Ján Jakubík
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AFFINE COMPLETENESS AND WREATH PRODUCT DECOMPOSITIONS OF LATTICE ORDERED GROUP

Ján Jakubík, Košice

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Abstract. Let Δ and H be a nonzero abelian linearly ordered group or a nonzero abelian lattice ordered group, respectively. In this paper we prove that the wreath product of Δ and H fails to be affine complete.

Keywords: lattice ordered group, wreath product, affine completeness

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1. Introduction

Affine completeness of algebraic structures was investigated in the monograph [6] by Kaarli and Pixley. A problem proposed in this monograph (and formulated also earlier in [2]) asks whether there exists a lattice ordered group $G \neq \{0\}$ which is affine complete; this problem remains open.

Some negative results in this direction (dealing with sufficient conditions under which G is not affine complete) were proved by Kaarli and Pixley [6], by Csontóová and the author [5] and by the author [2], [3], [4]. Cf. also Section 5 below.

In the present paper we prove

(∗) Assume that a lattice ordered group G can be represented as a wreath product of a nonzero abelian linearly ordered group and a nonzero abelian lattice ordered group. Then G is not affine complete.

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2. Preliminaries

For lattice ordered groups we apply the notation as in Conrad [1] (with some minor modifications). In particular, the group operation is always written additively, though it is not assumed to be commutative.

Let G be a lattice ordered group and let $P(G)$ be the set of all polynomials over G. If for each mapping $f\colon G^n \to G$ such that $n \in \mathbb{N}$ and f is compatible with all congruence relations on G the relation $f \in P(G)$ is valid then G is called affine complete.

We recall the definition of the wreath product (cf., e.g., [1]).

Let H be a lattice ordered group and let Δ be a linearly ordered group. For each $\delta \in \Delta$ let $G_\delta = H$. Consider the set-theoretical direct product

$$D = \Delta \times \prod_{\delta \in \Delta} G_\delta.$$

Suppose that

$$d_1 = (\alpha; \ldots, a_\delta, \ldots)_{\delta \in \Delta}, \quad d_2 = (\beta; \ldots, b_\delta, \ldots)_{\delta \in \Delta}$$

are elements of D. We define the operation $+$ on D by putting

$$d_1 + d_2 = (\alpha + \beta; c_\delta, \ldots)_{\delta \in \Delta}, \quad c_\delta = a_{\delta - \beta} + b_\delta.$$

Then $(D; +)$ is a group. The partial order on D is defined by putting $d_1 \geqslant 0$ if either $\alpha > 0$, or $\alpha = 0$ and $a_\delta \geqslant 0$ for each $\delta \in \Delta$. We obtain a lattice ordered group $(D; +, \leqslant)$ which will be denoted by $\Delta W H$. We say that this lattice ordered group is a wreath product of Δ and of H.

In what follows we assume that both Δ and H are nonzero and abelian.

3. Auxiliary results

Assume that G is a nonzero lattice ordered group. Let $p(x)$ be a polynomial over G with one variable x. It is well-known that then there exists a finite subset C of G such that $p(x)$ can be expressed in the form

$$p(x) = \bigwedge_{i \in I} \bigvee_{j \in J(i)} a_{ij}, \quad a_{ij} = \sum_{t \in T(i,j)} b_t^{ij}, \tag{1}$$

where $I \neq \emptyset$ is a finite set, $J(i) \neq \emptyset$ is a finite set for each $i \in I$, $T(i,j) \neq \emptyset$ is a finite set for each $i \in I$ and each $j \in J(i)$, and for each $i \in I$, $j \in J(i)$, $t \in T(i,j)$ we have either $b_t \in C$ or $b_t \in \{x, -x\}$.

Let D be as in Section 2 and let $d_1 = (\alpha; \ldots, a_\delta, \ldots)_{\delta\in\Delta}$ be an element of D. We denote

$$d_1^0 = (\alpha; \ldots, a_\delta^0 \ldots)_{\delta\in\Delta},$$

where $a_\delta^0 = 0$ for each $\delta \in \Delta$.

Further, we put

$$D^0 = \{d_1 \in D\colon\ d_1^0 = 0\},$$
$$d_1(\Delta) = \alpha, \quad d_1(G_\delta) = a_\delta \quad \text{for each } \delta \in \Delta.$$

For each $d_1 \in D$ we set

$$f(d_1) = d_1^0.$$

Let ϱ be a congruence relation on D and $d \in D$. We put $\varrho(d) = \{d' \in D\colon\ d\varrho d'\}$.

Lemma 3.1. *Let $d_1, d_2 \in D$, $d_1 \varrho d_2$. Then $f(d_1)\varrho f(d_2)$.*

Proof. For d_1 and d_2 we apply the notation as in Section 2. If $\alpha = \beta$, then $f(d_1) = f(d_2)$, whence $f(d_1)\varrho f(d_2)$.

Assume that $\alpha \neq \beta$. Then without loss of generality we can suppose that $\alpha < \beta$. Put $d_3 = d_2 - d_1$. We get $0\varrho d_3$ and $0 \leqslant |d_4| < d_3$ for each $d_4 \in D^0$. Hence $0\varrho d_4$. This yields $d_1^0 \varrho d_1$ and $d_2^0 \varrho d_2$. Thus $d_1^0 \varrho d_2^0$; hence $f(d_1)\varrho f(d_2)$. □

We have proved that the mapping f is compatible with all congruence relations on D. Thus in order to prove the assertion $(*)$ from Section 1 it remains to show that $f(x)$ does not belong to $P(G)$.

From the definition of the partial order in D we immediately obtain (under the notation as in Section 2)

Lemma 3.2. *If $\alpha < \beta$, then $d_1 \vee d_2 = d_2$. If $\alpha = \beta$, then $d_1 \vee d_2 = d'$, where $d' = (\alpha; \ldots, a_\delta \vee b_\delta, \ldots)_{\delta\in\Delta}$.*

The analogous result holds for $d_1 \wedge d_2$.

Let $p(x)$ and C be as above. For $d \in D$, the meanings of the expressions $p(d)$ and $a_{ij}(d)$ are obvious.

Lemma 3.3. *Let h be any element of H. There exists $d_0 \in D$ such that $d_0(\Delta) > 0$, $d_0(G_\delta) = h$ for each $\delta \in \Delta$ and*

$$d_0 > \sum_{i\in I,\, j\in J_i,\, t\in T^0(i,j)} b_t^{ij},$$

where $T^0(i,j)$ is the set of those $t \in T(i,j)$, for which the element b_t^{ij} belongs to C.

Proof. This is a consequence of the fact that the sets I, $J(i)$ and $T(i,j)$ are finite and that the linearly ordered group Δ is nonzero. □

We will deal with the element $f(d_0)$ of D. Below, in Section 4, we will apply specific conditions for choosing in an appropriate way the corresponding element h of G.

Again, let $p(x)$ be as above and let $i \in I$, $j \in J_i$. We denote by n_{ij}^1 and n_{ij}^2 the number of those $t \in T_{ij}$ for which we have $b_t^{ij} = x$ or $b_t^{ij} = -x$, respectively. Put $n_{ij} = n_{ij}^1 - n_{ij}^2$.

Lemma 3.4. *Let us apply the notation as above. Put $d_0(\Delta) = \alpha_0$. Then we have*

(i) $(a_{ij}(d_0))(\Delta) = n_{ij}\alpha_0 + \sum\limits_{t \in T^0(i,j)} b_t^{ij}(\Delta)$;

(ii) *for each $\delta \in \Delta$,*

$$(a_{ij}(d_0))(G_\delta) = n_{ij}h + c_{ij}^\delta,$$

where c_{ij}^δ is an element of C which is uniquely determined by a_{ij} and does not depend on the choice of h.

Proof. This is a consequence of the definition of the operation + in D and of the fact that Δ and H are abelian. □

4. Proof of (∗)

In proving (∗) we proceed by way of contradiction. Let $f(x)$ be as above. In view of 3.1, we have to prove that $f(x)$ does not belong to $P(D)$.

Suppose that there is $p(x) \in P(D)$ such that $p(x_0) = f(x_0)$ for each $x_0 \in D$. For $p(x)$, we apply the notation as above.

Let d_0 be as in Section 3.

Lemma 4.1. *Let $i_0 \in I$. Then there exists $j \in J_{i_0}$ such that $n_{i_0 j} \geqslant 1$.*

Proof. By way of contradiction, assume that $n_{i_0 j} < 1$ for each $j \in J_{i_0}$. Consider the element d_0^0 (cf. Section 3). Then in view of 3.3 we have $a_{i_0 j}(d_0) < d_0^0$ for each $j \in J_{i_0}$. By applying 3.2 we conclude that

$$\bigvee_{j \in J_{i_0}} a_{i_0 j}(d_0) < d_0^0.$$

This yields $p(d_0) < d_0^0 = f(d_0)$, which is a contradiction. □

Let $i \in I$. Put $J_i^0 = \{j \in J_i \colon n_{ij} \geqslant 1\}$. In view of 4.1 we have $J_i^0 \neq \emptyset$. Moreover, 3.2 yields

$$\bigvee_{j \in J_i} a_{ij}(d_0) = \bigvee_{j \in J_i^0} a_{ij}(d_0). \tag{1}$$

Let us denote this element by $\overline{a}_i(d_0)$. Hence

$$p(d_0) = \bigwedge_{i \in I} \overline{a}_i(d_0). \tag{2}$$

Denote $m_i = \max\{n_{ij}\}_{j \in J_i^0}$. Hence $m_i \geqslant 1$. Further, we put

$$J_i^{0m} = \{j \in J_i^0 : n_{ij} = m_i\}.$$

According to 3.2 we obtain

$$(\overline{a}_i(d_0))(\Delta) = m_i, \tag{3}$$

$$(\overline{a}_i(d_0))(G_\delta) = \bigvee_{j \in J_i^{0m}} a_{ij}(G_\delta). \tag{4}$$

Lemma 4.2. *Let $0 < k \in H$ and $\delta_0 \in \Delta$. There exists $h \in H$ such that $(a_{ij}(d_0))(G_{\delta_0}) \geqslant k$ for each $i \in I$ and each $j \in J_i^{0m}$.*

Proof. Let $i \in I$ and $j \in J_i^{0m}$. Then $n_{ij} \geqslant 1$. Let $c_{ij}^{\delta_0}$ be as in 3.4 (ii). Since the sets I and J_i are finite and $H \neq \{0\}$ there exists $h \in H$ such that

$$h \geqslant k - c_{ij}^{\delta_0}$$

for each $i \in I$ and $j \in J_i$; for such i and j we then have $h + c_{ij}^{\delta_0} \geqslant k$. In particular, if $j \in J_i^{0m}$, then $n_{ij}h + c_{ij}^{\delta_0} \geqslant h + c_{ij}^{\delta_0} \geqslant k$. □

In what follows let h be as in 4.2. Then according to (4) we obtain

$$(\overline{a}_i(d_0))(G_{\delta_0}) \geqslant k. \tag{5}$$

Now from the result analogous to 3.2 concerning the operation $\wedge$ and by applying (2), (5) we get

$$(p(d_0))(G_{\delta_0}) \geqslant k.$$

On the other hand, we have $f(d_0) = d_0^0$ and $d_0^0(G_\delta) = 0$ for each $\delta \in A$. Therefore $f(d_0) \neq p(d_0)$ and we arrived at a contradiction, concluding the proof of the assertion $(*)$.

5. On the relation between $(*)$ and the results of [2]–[6]

We denote by $\mathcal{C}_w$ the class of all nonzero lattice ordered groups which can be represented as a nontrivial wreath product.

Assume that G is a nonzero lattice ordered group; the following conditions are sufficient for G nit to be affine complete:

(a_1) G is complete. (Cf. [2].)

(a_2) G is abelian and projectable. (Cf. [5].)

(a_3) G can be represented as a nontrivial direct product. (Cf. [3].)

(a_4) G is abelian and can be represented as a nontrivial lexicographic product. (Cf. [4].)

(a_5) G can be represented as direct product $A \times B$, where A is a nonzero subdirectly irreducible lattice ordered group and B is any lattice ordered group. (Cf. [6].)

For $i \in \{1,2,3,4\}$ let $\mathcal{C}_i$ be the class of all nonzero lattice ordered groups satisfying the condition (a_i).

Now suppose that G is a lattice ordered group satisfying the assumption of $(*)$. Then G is nonzero. Further, we have

(i) G fails to be complete.

(ii) G fails to be projectable.

(iii) G is directly indecomposable.

Therefore for any lattice ordered group G, the assertion $(*)$ fails to be a consequence of (a_i) for $i = 1,2,3$.

Lemma 5.1. *Let G be as in $(*)$. Then G cannot be represented as a nontrivial lexicographic product.*

Proof. By way of contradiction, assume that G can be represented as a nontrivial lexicographic product. Thus without loss of generality we can suppose that G is a lexicographic product

$$G = \Gamma_{i \in I} K_i,$$

where I is a linearly ordered set having more than one element and all K_i are nonzero lattice ordered groups; moreover, if $i \in I$ and i is not the greatest element of I, then K_i is linearly ordered.

First suppose that I has no greatest element. Then G is linearly ordered. But since G satisfies the assumption of $(*)$ it is not linearly ordered, which is a contradiction. Hence I has the greatest element which will be denoted by i_1.

For each $i \in I$ let $\overline{K}_i$ be the set of all $g \in G$ such that $g(K_j) = 0$ whenever $j \in I$, $j \neq i$. If g_1 is an element of G which is incomparable with 0, then clearly $g_1 \in \overline{K}_{i_1}$. If, moreover, $i \in I$, $i \neq i_1$ and $g_2 \in \overline{K}_i$, then $g_1 + g_2 = g_2 + g_1$.

Since G satisfies the assumption of $(*)$ we can suppose that $G = D$, where D is as above. Choose $\delta_1 \in \Delta$; there exists $d \in D$ such that $d(\Delta) = 0$, $d(G_{\delta_1}) > 0$ and $d(G_\delta) = 0$ if $\delta \in \Delta$, $\delta \neq \delta_1$. Further, there exists $\delta_2 \in \Delta$ with $\delta_2 \neq \delta_1$ and there is $d' \in D$ with the properties analogous to those of d with the distinction that δ_1 is replaced by δ_2. Put $d_1 = d - d'$. Then d_1 is incomparable with 0, whence $d_1 \in \overline{K}_{i_1}$. Further, $d_1 \vee 0 = d$; since $\overline{K}_{i_1}$ is a sublattice of G, we obtain $d \in \overline{K}_{i_1}$.

Since $\Delta \neq \{0\}$, there exists $0 < g_1' \in D$ with $g_1'(\Delta) > 0$. Also, there exists $0 < g_2 \in D$ such that $g_2 > g_1'$ and $g_2 \in \overline{K}_i$ for some $i \neq i_1$. Then from the properties of D we infer that $g_1 + g_2 \neq g_2 + g_1$; we arrived at a contradiction. $\square$

Hence we have $\mathcal{C}_w \cap \mathcal{C}_4 = \emptyset$. Therefore for any lattice ordered group G, the assertion $(*)$ cannot be obtained as a consequence of (a_4).

The following example shows that a lattice ordered group satisfying the assumptions of $(*)$ can be subdirectly reducible. Let Z be the additive group of all integers with the natural linear order. Let $X = Y = \Delta = Z$ and put $G = \Delta W(X \times Y)$. Hence G satisfies the assumption of $(*)$. If $d \in G$ and (by using the notation as above)

$$d = (\alpha; \ldots, a_\delta, \ldots)_{\delta \in \Delta},$$

then $\alpha \in \Delta$ and $a_\delta = (x_\delta, y_\delta)$ with $x_\delta \in X$, $y_\delta \in Y$.

We denote by A_1 the set of all $d \in G$ such $\alpha = 0$ and $y_\delta = 0$ for each $\delta \in \Delta$. Similarly, let A_2 be the set of all $d \in G$ such that $\alpha = 0 = x_\delta$ for each $\delta \in \Delta$. Then both A_1 and A_2 are ℓ-ideals of G. We have $A_1 \cap A_2 = \{0\}$. Moreover, $G/A_1 \neq \{0\} \neq G/A_2$. Thus the lattice ordered group G is subdirectly reducible.

Therefore $(*)$ is not a consequence of (a_5).

References

[1] *P. Conrad*: Lattice Ordered Groups. Tulane University, New Orleans, 1970.

[2] *J. Jakubík*: Affine completeness of complete lattice ordered groups. Czechoslovak Math. J. *45* (1995), 571–576.

[3] *J. Jakubík*: On the affine completeness of lattice ordered groups. Czechoslovak Math. J. *54* (2004), 423–429.

[4] *J. Jakubík*: Affine completeness and lexicographic product decompositions of abelian lattice ordered groups. Czechoslovak Math. J. *55* (2005), 917–922.

[5] *J. Jakubík, M. Csontóová*: Affine completeness of projectable lattice ordered groups. Czechoslovak Math. J. *48* (1998), 359–363.

[5] *K. Kaarli, A. F. Pixley*: Polynomial Completeness in Algebraic Systems. Chapman-Hall, London-New York-Washington, 2000.

Author's address: Ján Jakubík, Matematický ústav SAV, Grešákova 6, 040 01 Košice, Slovakia.