\, 8)}}\frac{1+1/p^4}{(1+1/p^2)^2}\prod_{p|q}\left(1-\frac{1}{p^2}\right)^2.$$

So from (15), (16) we have

$$ {\sum_{n=1}^{\infty}}' n^{-2}\tau(2^m n)\sum_{t|n}\chi_{8_2}(t)$$

$$= \frac{(3m+4)\pi^4}{144}\prod_{\substack{p\nmid q\\ p\equiv 1,7 (\mathrm{mod}\, 8)}}\left(1+\frac{2}{p^2-1}\right)\prod_{\substack{p\nmid q\\ p\equiv 3,5 (\mathrm{mod}\, 8)}}\frac{p^4+1}{(p^2+1)^2}\prod_{p|q}\left(1-\frac{1}{p^2}\right)^2.$$

This proves the first formula. By the same method, one can obtain the other formulae. □

3. Proof of the theorem

In this section, we will complete the proof of the theorem. From Lemma 2, we can write

$$(17)\quad \sum_{a=1}^{[\frac{q}{2}]}(-1)^a\chi(a) = 2\sum_{a=1,2|a}^{[\frac{q}{2}]}\chi(a)-\sum_{a=1}^{[\frac{q}{2}]}\chi(a) = 2\chi(2)\sum_{a=1}^{[\frac{q}{4}]}\chi(a)-\sum_{a=1}^{[\frac{q}{2}]}\chi(a)$$

$$= \begin{cases} 2\chi(2)\dfrac{\tau(\chi)}{\pi}L(1,\bar{\chi}\chi_4) & \text{if } \chi(-1)=1\ ,\\[2ex] (2\chi(2)-1)\dfrac{\tau(\chi)}{\pi \mathrm{i}}L(1,\bar{\chi}) & \text{if } \chi(-1)=-1,\end{cases}$$

and

$$(18)\qquad \sum_{a=1}^{[\frac{q}{4}]}(-1)^a\chi(a) = 2\sum_{a=1,2|a}^{[\frac{q}{4}]}\chi(a) - \sum_{a=1}^{[\frac{q}{4}]}\chi(a)$$

$$= 2\chi(2)\sum_{a=1}^{[\frac{q}{8}]}\chi(a) - \sum_{a=1}^{[\frac{q}{4}]}\chi(a)$$

$$= \begin{cases} \sqrt{2}\chi(2)\dfrac{\tau(\chi)}{\pi}L(1,\bar{\chi}\chi_{8_1}) & \text{if } \chi(-1)=1, \\ \dfrac{\tau(\chi)}{\pi i}((2\chi(2)-1)L(1,\bar{\chi}) - \sqrt{2}\chi(2)L(1,\bar{\chi}\chi_{8_2})) & \text{if } \chi(-1)=-1. \end{cases}$$

First we prove Theorem 2. Noting that χ is a primitive character modulo q, we can get $|\tau(\chi)| = \sqrt{q}$. So from (17) and Lemma 5, we can obtain

$$\sum_{\chi(-1)=-1}^{*}\left|\sum_{a<\frac{q}{2}}(-1)^a\chi(a)\right|^4$$

$$= \sum_{\chi(-1)=-1}^{*}\left|(2\chi(2)-1)\frac{\tau(\chi)}{\pi}L(1,\bar{\chi})\right|^4$$

$$= \frac{q^2}{\pi^4}\sum_{\chi(-1)=-1}^{*}|2\chi(2)-1|^4|L(1,\bar{\chi})|^4$$

$$= \frac{q^2}{\pi^4}\sum_{\chi(-1)=-1}^{*}(33+4\chi(4)+4\bar{\chi}(4)-20\chi(2)-20\bar{\chi}(2))\,|L(1,\bar{\chi})|^4$$

$$= \frac{3J(q)q^2}{16}\prod_{p|q}\frac{(p^2-1)^3}{p^4(p^2+1)} + O\left(q^{2+\varepsilon}\right).$$

By formula (18) we can also write

$$\sum_{\chi(-1)=-1}^{*}\left|\sum_{a<\frac{q}{4}}(-1)^a\chi(a)\right|^4$$

$$= \sum_{\chi(-1)=-1}^{*}\left|\frac{\tau(\chi)}{\pi i}((2\chi(2)-1)L(1,\bar{\chi}) - \sqrt{2}\chi(2)L(1,\bar{\chi}\chi_{8_2}))\right|^4$$

$$
\begin{aligned}
&= \frac{q^2}{\pi^4} \sideset{}{^*}\sum_{\chi(-1)=-1} (33 + 4\chi(4) + 4\bar{\chi}(4) - 20\chi(2) - 20\bar{\chi}(2))\, |L(1,\bar{\chi})|^4 \\
&\quad + \frac{q^2}{\pi^4} \sideset{}{^*}\sum_{\chi(-1)=-1} (8 + 2\chi(4) - 8\chi(2)) L^2(1,\chi) L^2(1,\bar{\chi}\chi_{8_2}) \\
&\quad + \frac{q^2}{\pi^4} \sideset{}{^*}\sum_{\chi(-1)=-1} (8 + 2\bar{\chi}(4) - 8\bar{\chi}(2)) L^2(1,\bar{\chi}) L^2(1,\chi\chi_{8_2}) \\
&\quad - \frac{2\sqrt{2}q^2}{\pi^4} \sideset{}{^*}\sum_{\chi(-1)=-1} (12 - 9\chi(2) - 4\bar{\chi}(2) + 2\chi(4)) L^2(1,\chi) L(1,\bar{\chi}\chi_{8_2}) L(1,\bar{\chi}) \\
&\quad - \frac{2\sqrt{2}q^2}{\pi^4} \sideset{}{^*}\sum_{\chi(-1)=-1} (12 - 9\bar{\chi}(2) - 4\chi(2) + 2\bar{\chi}(4)) L^2(1,\bar{\chi}) L(1,\chi\chi_{8_2}) L(1,\chi) \\
&\quad + \frac{8q^2}{\pi^4} (5 - 2\bar{\chi}(2) - 2\chi(2)) |L(1,\bar{\chi})|^2 |L(1,\chi\bar{\chi}_{8_2})|^2 \\
&\quad - \frac{4\sqrt{2}q^2}{\pi^4} \sideset{}{^*}\sum_{\chi(-1)=-1} (2 - \bar{\chi}(2)) L(1,\bar{\chi}) L^2(1,\chi\chi_{8_2}) L(1,\bar{\chi}\chi_{8_2}) \\
&\quad - \frac{4\sqrt{2}q^2}{\pi^4} \sideset{}{^*}\sum_{\chi(-1)=-1} (2 - \chi(2)) L(1,\chi) L^2(1,\bar{\chi}\chi_{8_2}) L(1,\chi\chi_{8_2}) \\
&\quad + \frac{4q^2}{\pi^4} \sideset{}{^*}\sum_{\chi(-1)=-1} |L(1,\bar{\chi}\chi_{8_2})|^4 .
\end{aligned}
$$

By Lemma 5, Lemma 6 and Lemma 7, we can obtain

$$
\begin{aligned}
\sideset{}{^*}\sum_{\chi(-1)=-1} \left| \sum_{a<\frac{q}{4}} (-1)^a \chi(a) \right|^4 &= \frac{15J(q)q^2}{64} \prod_{p|q} \frac{(p^2-1)^3}{p^4(p^2+1)} \\
&\quad - \frac{3\sqrt{2}J(q)q^2}{8} \prod_{\substack{p\nmid q \\ p\equiv 1,7 \pmod 8}} \left(1 + \frac{2}{p^2-1}\right) \prod_{\substack{p\nmid q \\ p\equiv 3,5 \pmod 8}} \frac{p^4+1}{(p^2+1)^2} \prod_{p|q} \left(1 - \frac{1}{p^2}\right)^2 \\
&\quad + \frac{J(q)q^2}{4} \prod_{\substack{p\nmid q \\ p\equiv 1,7 \pmod 8}} \left(1 + \frac{2}{p^2-1}\right) \prod_{\substack{p\nmid q \\ p\equiv 3,5 \pmod 8}} \frac{p^2-1}{p^2+1} \prod_{p|q} \left(1 - \frac{1}{p^2}\right)^2 \\
&\quad + \frac{J(q)q^2}{8} \prod_{\substack{p\nmid q \\ p\equiv 1,7 \pmod 8}} \left(1 + \frac{2}{p^2-1}\right) \prod_{\substack{p\nmid q \\ p\equiv 3,5 \pmod 8}} \left(\frac{p^2-1}{p^2+1}\right)^3 \prod_{p|q} \left(1 - \frac{1}{p^2}\right)^2 + O(q^{2+\varepsilon}).
\end{aligned}
$$

Thus we complete the proof of Theorem 2. Similarly, combining (17), (18) and Lemma 5, we can use the same method to deduce Theorem 1.

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Authors' addresses: Ren Ganglian, Department of Mathematics, Xianyang normal university, Xianyang , Shaanxi,712000,China, e-mail: `rgl70718@sina.com`; `rgl70819@hotmail.com` Zhang Wenpeng, Department of Mathematics, Northwest University, Xi'an, Shaanxi, P.R. China, e-mail: `wpzhang@nwu.edu.cn`.