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A NON COMMUTATIVE GENERALIZATION OF $*$-AUTONOMOUS LATTICES

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Abstract. Pseudo $*$-autonomous lattices are non-commutative generalizations of $*$-autonomous lattices. It is proved that the class of pseudo $*$-autonomous lattices is a variety of algebras which is term equivalent to the class of dualizing residuated lattices. It is shown that the kernels of congruences of pseudo $*$-autonomous lattices can be described as their normal ideals.

Keywords: $*$-autonomous lattice, pseudo $*$-autonomous lattice, residuated lattice, ideal, normal ideal, congruence

MSC 2010: 03B47, 03B50, 06D35, 06F05, 06F15

1. Introduction

$*$-autonomous lattices (briefly $*$-lattices) were very intensively studied by F. Paoli in [8], [9], and [10]. They are an algebraic counterpart of the propositional linear logic without exponentials and without additive constants. The class of $*$-lattices contains as proper subclasses many classes of algebras, e.g. the classes of commutative Girard quantales, MV-algebras and Abelian lattice ordered groups.

In the present paper we introduce pseudo $*$-autonomous lattices (briefly pseudo $*$-lattices) which are non-commutative generalizations of $*$-lattices. As special cases of pseudo $*$-autonomous lattices one can view not only all $*$-autonomous lattices but also all (i.e. Abelian and non-Abelian) lattice ordered groups and pseudo MV-algebras (i.e., non-commutative generalizations of MV-algebras [4], [11]).

We describe properties of pseudo $*$-lattices and prove that they form a variety of algebras which is arithmetical. We compare the notion of a pseudo $*$-lattice with that

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of a residuated lattice and prove that the class of pseudo $*$-lattices is term equivalent to the class of dualizing residuated lattices.

Furthermore, ideals and normal ideals of pseudo $*$-lattices are introduced and it is shown that normal ideals are exactly the kernels of congruences of pseudo $*$-lattices.

2. Pseudo $*$-autonomous lattices

Definition 1. A *pseudo $*$-autonomous lattice* (or, briefly, a *pseudo $*$-lattice*) is an algebra $\mathscr{A} = (A, +, 0, ^-, ^\sim, \wedge, \vee)$ of type $\langle 2, 0, 1, 1, 2, 2\rangle$ such that

(P1) $(A, +, 0)$ is a monoid;

(P2) $(A, \wedge, \vee)$ is a lattice;

(P3) for any $x, y \in A$, $x \vee y = (x^- \wedge y^-)^\sim = (x^\sim \wedge y^\sim)^-$;

(P4) for any $x, y \in A$ we have $x \leqslant y$ iff $0^- \leqslant x^- + y$ iff $0^\sim \leqslant y + x^\sim$,

where "$\leqslant$" denotes the induced lattice order of the reduct $(A, \wedge, \vee)$.

Example 1. $*$-autonomous lattices were investigated by F. Paoli in [8], [9] and [10] as algebras $\mathscr{A} = (A, +, -, 0, \wedge, \vee)$ of type $\langle 2, 1, 0, 2, 2\rangle$ such that $(A, +, 0)$ is a commutative monoid, $(A, \wedge, \vee)$ is an involutive lattice and for any $x, y \in A$ we have $x \leqslant y$ iff $-0 \leqslant -x + y$. The $*$-autonomous lattices are algebraic models of linear logic without exponentials and without additive constants. It is easy to check that $*$-autonomous lattices are special cases of pseudo $*$-autonomous lattices where "$+$" is commutative and "$^-$" and "$^\sim$" coincide with "$-$".

Example 2. GMV-algebras (or pseudo MV-algebras) were introduced and studied by the second author in [11] as well as by G. Georgescu and A. Iorgulescu in [4] as non-commutative generalizations of MV-algebras. A GMV-algebra is an algebra $\mathscr{A} = (A, \oplus, \neg, \sim, 0, 1)$ of type $\langle 2, 1, 1, 0, 0\rangle$ such that $(A, \oplus, 0)$ is a monoid and for any $x, y \in A$,

$x \oplus 1 = 1 = 1 \oplus x$;

$\neg 1 = 0 = \sim 1$;

$\neg(\sim x \oplus \sim y) = \sim(\neg x \oplus \neg y)$;

$x \oplus (y \odot \sim x) = y \oplus (x \odot \sim y) = (\neg y \odot x) \oplus y = (\neg x \odot y) \oplus x$;

$(\neg x \oplus y) \odot x = y \odot (x \oplus \sim y)$ (where $x \odot y = \sim(\neg x \oplus \neg y)$);

$\sim \neg x = x$.

If we put $x \leqslant y$ if and only if $\neg x \oplus y = 1$ then "$\leqslant$" is an order on A. Moreover, $(A, \leqslant)$ is a bounded distributive lattice in which 0 is the least and 1 the greatest element in $\mathscr{A}$. Clearly, if we put $x + y := x \oplus y$, $x^- := \neg x$ and $x^\sim := \sim x$, then $(A, +, 0, ^-, ^\sim, \wedge, \vee)$ is a pseudo $*$-autonomous lattice.

By [7], GMV-algebras are an algebraic counterpart of the non-commutative Łukasiewicz propositional logic.

Example 3. Let $\mathscr{G} = (G, +, 0, -, \wedge, \vee)$ be an arbitrary ℓ-group (i.e. $\mathscr{G}$ need not be commutative). Then one can easily verify that $\mathscr{G}$ has the properties of the pseudo $*$-autonomous lattice where "$^-$" and "$^\sim$" coincide with "$-$" and $-0 = 0$.

Lemma 1. *Let $\mathscr{A} = (A, +, 0, ^-, ^\sim, \wedge, \vee)$ be a pseudo $*$-lattice and $x, y, z \in A$. Then the following conditions are satisfied:*

(i) $x^{-\sim} = x^{\sim -} = x$;

(ii) $0^- = 0^\sim$;

(iii) $x \leqslant y + z$ *iff* $z^- \leqslant x^- + y$ *iff* $y^\sim \leqslant z + x^\sim$;

(iv) $x \leqslant y$ *iff* $y^- \leqslant x^-$ *iff* $y^\sim \leqslant x^\sim$;

(v) $(x \vee y)^- = x^- \wedge y^-$, $(x \vee y)^\sim = x^\sim \wedge y^\sim$;

(vi) $(x \wedge y)^- = x^- \vee y^-$, $(x \wedge y)^\sim = x^\sim \vee y^\sim$;

(vii) $x \wedge y = (x^- \vee y^-)^\sim = (x^\sim \vee y^\sim)^-$.

Proof. (i) Due to (P3) we have $x = x \vee x = (x^- \wedge x^-)^\sim = x^{-\sim}$, $x = x \vee x = (x^\sim \wedge x^\sim)^- = x^{\sim -}$.

(ii) According to (P4) and (i) we obtain $0^\sim \leqslant 0^-$ iff $0^- \leqslant 0^{\sim -} + 0^- = 0^-$, i.e. $0^\sim \leqslant 0^-$. Analogously we can show that $0^- \leqslant 0^\sim$; thus $0^- = 0^\sim$.

(iii) $x \leqslant y + z$ iff $0^- \leqslant x^- + (y + z) = (x^- + y) + z^{-\sim}$ iff $z^- \leqslant x^- + y$. Similarly, $x \leqslant y + z$ iff $0^\sim \leqslant (y + z) + x^\sim = y^{\sim -} + (z + x^\sim)$ iff $y^\sim \leqslant z + x^\sim$.

(iv) The assertion follows from (iii) for $y = 0$ and $z = 0$ respectively.

(v) The identities (P3) and (i) yield $(x \vee y)^- = (x^- \wedge y^-)^{\sim -} = x^- \wedge y^-$. Analogously, $(x \vee y)^\sim = x^\sim \wedge y^\sim$.

(vi) Using (i) and (P3) again we get $(x \wedge y)^- = (x^{-\sim} \wedge y^{-\sim})^- = x^- \vee y^-$. Similarly we can prove the second part of the assertion.

(vii) $x \wedge y = (x \wedge y)^{-\sim} = (x^- \vee y^-)^\sim$ by (i) and (vi). □

Definition 2. We introduce the following abbreviations for the pseudo $*$-lattice $\mathscr{A} = (A, +, 0, ^-, ^\sim, \wedge, \vee)$:

$$1 := 0^- = 0^\sim;$$

$$x \to y := x^- + y, \qquad x \rightsquigarrow y := y + x^\sim;$$

$$\neg_1 x := (x + 1)^-, \qquad \neg_2 x := (1 + x)^\sim;$$

$$\sigma_1(x, y) := ((x \to y) \rightsquigarrow 0) \vee 0, \qquad \sigma_2(x, y) := ((x \rightsquigarrow y) \to 0) \vee 0;$$

$$\delta_1(x, y) := \sigma_1(x, y) \vee \sigma_1(y, x), \qquad \delta_2(x, y) := \sigma_2(x, y) \vee \sigma_2(y, x);$$

$$x \rhd_1 y := (x^- + y)^\sim, \qquad x \rhd_2 y := (y + x^\sim)^-.$$

Lemma 2. *Let $\mathscr{A} = (A, +, 0, ^-, ^\sim, \wedge, \vee)$ be a pseudo $*$-lattice and $a, b, c, d \in A$. Then the following conditions are fulfilled:*

(i) $1 \leqslant a \to a$, $1 \leqslant a \rightsquigarrow a$;

(ii) $a \leqslant (a \to b) \rightsquigarrow b$, $a \leqslant (a \rightsquigarrow b) \to b$;

(iii) $(b \to b) \rightsquigarrow a \leqslant a$, $(b \rightsquigarrow b) \to a \leqslant a$;

(iv) $a \leqslant b, c \leqslant d \Rightarrow a + c \leqslant b + d$;

(v) $a + (b \wedge c) = (a + b) \wedge (a + c)$, $(b \wedge c) + a = (b + a) \wedge (c + a)$;

(vi) $(a + b) \vee (a + c) \leqslant a + (b \vee c)$, $(b + a) \vee (c + a) \leqslant (b \vee c) + a$;

(vii) $a \leqslant b \Rightarrow \neg_1 b \leqslant \neg_1 a, \neg_2 b \leqslant \neg_2 a$;

(viii) $\neg_1(a \wedge b) = \neg_1 a \vee \neg_1 b$, $\neg_2(a \wedge b) = \neg_2 a \vee \neg_2 b$;

(ix) $\neg_1(a \vee b) \leqslant \neg_1 a \wedge \neg_1 b$, $\neg_2(a \vee b) \leqslant \neg_2 a \wedge \neg_2 b$;

(x) $0 \leqslant \neg_1 a + a$, $0 \leqslant a + \neg_2 a$;

(xi) $\neg_1 \neg_2 a \leqslant a$, $\neg_2 \neg_1 a \leqslant a$;

(xii) $\sigma_1(a, a) = \sigma_2(a, a) = 0$;

(xiii) $a \leqslant b \Leftrightarrow \sigma_1(a, b) = 0 \Leftrightarrow \sigma_2(a, b) = 0$;

(xiv) $a = b \Leftrightarrow \delta_1(a, b) = \delta_2(a, b) = 0$;

(xv) $a \to b = b^- \rightsquigarrow a^-$, $a \rightsquigarrow b = b^\sim \to a^\sim$.

P r o o f. (i) It follows from (P4) and the reflexivity of "$\leqslant$".

(ii) Due to (i) we have $1 \leqslant (a^- + b) \rightsquigarrow (a^- + b) = (a^- + b) + (a^- + b)^\sim = a^- + (b + (a^- + b)^\sim)$. Hence by (P4) we obtain $a \leqslant b + (a^- + b)^\sim = (a \to b) \rightsquigarrow b$. Analogously, $a \leqslant (b + a^\sim)^- + b = (a \rightsquigarrow b) \to b$.

(iii) Using (ii) and (i) we get $((b \to b) \rightsquigarrow a) \to a \geqslant b \to b \geqslant 1$. Thus $(b \to b) \rightsquigarrow a \leqslant a$ by (P4). Similarly for the second part.

(iv) Suppose that $a \leqslant b$. Then according to Lemma 1 (iv) and Lemma 2 (ii) we have $b^\sim \leqslant a^\sim \leqslant (a^\sim \to c) \rightsquigarrow c$. Further, $a + c \leqslant b + c$ iff $b^\sim \leqslant c + (a + c)^\sim = (a + c) \rightsquigarrow c = (a^{\sim -} + c) \rightsquigarrow c = (a^\sim \to c) \rightsquigarrow c$. Hence $a \leqslant b$ implies $a + c \leqslant b + c$. Analogously, if $c \leqslant d$ then $d^- \leqslant c^- \leqslant (c^- \rightsquigarrow b) \to b$ and since $b + c \leqslant b + d$ iff $d^- \leqslant (b + c) \to b = (c^- \rightsquigarrow b) \to b$ we get that $c \leqslant d$ implies $b + c \leqslant b + d$. Using transitivity the proof is completed.

(v) From $b \wedge c \leqslant b, c$ we obtain by (iv) $a + (b \wedge c) \leqslant (a + b) \wedge (a + c)$. Suppose now that $x \leqslant a + b, a + c$, i.e. $1 \leqslant x^- + a + b$, $1 \leqslant x^- + a + c$. This implies $(x^- + a)^\sim \leqslant b + 1^\sim = b + 0 = b$, $(x^- + a)^\sim \leqslant c + 1^\sim = c$. Thus $(x^- + a)^\sim \leqslant b \wedge c$, which yields $1 \leqslant (x^- + a) + (b \wedge c) = x^- + (a + (b \wedge c))$ and $x \leqslant a + (b \wedge c)$. Altogether we get $a + (b \wedge c) = (a + b) \wedge (a + c)$. Similarly we can prove $(b \wedge c) + a = (b + a) \wedge (c + a)$.

(vi) $b, c \leqslant b \vee c$ implies $a + b, a + c \leqslant a + (b \vee c)$, hence $(a + b) \vee (a + c) \leqslant a + (b \vee c)$. Analogously for the second inequality.

(vii) Let $a \leqslant b$. Then $a + 1 \leqslant b + 1$ and $\neg_1 b = (b + 1)^- \leqslant (a + 1)^- = \neg_1 a$. Similarly for the second implication.

(viii) Using (v) and Lemma 1 (vi) we get $\neg_1(a \wedge b) = ((a \wedge b) + 1)^- = ((a+1) \wedge (b+1))^- = (a+1)^- \vee (b+1)^- = \neg_1 a \vee \neg_1 b$. Analogously for "$\neg_2$".

(ix) According to (vi) we have $\neg_1(a \vee b) = ((a \vee b) + 1)^- \leqslant ((a+1) \vee (b+1))^- = (a+1)^- \wedge (b+1)^- = \neg_1 a \wedge \neg_1 b$. Analogously we can show that the second inequality also holds.

(x) $a+1 \leqslant a+1$ implies $1^- \leqslant (a+1)^- + a$, i.e. $0 \leqslant \neg_1 a + a$. Analogously for the second part.

(xi) From $1 + a \leqslant 1 + a$ we get $1^\sim \leqslant a + (1+a)^\sim$. Thus $a^\sim \leqslant (1+a)^\sim + 1^{\sim\sim} = (1+a)^\sim + 1$ and finally $\neg_1\neg_2 a = ((1+a)^\sim + 1)^- \leqslant a^{\sim -} = a$. Similarly, $\neg_2\neg_1 a \leqslant a$.

(xii) Clearly, $(a^- + a)^\sim \leqslant 0$, hence $\sigma_1(a,a) = (a^- + a)^\sim \vee 0 = 0$. Analogously, $\sigma_2(a,a) = 0$.

(xiii) Let $a \leqslant b$. Then $1 \leqslant a^- + b$ and $(a^- + b)^\sim \leqslant 0$, which yields $\sigma_1(a,b) = (a^- + b)^\sim \vee 0 = 0$. Conversely, assume $\sigma_1(a,b) = 0$. Then $(a^- + b)^\sim \leqslant 0$, thus $1 \leqslant a^- + b$ and $a \leqslant b$. Similarly for $\sigma_2(a,b)$.

(xiv) Suppose $a = b$. Then $\sigma_1(a,b) = 0$ and $\sigma_1(b,a) = 0$ by (xiii), hence $\delta_1(a,b) = 0 \vee 0 = 0$. Conversely, let $\delta_1(a,b) = 0$. Then $\sigma_1(a,b) \vee \sigma_1(b,a) = 0$, which implies $\sigma_1(a,b) \leqslant 0$, $\sigma_1(b,a) \leqslant 0$. The first inequality yields $(a^- + b)^\sim \vee 0 = 0$, thus $(a^- + b)^\sim \leqslant 0$, $1 \leqslant a^- + b$ and $a \leqslant b$. Analogously, $\sigma_1(b,a) \leqslant 0$ implies $b \leqslant a$. Altogether we obtain $a = b$. Similarly for $\delta_2(a,b)$.

(xv) By Definition 2 and Lemma 1 (i) we have $a \rightarrow b = a^- + b = a^- + b^{-\sim} = b^- \rightsquigarrow a^-$. Analogously, $a \rightsquigarrow b = b^\sim \rightarrow a^\sim$. $\square$

Lemma 3. *Let $\mathscr{A} = (A, +, 0, ^-, ^\sim, \wedge, \vee)$ be a pseudo $*$-lattice and $a \in A$. Then $a^\sim$ is the least element $c \in A$ such that $a + c \geqslant 1$ and a^- is the least element $d \in A$ such that $d + a \geqslant 1$.*

Proof. Let $1 \leqslant a + c$, i.e. $(a+c) \wedge 1 = 1$, that means $(a+c)^\sim \vee 0 = 0$. Then we get by Lemma 2 (ii) $a^\sim = (c + (a^{\sim -} + c)^\sim) \wedge a^\sim = (c + (a+c)^\sim) \wedge a^\sim$. By virtue of $(a+c)^\sim \leqslant 0$ we have $x = c + (a+c)^\sim \leqslant c$. Therefore $a^\sim = (c + (a+c)^\sim) \wedge a^\sim = x \wedge a^\sim$, i.e. $a^\sim \leqslant x \leqslant c$. Similarly it can be shown that a^- is the least element $d \in A$ such that $d + a \geqslant 1$. $\square$

Lemma 4. *Let $\mathscr{A} = (A, +, 0, ^-, ^\sim, \wedge, \vee)$ be a pseudo $*$-lattice. Then $\mathscr{A}$ satisfies the following conditions:*

(1) $(A, +, 0)$ *is a monoid;*
(2) $(A, \wedge, \vee)$ *is a lattice;*
(3) $(x^- + x)^\sim \vee 0 = 0$, $(x + x^\sim)^- \vee 0 = 0$ *for any* $x \in A$*;*
(4) $(y + (x^- + y)^\sim) \wedge x = x$, $((y + x^\sim)^- + y) \wedge x = x$ *for any* $x, y \in A$*;*
(5) $x \vee y = (x^- \wedge y^-)^\sim = (x^\sim \wedge y^\sim)^-$ *for any* $x, y \in A$*;*
(6) $x + (y \wedge z) = (x+y) \wedge (x+z)$, $(y \wedge z) + x = (y+x) \wedge (z+x)$ *for any* $x, y, z \in A$.

P r o o f. Let $\mathscr{A} = (A, +, 0, ^-, ^\sim, \wedge, \vee)$ be a pseudo $*$-lattice. Then we get the conditions (1), (2) and (5) immediately from Definition 1. The condition (3) follows from Lemma 2 (xii), the condition (4) from Lemma 2 (ii) and (6) is an immediate consequence of Lemma 2 (v). $\square$

Lemma 5. *Let* $\mathscr{A} = (A, +, 0, ^-, ^\sim, \wedge, \vee)$ *be an algebra of type* $\langle 2, 0, 1, 1, 2, 2\rangle$ *satisfying the conditions* (1)–(6)*. Then in* $\mathscr{A}$ *the following assertions hold:*

(i) $x^{-\sim} = x^{\sim -} = x$;

(ii) $(x \vee y)^- = x^- \wedge y^-$, $(x \vee y)^\sim = x^\sim \wedge y^\sim$;

(iii) $(x \wedge y)^- = x^- \vee y^-$, $(x \wedge y)^\sim = x^\sim \vee y^\sim$;

(iv) $x \wedge y = (x^- \vee y^-)^\sim = (x^\sim \vee y^\sim)^-$;

(v) $x \leqslant y$ *iff* $y^- \leqslant x^-$ *iff* $y^\sim \leqslant x^\sim$;

(vi) $a \leqslant b, c \leqslant d \Rightarrow a + c \leqslant b + d$.

P r o o f. (i) Due to (5) we have $x = x \vee x = (x^- \wedge x^-)^\sim = x^{-\sim}$, $x = x \vee x = (x^\sim \wedge x^\sim)^- = x^{\sim -}$.

(ii) The identities (5) and (i) yield $(x \vee y)^- = (x^- \wedge y^-)^{\sim -} = x^- \wedge y^-$. Analogously, $(x \vee y)^\sim = x^\sim \wedge y^\sim$.

(iii) Using (i) and (5) again we get $(x \wedge y)^- = (x^{-\sim} \wedge y^{-\sim})^- = x^- \vee y^-$. Similarly we can prove the second part of the claim.

(iv) $x \wedge y = (x \wedge y)^{-\sim} = (x^- \vee y^-)^\sim$ by (i) and (iii).

(v) Clearly by (ii), $x \leqslant y$ iff $x \wedge y = x$ iff $x^- \vee y^- = x^-$ iff $y^- \leqslant x^-$. Analogously, $x \leqslant y$ iff $y^\sim \leqslant x^\sim$.

(vi) Suppose $a, b, u \in A$ with $a \leqslant b$. Then $u + (a \wedge b) = u + a$ and using (6) we obtain $(u + a) \wedge (u + b) = u + a$, i.e. $u + a \leqslant u + b$. Analogously, $a \leqslant b$ implies $a + u \leqslant b + u$. Now, let $c, d \in A$ with $c \leqslant d$. Then $a + c \leqslant b + c$, $b + c \leqslant b + d$ and $a + c \leqslant b + d$ by transitivity. $\square$

Lemma 6. *Let* $\mathscr{A} = (A, +, 0, ^-, ^\sim, \wedge, \vee)$ *be an algebra of type* $\langle 2, 0, 1, 1, 2, 2\rangle$ *satisfying the conditions* (1)–(6) *and* $a \in A$*. Then* $a^\sim$ *is the least element* $c \in A$ *such that* $a + c \geqslant 1$ *and* a^- *is the least element* $d \in A$ *such that* $d + a \geqslant 1$.

P r o o f. Let $1 \leqslant a + c$, i.e. $(a + c) \wedge 1 = 1$, that means $(a + c)^\sim \vee 0 = 0$ by Lemma 5 (iii). Then we get $a^\sim = (c + (a^{\sim -} + c)^\sim) \wedge a^\sim = (c + (a + c)^\sim) \wedge a^\sim$ by the identity (4) and Lemma 5 (i). By Lemma 5 (v) we have $(a + c)^\sim \leqslant 0$, thus $x = c + (a + c)^\sim \leqslant c$ according to Lemma 5 (vi). Therefore $a^\sim = (c + (a + c)^\sim) \wedge a^\sim = x \wedge a^\sim$, i.e. $a^\sim \leqslant x \leqslant c$. Similarly it can be shown that a^- is the least element $d \in A$ such that $d + a \geqslant 1$. $\square$

Theorem 1. *Let $\mathscr{A} = (A, +, 0, ^-, ^\sim, \wedge, \vee)$ be an algebra of type $\langle 2,0,1,1,2,2\rangle$. Then $\mathscr{A}$ is a pseudo $*$-lattice if and only if it satisfies the conditions* (1)–(6).

P r o o f. According to Lemma 4 it remains to prove the converse implication.

Let $\mathscr{A}$ satisfy (1)–(6). Clearly, it suffices to prove (P4).

Suppose $x, y \in A$, $x \leqslant y$, i.e. $x = x \wedge y$. Then due to (3), (6) and Lemma 5 (iii) we obtain $0 = (x^- + x)^\sim \vee 0 = (x^- + (x \wedge y))^\sim \vee 0 = ((x^- + x) \wedge (x^- + y))^\sim \vee 0 = ((x^- + x)^\sim \vee (x^- + y)^\sim) \vee 0 = ((x^- + x)^\sim \vee 0) \vee (x^- + y)^\sim = (x^- + y)^\sim \vee 0$. Thus $0^- = (x^- + y)^{\sim -} \wedge 0^-$ by (5) and Lemma 5 (i), i.e. $0^- = (x^- + y) \wedge 0^-$ and we have $0^- \leqslant x^- + y$. Similarly we can get $0^\sim \leqslant y + x^\sim$.

Conversely, let $0^- \leqslant x^- + y$. Then according to (3), (6) and Lemma 5 (iii) we have $(x^- + (x \wedge y))^\sim \vee 0 = 0$, hence $(x^- + (x \wedge y)) \wedge 0^- = 0^-$, i.e. $0^- \leqslant x^- + (x \wedge y)$. By Lemma 6 we know that $x^{-\sim} = x$ is the least element $z \in A$ such that $x^- + z \geqslant 0^-$. Thus $x \leqslant x \wedge y$, which gives $x = x \wedge y$ and $x \leqslant y$. Analogously we can prove that $0^\sim \leqslant y + x^\sim$ yields $x \leqslant y$. □

Due to the previous theorem it is evident that the class of all pseudo $*$-lattices forms a variety (we will denote it by $\mathscr{PL}$); moreover, it is possible to show that the variety is arithmetical, i.e., it is congruence permutable and distributive [2].

Theorem 2. *The variety $\mathscr{PL}$ is arithmetical.*

P r o o f. Let $d_1(x,y,z) = ((x \rightarrow y) \rightsquigarrow z) \vee z$, $m_1(x,y,z) = d_1(x,y,y) \wedge d_1(y,z,z) \wedge d_1(z,x,x)$. Then $m_1(x,x,z) = d_1(x,x,x) \wedge d_1(x,z,z) \wedge d_1(z,x,x) = x$ because $d_1(z,x,x) \geqslant x$, $d_1(x,z,z) \geqslant x$ by Lemma 2 (ii) and $d_1(x,x,x) = x$ by Lemma 2 (ii), (iii). Similarly, $m_1(x,z,z) = d_1(x,z,z) \wedge d_1(z,z,z) \wedge d_1(z,x,x) = z$, $m_1(x,z,x) = d_1(x,z,z) \wedge d_1(z,x,x) \wedge d_1(x,x,x) = x$. It means that $m_1(x,y,z)$ is a majority term and the variety $\mathscr{PL}$ is distributive. Note that another majority term of $\mathscr{PL}$ is $m_2(x,y,z) = d_2(x,y,y) \wedge d_2(y,z,z) \wedge d_2(z,x,x)$ where $d_2(x,y,z) = ((x \rightsquigarrow y) \rightarrow z) \vee z$.

Further, let $p_1(x,y,z) = d_1(x,y,z) \wedge d_1(z,y,x)$. Then $p_1(x,x,z) = d_1(x,x,z) \wedge d_1(z,x,x) = z$ and $p_1(x,z,z) = d_1(x,z,z) \wedge d_1(z,z,x) = x$. Thus $p_1(x,y,z)$ is Malcev's term and the variety is permutable. Another Malcev's term is $p_2(x,y,z) = d_2(x,y,z) \wedge d_2(z,y,x)$. □

Definition 3. A *residuated lattice* is an algebra $\mathscr{L} = (L, *, \rightarrow_1, \rightarrow_2, \wedge, \vee, e)$ of type $\langle 2,2,2,2,2,0\rangle$ such that $(L, \wedge, \vee)$ is a lattice, $(L, *, e)$ is a monoid and the following *residuation laws* are satisfied for all $a, b, c \in L$: $a * b \leqslant c$ iff $a \leqslant b \rightarrow_2 c$ iff $b \leqslant a \rightarrow_1 c$.

Definition 4. By a *dualizing residuated lattice* we mean an algebra $\mathscr{D} = (D, *, \rightarrow_1, \rightarrow_2, \wedge, \vee, e, d)$ of type $\langle 2,2,2,2,2,0,0\rangle$, where $(D, *, \rightarrow_1, \rightarrow_2, \wedge, \vee, e)$ is a residuated lattice and d is a *dualizing element* of $\mathscr{D}$, i.e. $(a \to_1 d) \to_2 d = a$, $(a \to_2 d) \to_1 d = a$ holds for any $a \in D$.

Remark 1. Let us recall some well-known properties of the residuated lattices (see e.g. [1], [3] and [6]) which will be useful for our subsequent investigation of the pseudo $*$-lattices. For example, for any residuated lattice $\mathscr{L} = (L, *, \to_1, \to_2, \wedge, \vee, e)$ and $a, b, c \in L$ we have

(α) $(a * b) \to_1 c = b \to_1 (a \to_1 c)$, $(b * a) \to_2 c = b \to_2 (a \to_2 c)$;

(β) $a \to_1 (b \to_2 c) = b \to_2 (a \to_1 c)$;

(γ) $(a \vee b) \to_1 c = (a \to_1 c) \wedge (b \to_1 c)$, $(a \vee b) \to_2 c = (a \to_2 c) \wedge (b \to_2 c)$.

Lemma 7. *$\mathscr{D} = (D, *, \to_1, \to_2, \wedge, \vee, e, d)$ be a dualizing residuated lattice with a dualizing element d. Then $((a \to_1 d) * (b \to_1 d)) \to_2 d = ((a \to_2 d) * (b \to_2 d)) \to_1 d$ for any $a, b \in D$.*

Proof. Applying Remark 1 (α), (β) we can compute: $((a \to_1 d) * (b \to_1 d)) \to_2 d = (a \to_1 d) \to_2 ((b \to_1 d) \to_2 d) = (a \to_1 d) \to_2 ((b \to_2 d) \to_1 d) = (b \to_2 d) \to_1 ((a \to_1 d) \to_2 d) = (b \to_2 d) \to_1 ((a \to_2 d) \to_1 d) = ((a \to_2 d) * (b \to_2 d)) \to_1 d$. □

Lemma 8. *Let $\mathscr{A} = (A, +, 0, ^-, ^\sim, \wedge, \vee)$ be a pseudo $*$-lattice and $x, y \in A$. Then $x \rhd_1 y$ is the least element $u \in A$ with the property $y + u \geqslant x$ and $x \rhd_2 y$ is the least element $v \in A$ with the property $v + y \geqslant x$.*

Proof. Clearly, we have $y + (x \rhd_1 y) = y + (x^- + y)^\sim \geqslant x$ by Lemma 4 (4). Now, suppose $y + u \geqslant x$. Then using Lemma 2 (iv), (i) we get $x^- + (y + u) \geqslant x^- + x \geqslant 1$. Thus $1 \leqslant x^- + y + u = (x^- + y)^{\sim -} + u$ and therefore we obtain $(x^- + y)^\sim \leqslant u$ according to Definition 1 , i.e. $x \rhd_1 y \leqslant u$. Similarly we can show that $x \rhd_2 y$ is the least element $v \in A$ with the property $v + y \geqslant x$. □

Lemma 9. *Let $\mathscr{A} = (A, +, 0, ^-, ^\sim, \wedge, \vee)$ be a pseudo $*$-lattice, $x, y \in A$ and define $x \cdot y := (y^\sim + x^\sim)^-$. Then $x \rightsquigarrow y$ is the greatest element $u \in A$ with the property $u \cdot x \leqslant y$ and $x \to y$ is the greatest element $v \in A$ with the property $x \cdot v \leqslant y$.*

Proof. We compute $(x \rightsquigarrow y) \cdot x = (x^\sim + (x \rightsquigarrow y)^\sim)^- = (x^\sim + (y + x^\sim)^\sim)^- = (x^\sim + (y^{\sim -} + x^\sim)^\sim)^-$. Applying Lemma 4 (4) we obtain $x^\sim + (y^{\sim -} + x^\sim)^\sim \geqslant y^\sim$ and therefore we have $(x \rightsquigarrow y) \cdot x \leqslant y^{\sim -} = y$. Now, assume $u \cdot x \leqslant y$. Then $(x^\sim + u^\sim)^- \leqslant y$, which implies $x^\sim + u^\sim \geqslant y^\sim$ and $u^\sim \geqslant y^\sim \rhd_1 x^\sim$ due to Lemma 8. Hence $u \leqslant (y^\sim \rhd_1 x^\sim)^-$, i.e. $u \leqslant (y^{\sim -} + x^\sim)^{\sim -} = y + x^\sim$, which implies $u \leqslant x \rightsquigarrow y$. Analogously it can be proved that $x \to y$ is the greatest element $v \in A$ such that $x \cdot v \leqslant y$. □

Let us denote the class of all dualizing residuated lattices by $\mathscr{DRL}$.

Theorem 3. *$\mathscr{PL}$ is term equivalent to $\mathscr{DRL}$.*

Proof. Let $\mathscr{A} = (A, +, 0, ^-, ^\sim, \wedge, \vee)$ be a pseudo $*$-lattice and $x \cdot y := (y^\sim + x^\sim)^-$. Then we shall show that $\mathscr{A}^+ = (A, \cdot, \rightarrow, \rightsquigarrow, \wedge, \vee, 1, 0)$ is a dualizing residuated lattice with the dualizing element 0.

Clearly, the residuation laws are satisfied due to Lemma 9. Further, for an arbitrary $a \in A$ we have $(a \rightarrow 0) \rightsquigarrow 0 = 0 + (a^- + 0)^\sim = a^{-\sim} = a$ $(a \rightsquigarrow 0) \rightarrow 0 = (0 + a^\sim)^- + 0 = a^{\sim-} = a$ and 0 is the dualizing element of $\mathscr{A}^+$. Now, let us show that $(A, \cdot, 1)$ is a monoid. We compute $(x \cdot y) \cdot z = (y^\sim + x^\sim)^- \cdot z = (z^\sim + (y^\sim + x^\sim)^{-\sim})^- = (z^\sim + (y^\sim + x^\sim))^- = ((z^\sim + y^\sim) + x^\sim)^- = ((z^\sim + y^\sim)^{-\sim} + x^\sim)^- = x \cdot (y \cdot z)$. Finally, $x \cdot 1 = (1^\sim + x^\sim)^- = (0 + x^\sim)^- = x$ and $1 \cdot x = (x^\sim + 1^\sim)^- = (x^\sim + 0)^- = x$.

Conversely, let $(D, *, \rightarrow_1, \rightarrow_2, \wedge, \vee, e, d)$ be a dualizing residuated lattice and let $^{-_d}$, $^{\sim_d}$, $+_d$ be such that for any $a, b \in D$ we have $a^{-_d} = a \rightarrow_2 d$, $a^{\sim_d} = a \rightarrow_1 d$, $a +_d b = ((a \rightarrow_1 d) * (b \rightarrow_1 d)) \rightarrow_2 d$ $(= ((a \rightarrow_2 d) * (b \rightarrow_2 d)) \rightarrow_1 d$ by Lemma 7).

Then we can prove that $\mathscr{D}_+ = (D, +_d, d, ^{-_d}, ^{\sim_d}, \wedge, \vee)$ is a pseudo $*$-lattice. Indeed, according to Remark 1 (α) we have $a +_d b = ((a \rightarrow_1 d) * (b \rightarrow_1 d)) \rightarrow_2 d = (a \rightarrow_1 d) \rightarrow_2 ((b \rightarrow_1 d) \rightarrow_2 d) = (a \rightarrow_1 d) \rightarrow_2 b$. Similarly, $a +_d b = ((a \rightarrow_2 d) * (b \rightarrow_2 d)) \rightarrow_1 d = (b \rightarrow_2 d) \rightarrow_1 ((a \rightarrow_2 d) \rightarrow_1 d) = (b \rightarrow_2 d) \rightarrow_1 a$. Due to this argument and Remark 1 (β) we can write for $a, b, c \in D$: $(a +_d b) +_d c = (c \rightarrow_2 d) \rightarrow_1 ((a \rightarrow_1 d) \rightarrow_2 b) = (a \rightarrow_1 d) \rightarrow_2 ((c \rightarrow_2 d) \rightarrow_1 b) = a +_d (b +_d c)$.

Further, applying Remark 1 (α), $a +_d d = ((a \rightarrow_1 d) * (d \rightarrow_1 d)) \rightarrow_2 d = (a \rightarrow_1 d) \rightarrow_2 ((d \rightarrow_1 d) \rightarrow_2 d) = (a \rightarrow_1 d) \rightarrow_2 d = a$ and $d +_d a = ((d \rightarrow_2 d) * (a \rightarrow_2 d) \rightarrow_1 d = (a \rightarrow_2 d) \rightarrow_1 ((d \rightarrow_2 d) \rightarrow_1 d) = (a \rightarrow_2 d) \rightarrow_1 d = a$.

To prove (P3) we compute using Remark 1 (γ): $(a^{-_d} \wedge b^{-_d})^{\sim_d} = ((a \rightarrow_2 d) \wedge (b \rightarrow_2 d)) \rightarrow_1 d = ((a \vee b) \rightarrow_2 d) \rightarrow_1 d = a \vee b$. Analogously, $(a^{\sim_d} \wedge b^{\sim_d})^{-_d} = a \vee b$.

Using the properties of the residuated lattice again we verify (P4): We have $a \leqslant b$ iff $(a \rightarrow_2 d) \rightarrow_1 d \leqslant (b \rightarrow_1 d) \rightarrow_2 d$ iff $((a \rightarrow_2 d) \rightarrow_1 d) * (b \rightarrow_1 d) \leqslant d$ iff $d \rightarrow_2 d \leqslant (((a \rightarrow_2 d) \rightarrow_1 d) * (b \rightarrow_1 d)) \rightarrow_2 d$ iff $d^{-_d} \leqslant a^{-_d} +_d b$. Analogously we can prove the second part of (P4).

Finally, it can be seen that $\mathscr{A}$ coincides with $(\mathscr{A}^+)_+$ and $\mathscr{D}$ coincides with $(\mathscr{D}_+)^+$. □

Lemma 10. *Let $\mathscr{A} = (A, +, 0, ^-, ^\sim, \wedge, \vee)$ be a pseudo $*$-lattice and $x \cdot y := (y^\sim + x^\sim)^-$. Then*

(i) $(x^- \cdot y^-)^\sim = (x^\sim \cdot y^\sim)^-$;

(ii) $(x^- + y^-)^\sim = (x^\sim + y^\sim)^-$.

Proof. (i) According to Theorem 3 we can use the properties of the dualizing residuated lattice $\mathscr{A}^+ = (A, \cdot, \rightarrow, \rightsquigarrow, \wedge, \vee, 1, 0)$, especially the condition (α) from

Remark 1, and we can write $(x^- \cdot y^-)^\sim = ((x \to 0) \cdot (y \to 0)) \rightsquigarrow 0 = (x \to 0) \rightsquigarrow ((y \to 0) \rightsquigarrow 0) = x^- \rightsquigarrow y$.

Analogously, $(x^\sim \cdot y^\sim)^- = ((x \rightsquigarrow 0) \cdot (y \rightsquigarrow 0)) \to 0 = (y \rightsquigarrow 0) \to ((x \rightsquigarrow 0) \to 0) = y^\sim \to x$.

Due to Lemma 2 (xv) we have $x^- \rightsquigarrow y = y^\sim \to x^{-\sim} = y^\sim \to x$, i.e. $(x^- \cdot y^-)^\sim = (x^\sim \cdot y^\sim)^-$.

(ii) Clearly, $x \cdot y = (y^\sim + x^\sim)^-$ yields $(x \cdot y)^\sim = y^\sim + x^\sim$, hence $x + y = x^{-\sim} + y^{-\sim} = (y^- \cdot x^-)^\sim$. By virtue of (i) this implies $(x^- + y^-)^\sim = (y^{--} \cdot x^{--})^{\sim\sim} = (y^{-\sim} \cdot x^{-\sim})^{-\sim} = y \cdot x = (x^\sim + y^\sim)^-$. □

Definition 5. A *coresiduated lattice* is an algebra $\mathscr{L} = (L, \bullet, \triangleright, \triangleleft, \wedge, \vee, n)$ of type $\langle 2,2,2,2,2,0\rangle$ such that $(L, \wedge, \vee)$ is a lattice, $(L, \bullet, n)$ is a monoid and the following *coresiduation laws* hold for all $a, b, c \in L$: $a \leqslant b \bullet c$ iff $a \triangleright b \leqslant c$ iff $c \triangleleft a \leqslant b$.

Definition 6. A *codualizing coresiduated lattice* is an algebra $\mathscr{C} = (C, \bullet, \triangleright, \triangleleft, \wedge, \vee, n, c)$ of type $\langle 2,2,2,2,2,0,0\rangle$ where $(C, \bullet, \triangleright, \triangleleft, \wedge, \vee, n)$ is a coresiduated lattice and c is a *codualizing element* of $\mathscr{C}$, i.e. $c \triangleright (c \triangleleft a) = c \triangleleft (c \triangleright a) = a$ for any $a \in C$.

Lemma 11. *Let $\mathscr{A} = (A, +, 0, ^-, ^\sim, \wedge, \vee)$ be a pseudo $*$-lattice. Then $\mathscr{A}^{++} = (A, +, \triangleright_1, \triangleright_2, \wedge, \vee, 0, 1)$ is a codualizing coresiduated lattice.*

P r o o f. To verify the coresiduation laws we suppose that $a, b, c \in A$ and $a \leqslant b+c$. By Lemma 1 (iii), (iv) we have $a \leqslant b+c$ iff $c^- \leqslant a^- + b$ iff $(a^- + b)^\sim \leqslant c$, i.e. $a \triangleright_1 b \leqslant c$. Analogously we can show that $a \leqslant b + c$ iff $c \triangleright_2 a \leqslant b$.

Further, $1 \triangleright_1 (1 \triangleright_2 a) = 1 \triangleright_1 (a + 1^\sim)^- = (1^- + a^-)^\sim = a^{-\sim} = a$, $1 \triangleright_2 (1 \triangleright_1 a) = 1 \triangleright_2 (1^- + a)^\sim = (a^\sim + 1^\sim)^- = a^{\sim-} = a$ and 1 is the codualizing element of $\mathscr{A}^{++}$. □

Lemma 12. *Let $\mathscr{C} = (C, \bullet, \triangleright, \triangleleft, \wedge, \vee, n, c)$ be a codualizing coresiduated lattice and define $x^{-_c} := c \triangleleft x$, $x^{\sim_c} := c \triangleright x$. Then $\mathscr{C}_{++} = (C, \bullet, n, ^{-_c}, ^{\sim_c}, \wedge, \vee)$ is a pseudo $*$-lattice.*

P r o o f. To prove (P3) we will use a property of the coresiduated lattices which is analogous to the condition (γ) of Remark 1: $(x^{-_c} \wedge y^{-_c})^{\sim_c} = ((c \triangleleft x) \wedge (c \triangleleft y))^{\sim_c} = c \triangleright ((c \triangleleft x) \wedge (c \triangleleft y)) = (c \triangleright (c \triangleleft x)) \vee (c \triangleright (c \triangleleft y)) = x \vee y$. Similarly, $(x^{\sim_c} \wedge y^{\sim_c})^{-_c} = x \vee y$.

Now, we will prove (P4). Applying Definition 6 we have $x \leqslant y$ iff $c \triangleright (c \triangleleft x) \leqslant y$ iff $c \leqslant (c \triangleleft x) \bullet y$. Clearly, $c \triangleleft n = c$, thus $n^{-_c} \leqslant x^{-_c} \bullet y$. Further, $x \leqslant y$ iff $c \triangleright y \leqslant c \triangleright x$ iff $c \leqslant y \bullet (c \triangleright x)$ and since $c \triangleright n = c$ we obtain $n^{\sim_c} \leqslant y \bullet x^{\sim_c}$. □

3. Ideals

Definition 7. Let $\mathscr{A}$ be a pseudo $*$-autonomous lattice, $x \in A$. By an *absolute value* of x we mean an element $|x| := \delta_2(x, 0) = x \vee \neg_1 x \vee 0$.

Lemma 13. *For an arbitrary pseudo $*$-lattice $\mathscr{A}$ and $a, b \in A$ the following conditions hold:*

(i) $0 \leqslant |a|$, $a \leqslant |a|$, $\neg_1 a \leqslant |a|$;

(ii) $0 \leqslant a \Rightarrow |a| = a$, $a \leqslant 0 \Rightarrow |a| = \neg_1 a$;

(iii) $a = 0 \Leftrightarrow |a| = 0$;

(iv) $\||a|\| = |a|$;

(v) $|a \vee b| \leqslant |a| \vee |b|$;

(vi) $|a| \vee |b| \leqslant |a| + |b|$.

Proof. (i) It evidently follows from Definition 7.

(ii) For any $a \in A$, $0 \leqslant a$ we have $a \vee 0 = a$, i.e. $|a| = a \vee \neg_1 a$. Further, $1 = 0 + 1 \leqslant a + 1$, which implies $(a+1)^- \leqslant 0$, thus $\neg_1 a \leqslant 0$ and using transitivity we obtain $\neg_1 a \leqslant a$ and $|a| = a \vee \neg_1 a = a$. Similarly, for any $a \in A$, $a \leqslant 0$ we have $0 = \neg_1 0 \leqslant \neg_1 a$. Hence $|a| = \neg_1 a$.

(iii) Clearly, $a = 0$ implies $|a| = 0$.

Conversely, if $|a| = a \vee \neg_1 a \vee 0 = 0$ then $a \vee \neg_1 a \leqslant 0$. Consequently, $a \leqslant 0$, which implies $0 \leqslant \neg_1 a$, thus $\neg_1 a = 0$ and $a = 0$.

(iv) We have $|a| \geqslant 0$ for any $a \in A$ due to (i) and according to (ii) we obtain the claim.

(v) $|a \vee b| = (a \vee b) \vee \neg_1(a \vee b) \vee 0 = (a \vee b \vee 0) \vee \neg_1(a \vee b)$, $|a| \vee |b| = (a \vee \neg_1 a \vee 0) \vee (b \vee \neg_1 b \vee 0) = (a \vee b \vee 0) \vee (\neg_1 a \vee \neg_1 b)$. Using Lemma 2 (ix) we are done.

(vi) With respect to Lemma 2 (vi) we have $|a| \leqslant (|a| + b) \vee (|a| + \neg_1 b) \vee (|a| + 0) \leqslant |a| + (b \vee \neg_1 b \vee 0) = |a| + |b|$.

Analogously we can show that $|b| \leqslant |a| + |b|$. These two inequalities give the proposition. $\square$

Definition 8. Let $\mathscr{A}$ be a pseudo $*$-autonomous lattice and $\emptyset \neq J \subseteq A$. Then J is called an *ideal* of A if for any $a, b \in A$ the following conditions are satisfied:

(I1) $a, b \in J$ imply $a + b \in J$;

(I2) $a \in J$ implies $\neg_1 a \in J$;

(I3) $a \in J$, $b \in A$, $|b| \leqslant |a|$ imply $b \in J$.

The set of all ideals of $\mathscr{A}$ will be denoted by $\mathscr{I}(\mathscr{A})$.

Lemma 14. *In any pseudo $*$-lattice $\mathscr{A}$:*

(i) *$\{0\}$ is the smallest ideal of A;*

(ii) *if $J \in \mathscr{I}(\mathscr{A})$ and $a \in A$ then $a \in J$ iff $|a| \in J$;*

(iii) *if $J \in \mathscr{I}(\mathscr{A})$ then J is a convex sublattice of $(A, \wedge, \vee)$.*

Proof. (i) Due to Lemma 13 (i), (iii) we have $|0| \leqslant |a|$ for an arbitrary $a \in A$. Thus $0 \in J$ for any $J \in \mathscr{I}(\mathscr{A})$ by (I3). Evidently, $\{0\}$ satisfies (I1)–(I3), i.e. $\{0\}$ is the smallest ideal of A.

(ii) Let $a \in J$. Then $\|a\| = |a| \leqslant |a|$ according to Lemma 13 (iv) and by (I3) we get $|a| \in J$.

Conversely, if $|a| \in J$ then $|a| \leqslant |a| = \|a\|$ and using (I3) again we obtain $a \in J$.

(iii) Let $J \in \mathscr{I}(\mathscr{A})$, $a, b \in J$. Then $|a \vee b| \leqslant |a| \vee |b| \leqslant |a| + |b| = \big||a| + |b|\big|$ due to Lemma 13 (v), (vi) and by virtue of $|a| + |b| \geqslant 0$. Clearly $|a| + |b| \in J$ by (ii) and (I1). Hence (I3) gives $a \vee b \in J$.

Further, according to Lemma 2 (viii) we have $|a \wedge b| = (a \wedge b) \vee \neg_1(a \wedge b) \vee 0 = (a \wedge b) \vee (\neg_1 a \vee \neg_1 b) \vee 0 \leqslant (a \vee b) \vee (\neg_1 a \vee \neg_1 b) \vee 0 = |(a \vee b) \vee (\neg_1 a \vee \neg_1 b) \vee 0|$. Thus $a \wedge b \in J$ by (I3) and we conclude that J is a sublattice of $(A, \wedge, \vee)$.

To prove the convexity of J we suppose that $a, b \in J$, $x \in A$ and $a \leqslant x \leqslant b$. Then $x \vee 0 \leqslant b \vee 0$ and taking into account $b \vee 0 \in J$ and $|x \vee 0| = x \vee 0 \leqslant b \vee 0 = |b \vee 0|$ we get $x \vee 0 \in J$ by (I3). Further, $|x| = (x \vee 0) \vee \neg_1 x \leqslant (x \vee 0) \vee \neg_1 a = |(x \vee 0) \vee \neg_1 a|$. Since $(x \vee 0) \vee \neg_1 a \in J$ we obtain $x \in J$. $\square$

4. Homomorphisms and congruences

Definition 9. An ideal J of a pseudo $*$-lattice $\mathscr{A}$ is said to be *normal* if it satisfies the following condition for each $a, b \in A$:

$$\sigma_1(a, b) \in J \quad \text{iff} \quad \sigma_2(a, b) \in J.$$

The set of all normal ideals of $\mathscr{A}$ will be denoted by $\mathcal{N}(\mathcal{A})$ and the set of all congruences on $\mathcal{A}$ by $\mathrm{Con}(\mathcal{A})$.

Lemma 15. *If $J \in \mathscr{N}(\mathscr{A})$ then for each $a, b \in A$ we have*

$$\delta_1(a, b) \in J \quad \textit{iff} \quad \delta_2(a, b) \in J.$$

Proof. Let $J \in \mathscr{N}(\mathscr{A})$ and $\delta_1(a, b) \in J$. Then $\sigma_1(a, b) \vee \sigma_1(b, a) \in J$ and since $0 \leqslant \sigma_1(a, b), \sigma_1(b, a) \leqslant \sigma_1(a, b) \vee \sigma_1(b, a)$ we get $\sigma_1(a, b), \sigma_1(b, a) \in J$ by the convexity of J. Hence also $\sigma_2(a, b), \sigma_2(b, a) \in J$ by the normality of J and $\delta_2(a, b) \in J$. The converse is analogous. $\square$

Definition 10. Let $\mathscr{A}$, $\mathscr{B}$ be two pseudo $*$-lattices and let h be a homomorphism from $\mathscr{A}$ to $\mathscr{B}$. The set $\operatorname{Ker} h = \{a \in A;\ h(a) = 0^B\}$ is called the *kernel* of h.

Lemma 16. *Let $h\colon A \to B$ be a homomorphism of pseudo $*$-lattices $\mathscr{A}$ and $\mathscr{B}$. Then for each $a, b \in A$ the following assertions are valid:*
(i) $h(a) \leqslant^B h(b)$ *iff* $\sigma_1^A(a,b) \in \operatorname{Ker} h$;
(ii) $\operatorname{Ker} h = \{0^A\}$ *iff h is an injection;*
(iii) $\operatorname{Ker} h \in \mathscr{N}(\mathscr{A})$.

P r o o f. (i) According to Lemma 2 (xiii) we have $h(a) \leqslant^B h(b) \Leftrightarrow \sigma_1(h(a), h(b)) = 0^B$ but $\sigma_1(h(a), h(b)) = h(\sigma_1(a,b))$ and we are done.

(ii) Let h be an injection from $\mathscr{A}$ to $\mathscr{B}$. Then obviously $\operatorname{Ker} h = \{0^A\}$.

Conversely, let $\operatorname{Ker} h = \{0^A\}$ and let $a, b \in A$ be such that $h(a) = h(b)$. Then by Lemma 2 (xiv) we have $\delta_1(h(a), h(b)) = 0^B$, i.e. $h(\delta_1(a,b)) = 0^B$ and $\delta_1(a,b) \in \operatorname{Ker} h$. Hence $\delta_1(a,b) = 0^A$ and using Lemma 2 (xiv) again we get $a = b$.

(iii) To check (I1) we suppose that $a, b \in \operatorname{Ker} h$, i.e. $h(a) = h(b) = 0^B$. Then $h(a+b) = h(a) + h(b) = 0^B + 0^B = 0^B$ and $a + b \in \operatorname{Ker} h$.

Further, for $a \in \operatorname{Ker} h$ we have $h(\neg_1 a) = h((a+1)^-) = (h(a+1))^- = (h(a) + h(1))^- = (0^B + h(1))^- = (h(1))^- = h(1^-) = h(0^A) = 0^B$, i.e. $\neg_1 a \in \operatorname{Ker} h$.

Now, we will prove the condition (I3). Let $a \in \operatorname{Ker} h$, $b \in A$ and $|b| \leqslant |a|$. Then $h(b) \vee \neg_1 h(b) \vee h(0) = h(b \vee \neg_1 b \vee 0) \leqslant h(a \vee \neg_1 a \vee 0) = h(a) \vee \neg_1 h(a) \vee h(0)$. Consequently, $|h(b)| \leqslant |h(a)| = 0$. This implies $|h(b)| = 0$, $h(b) = 0$ and $b \in \operatorname{Ker} h$.

It remains to prove that $\operatorname{Ker} h$ is normal. For this purpose we compute $\sigma_1(x,y) \in \operatorname{Ker} h \Leftrightarrow h((x^- + y)^\sim \vee 0) = 0 \Leftrightarrow (h(x)^- + h(y))^\sim \vee 0 = 0 \Leftrightarrow (h(x)^- + h(y))^\sim \leqslant 0 \Leftrightarrow h(x)^- + h(y) \geqslant 1 \Leftrightarrow h(x) \leqslant h(y) \Leftrightarrow h(y) + h(x)^\sim \geqslant 1 \Leftrightarrow (h(y) + h(x)^\sim)^- \leqslant 0 \Leftrightarrow (h(y) + h(x)^\sim)^- \vee 0 = 0 \Leftrightarrow h(y + x^\sim)^- \vee 0 = 0 \Leftrightarrow \sigma_2(x,y) \in \operatorname{Ker} h$. $\square$

Definition 11. Let $J \in \mathscr{I}(\mathscr{A})$. The binary relation $f_1(J) \subseteq A \times A$ is defined as follows: $\langle a,b \rangle \in f_1(J)$ iff $\delta_1(a,b) \in J$.

Lemma 17. *Let $\mathscr{A}$ be a pseudo $*$-lattice and $J \in \mathscr{I}(\mathscr{A})$. Then the following conditions are equivalent for any $a, b \in A$:*
(a) $\langle a,b \rangle \in f_1(J)$;
(b) *there exists $c \in J$, $c \geqslant 0$ such that $a \leqslant b + c$ and $b \leqslant a + c$;*
(c) $\sigma_1(a,b) \in J$ *and* $\sigma_1(b,a) \in J$.

P r o o f. (a) $\Rightarrow$ (b): Due to Lemma 2 (ii) we have $a \leqslant (a \to b) \rightsquigarrow b = b + (a^- + b)^\sim \leqslant b + ((a^- + b)^\sim \vee 0) = b + \sigma_1(a,b) \leqslant b + \delta_1(a,b)$. Since $\langle a,b \rangle \in f_1(J)$ we have $\delta_1(a,b) \in J$. Similarly we can show that $b \leqslant a + \delta_1(a,b)$.

(b) $\Rightarrow$ (c): Let $a \leqslant b + c$ where $c \in J$, $c \geqslant 0$. Then $c^- \leqslant a^- + b$, which implies $(a^- + b)^\sim \leqslant c$ and consequently $\sigma_1(a, b) = (a^- + b)^\sim \vee 0 \leqslant c \vee 0 \leqslant |c|$. Applying (I3) we obtain $\sigma_1(a, b) \in J$. Analogously one can prove that $b \leqslant a + c$ entails $\sigma_1(b, a) \in J$.

(c) $\Rightarrow$ (a): This implication follows immediately from Lemma 14 (iii). $\square$

Remark 2. Analogously we can define the relation $f_2(J) \subseteq A \times A$ such that $\langle a, b\rangle \in f_2(J)$ iff $\delta_2(a, b) \in J$.

Then we can get equivalent conditions similarly to the previous lemma:

(a)* $\langle a, b\rangle \in f_2(J)$;

(b)* there exists $d \in J$, $d \geqslant 0$, such that $a \leqslant d + b$, $b \leqslant d + a$;

(c)* $\sigma_2(a, b) \in J$ and $\sigma_2(b, a) \in J$.

Obviously, we can take $d = \delta_2(a, b)$.

Remark 3. Clearly, if $J \in \mathscr{N}(\mathscr{A})$ then we have $\langle a, b\rangle \in f_1(J)$ iff $\langle a, b\rangle \in f_2(J)$ iff there exists $0 \leqslant u = \delta_1(a, b) \vee \delta_2(a, b)$ such that $a \leqslant b + u$, $b \leqslant a + u$, $a \leqslant u + b$, $b \leqslant u + a$. It means that for $J \in \mathscr{N}(\mathscr{A})$ we have $f_1(J) = f_2(J)$ and therefore we will denote this relation simply by $f(J)$.

Lemma 18. *Let $J \in \mathscr{I}(\mathscr{A})$. Then $f_1(J)$ and $f_2(J)$ are equivalence relations on $\mathscr{A}$.*

Proof. It is obvious that $f_1(J)$ is reflexive and symmetric. Let us prove transitivity applying the previous lemma. Suppose that $\langle a, b\rangle, \langle b, c\rangle \in f_1(J)$. Then there exist $u, v \in J$, $0 \leqslant u, v$ such that $a \leqslant b + u$, $b \leqslant a + u$, $b \leqslant c + v$, $c \leqslant b + v$. This entails $a \leqslant a \vee c \leqslant (b + u) \vee (b + v) \leqslant b + (u \vee v) \leqslant (c + v) + (u \vee v) = c + (v + (u \vee v))$. Similarly it can be shown that $c \leqslant a + (u + (u \vee v))$ and we conclude that there exists $w = (v + (u \vee v)) \vee (u + (u \vee v)) \in J$ such that $a \leqslant c + w$, $c \leqslant a + w$. Hence $\langle a, c\rangle \in f_1(J)$ by Lemma 17 and $f_1(J)$ is transitive. Analogously for $f_2(J)$. $\square$

Lemma 19. *Let $J \in \mathscr{N}(\mathscr{A})$. Then $f(J)$ is a congruence relation on $\mathscr{A}$.*

Proof. Assume that $J \in \mathscr{N}(\mathscr{A})$ and $\langle a, b\rangle \in f(J)$. Then by Lemma 17 and Remark 2 there exists $x \in J$, $x \geqslant 0$ such that $a \leqslant b + x$, $b \leqslant a + x$, $a \leqslant x + b$ and $b \leqslant x + a$. Then $a^- \leqslant b^- + x$ and $b^- \leqslant a^- + x$, hence $\langle a^-, b^-\rangle \in f(J)$. Further, $a^\sim \leqslant x + b^\sim$ and $b^\sim \leqslant x + a^\sim$. Thus $\langle a^\sim, b^\sim\rangle \in f(J)$.

To prove that $f(J)$ satisfies the substitution property under $+$ and $\wedge$ we suppose $u \in A$. Then $a + u \leqslant (x + b) + u = x + (b + u)$ and $b + u \leqslant (x + a) + u = x + (a + u)$, i.e. $\langle a + u, b + u\rangle \in f(J)$. Analogously it can be shown that $\langle a, b\rangle \in f(J)$ yields $\langle u + a, u + b\rangle \in f(J)$.

Similarly, $a \wedge u \leqslant (x + b) \wedge u \leqslant (x + b) \wedge (x + u) = x + (b \wedge u)$ because $0 \leqslant x$ implies $u \leqslant x + u$. Hence $\langle a \wedge u, b \wedge u\rangle \in f(J)$. Now, let $\langle c, d\rangle \in f(J)$. Then

$\langle a+c, b+c\rangle, \langle b+c, b+d\rangle, \langle a\wedge c, b\wedge c\rangle, \langle b\wedge c, b\wedge d\rangle \in f(J)$ and $\langle a+c, b+d\rangle, \langle a\wedge c, b\wedge d\rangle \in f(J)$ by the transitivity.

Compatibility of $f(J)$ with $\vee$ follows from the fact that $a \vee c = (a^- \wedge c^-)^\sim$ and $b \vee d = (b^- \wedge d^-)^\sim$. $\square$

Definition 12. Let Θ be a congruence on $\mathscr{A}$. We define $g(\Theta)$ as the coset of 0 modulo Θ, i.e. $g(\Theta) = 0/\Theta = \{x \in A;\ \langle x, 0\rangle \in \Theta\}$.

Lemma 20. *If Θ is a congruence on $\mathscr{A}$, then $g(\Theta) \in \mathscr{N}(\mathscr{A})$.*

P r o o f. (I1): Let $\Theta \in \mathrm{Con}(\mathscr{A})$ and $a, b \in g(\Theta)$. Then $\langle a, 0\rangle, \langle b, 0\rangle \in \Theta$, thus $\langle a+b, 0\rangle \in \Theta$, i.e. $a + b \in g(\Theta)$.

(I2): Clearly, $a \in g(\Theta)$ implies $\langle a, 0\rangle \in \Theta$, $\langle a+1, 0+1\rangle \in \Theta$, $\langle (a+1)^-, 1^-\rangle \in \Theta$, i.e. $\langle \neg_1 a, 0\rangle \in \Theta$ and $\neg_1 a \in g(\Theta)$.

(I3): Suppose $a \in g(\Theta)$, $b \in A$ and $|b| \leqslant |a|$. Then $\langle a, 0\rangle, \langle \neg_1 a, 0\rangle \in \Theta$, which yields $\langle a \vee \neg_1 a \vee 0, 0\rangle \in \Theta$, i.e. $\langle |a|, 0\rangle \in \Theta$ and $|a| \in g(\Theta)$. Now we have $0 \leqslant |b| \leqslant |a| \in g(\Theta)$ and using the convexity of the sublattice $(g(\Theta), \wedge, \vee)$ we conclude $|b| \in g(\Theta)$. We will show that $|b| \in g(\Theta)$ implies $b \in g(\Theta)$. Obviously, $\langle b \vee \neg_1 b \vee 0, 0\rangle \in \Theta$ entails $\langle b \wedge (b \vee (\neg_1 b \vee 0)), b \wedge 0\rangle \in \Theta$, i.e. $\langle b, b \wedge 0\rangle \in \Theta$. This implies $\langle b \vee 0, (b \wedge 0) \vee 0\rangle \in \Theta$, i.e. $\langle b \vee 0, 0\rangle \in \Theta$ and $b \vee 0 \in g(\Theta)$. Analogously, $\neg_1 b \vee 0 \in g(\Theta)$ and consequently $\neg_2(\neg_1 b \vee 0) \in g(\Theta)$. Further, $b \wedge 0 \leqslant b$ entails $\neg_1 b \leqslant \neg_1(b \wedge 0) = \neg_1 b \vee \neg_1 0 = \neg_1 b \vee 0$. Now, applying Lemma 2 (xi), we have $g(\Theta) \ni \neg_2(\neg_1 b \vee 0) \leqslant \neg_2 \neg_1 b \leqslant b \leqslant b \vee 0 \in g(\Theta)$. Hence we get $b \in g(\Theta)$ by the convexity of $g(\Theta)$.

To show the normality of $g(\Theta)$ it suffices to use Lemma 16 and to realize that $g(\Theta)$ is the kernel of the canonical homomorphism $\nu\colon\ a \mapsto a/\Theta$. $\square$

Theorem 4. *Let $\mathscr{A}$ be a pseudo $*$-lattice. Then the lattices $\mathscr{N}(\mathscr{A})$ and $\mathrm{Con}(\mathscr{A})$ are isomorphic.*

P r o o f. Obviously, it suffices to prove the following properties of the correspondences f, g from Remark 3 and Definition 12: (A) $g(f(J)) = J$, (B) $f(g(\Theta)) = \Theta$, (C) both f and g are order preserving.

(A) Applying Lemma 14 (ii) we get $g(f(J)) = \{x \in A;\ \langle x, 0\rangle \in f(J)\} = \{x \in A;\ \delta_2(x, 0) \in J\} = \{x \in A;\ |x| \in J\} = J$.

(B) Due to Lemma 17 we have $f(g(\Theta)) = \{\langle a, b\rangle \subseteq A \times A;\ \delta_2(a, b)/\Theta = 0/\Theta\} = \{\langle a, b\rangle;$ there exists $c \in J$ such that $\langle c, 0\rangle \in \Theta,\ a \leqslant b + c,\ b \leqslant a + c\}$. First, we will show $\Theta \subseteq f(g(\Theta))$. Let $\langle a, b\rangle \in \Theta$. Then $\langle (a^- + a)^\sim \vee 0, (a^- + b)^\sim \vee 0\rangle \in \Theta$, i.e. $\langle \sigma_1(a, a), \sigma_1(a, b)\rangle \in \Theta$ and since $\sigma_1(a, a) = 0$ by Lemma 2 (xii) we obtain $\langle \sigma_1(a, b), 0\rangle \in \Theta$. Similarly it can be shown that $\langle \sigma_1(b, a), 0\rangle \in \Theta$, thus $\langle \delta_1(a, b), 0\rangle \in \Theta$ and $\langle a, b\rangle \in f(g(\Theta))$.

Conversely, let $\langle a, b\rangle \in f(g(\Theta))$, i.e. there exists $c \in 0/\Theta$ such that $a \leqslant b + c$, $b \leqslant a + c$. Hence $\langle c, 0\rangle \in \Theta$, which entails $\langle b + c, b\rangle, \langle a + c, a\rangle \in \Theta$ and consequently $a/\Theta = (a \wedge (b+c))/\Theta = (a \wedge b)/\Theta = (b \wedge a)/\Theta = (b \wedge (a+c))/\Theta = b/\Theta$, i.e. $\langle a, b\rangle \in \Theta$ and $f(g(\Theta)) \subseteq \Theta$.

(C) Assume $I \subseteq J$ and $\langle a, b\rangle \in f(I)$, i.e. $\delta_1(a, b) \in I \subseteq J$. Hence $\delta_1(a, b) \in J$, $\langle a, b\rangle \in f(J)$ and we conclude $f(I) \subseteq f(J)$.

Finally, let $\Theta, \Phi \in \mathrm{Con}(\mathscr{A})$ with $\Theta \subseteq \Phi$ and let $a \in g(\Theta)$, i.e. $\langle a, 0\rangle \in \Theta \subseteq \Phi$. Thus $a \in g(\Phi)$ and $g(\Theta) \subseteq g(\Phi)$. □

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