Ján Jakubík

On some types of radical classes

Czechoslovak Mathematical Journal, Vol. 58 (2008), No. 3, 833–848

Persistent URL: http://dml.cz/dmlcz/140424

Czechoslovak Mathematical Journal, 58 (133) (2008), 833–848

ON SOME TYPES OF RADICAL CLASSES

Ján Jakubík, Košice

(Received September 18, 2006)

Abstract. Let $\mathfrak{m}$ be an infinite cardinal. We denote by $C_{\mathfrak{m}}$ the collection of all $\mathfrak{m}$-representable Boolean algebras. Further, let $C_{\mathfrak{m}}^0$ be the collection of all generalized Boolean algebras B such that for each $b \in B$, the interval $[0, b]$ of B belongs to $C_{\mathfrak{m}}$. In this paper we prove that $C_{\mathfrak{m}}^0$ is a radical class of generalized Boolean algebras. Further, we investigate some related questions concerning lattice ordered groups and generalized MV-algebras.

Keywords: Boolean algebra, generalized Boolean algebra, $\mathfrak{m}$-representability, lattice ordered group, generalized MV-algebra, radical class

MSC 2010: 06E05, 06F20, 06D35

1. Introduction

In the present paper we deal with some types of radical classes of generalized Boolean algebras, lattice ordered groups and generalized MV-algebras.

Let $\mathfrak{m}$ be an infinite cardinal. The $\mathfrak{m}$-representability of Boolean algebras was investigated by Chang [1], Scott [16], Pierce [14], Smith [17] and Sikorski [18], [19]; cf. also the monograph Sikorski [20].

The notion of radical class of generalized Boolean algebras was studied by the author [9]; for the analogous notion concerning lattice ordered groups cf. the author [8], Conrad [2], Darnel [4] and Ton [21]; for MV-algebras cf. the author [10].

The collection of all $\mathfrak{m}$-representable Boolean algebras will be denoted by $C_{\mathfrak{m}}$. Let $C_{\mathfrak{m}}^0$ be the collection of all generalized Boolean algebras B such that for each $b \in B$, the interval $[0, b]$ of B is an element of $C_{\mathfrak{m}}$.

This work was supported by Science and Technology Assistance Agency under the contract No. APVT-51-032002.
This work has been partially supported by the Slovak Academy of Sciences via the project Center of Excellence-Physics of Information.

In the present paper we prove that $C^0_{\mathfrak{m}}$ is a radical class of generalized Boolean algebras.

For dealing with the $\mathfrak{m}$-representability of Boolean algebras in Theorem 29.3 of [20], several equivalent conditions concerning a Boolean algebra $\mathfrak{A}$ and the infinite cardinal $\mathfrak{m}$ have been considered. Each of these conditions is necessary and sufficient for the $\mathfrak{m}$-representability of $\mathfrak{A}$.

Only two from these conditions (namely, (r_1) nad (r_2)) are expressed in terms of elements of $\mathfrak{A}$; the other conditions deal with filters of $\mathfrak{A}$ or with topological notions related to the Stone space of $\mathfrak{A}$.

The detailed definitions of (r_1) and (r_2) are given in Section 2 below. According to the mentioned result of Sikorski, we have

$$(1) \qquad (\mathrm{r}_1) \Leftrightarrow (\mathrm{r}_2)$$

for each Boolean algebra $\mathfrak{A}$.

The conditions (r_1) and (r_2) can be applied also for generalized MV-algebras. We prove that (1) is valid for projectable generalized MV-algebras. This is a generalization of the mentioned Sikorski's result, since each Boolean algebra is the underlying lattice of some projectable MV-algebra; cf. Section 6 for a more detailed description of this fact.

We prove that the collection of all generalized MV-algebras satisfying the condition (r_1) is a radical class. The analogous assertion concerning (r_2) fails to be valid.

Some related questions concerning lattice ordered groups are studied.

2. Preliminaries

For Boolean algebras we apply the terminology and the notation as in Sikorski [20] with the distinctions that 0 and 1 denote the least and the greatest element of a Boolean algebra, respectively, and that symbols $\wedge, \vee$ are used for denoting lattice operations.

We recall some definitions which will be frequently used in the sequel. Let $\mathfrak{m}$ be an infinite cardinal.

A Boolean algebra $\mathfrak{A}$ is $\mathfrak{m}$-*representable* if it is isomorphic to a quotient algebra $\mathfrak{F}/\Delta$, where $\mathfrak{F}$ is an $\mathfrak{m}$-field of sets and Δ is an $\mathfrak{m}$-ideal of $\mathfrak{F}$.

We remark that the classical result proved independently by Loomis [13] and Sikorski [18] can be formulated as follows:

(LS) Each Boolean σ-algebra is σ-representable.

It was remarked in [20] that in the case $\mathfrak{m} = \aleph_0$, Theorem 29.3 (dealing with the conditions (r_1) and (r_2)) yields the following generalization of Theorem (LS):

(LS_0) (Cf. [20], 29.4.) Every Boolean algebra is σ-representable.

The detailed formulations of the conditions (r_1) and (r_2) concerning a Boolean algebra $\mathfrak{A}$ and an infinite cardinal $\mathfrak{m}$ are as follows:

(r_1) If $(a_{t,s})_{t\in T, s\in S}$ is an $\mathfrak{m}$-indexed set of elements of $\mathfrak{A}$ (i.e., $\operatorname{card} T = \operatorname{card} S = \mathfrak{m}$) such that

(1) there exist all joins $\bigvee_{s\in S} a_{t,s}$ and the meet $\bigwedge_{t\in T}\bigvee_{s\in S} a_{t,s}$,

and if

(2) $\bigwedge_{t\in T}\bigvee_{s\in S} a_{t,s} \neq 0$,

then there exists a function $\varphi \in S^T$ with

(3) $\bigwedge_{t\in T'} a_{t,\varphi(t)} \neq 0$ for each finite subset T' of T.

(r_2) If $(a_{t,s})_{t\in T, s\in S}$ is an $\mathfrak{m}$-indexed set of elements of $\mathfrak{A}$ such that (1) is valid and if

(4) $\bigwedge_{t\in T}\bigvee_{s\in S} a_{t,s} = 1$,

then for each $a \in \mathfrak{A}$ with $a \neq 0$ there exists $\varphi \in S^T$ such that

(5) $a \wedge \bigwedge_{t\in T'} a_{t,\varphi(t)} \neq 0$ for each finite subset T' of T.

Theorem 2.1 (Cf. [20], 29.3.). *Let $\mathfrak{A}$ be a Boolean algebra and $i \in \{1,2\}$. Then $\mathfrak{A}$ is $\mathfrak{m}$-representable if and only if the condition (r_i) is satisfied.*

Lemma 2.2. *Assume that $\mathfrak{A}$ is a Boolean algebra satisfying the condition (r_1). Let $b_1, b_2 \in \mathfrak{A}$, $b_1 \leqslant b_2$. Then the interval $[b_1, b_2]$ of $\mathfrak{A}$ also satisfies the condition (r_1).*

Proof. From the definition of (r_1) we immediately obtain that if $x \in \mathfrak{A}$, then the interval $[0, x]$ of $\mathfrak{A}$ satisfies the condition (r_1). Let b_3 be the complement of b_1 in $\mathfrak{A}$. Then the interval $[b_1, b_2]$ is isomorphic to the interval $[0, b_2 \wedge b_3]$. Thus $[b_1, b_2]$ satisfies (r_1). □

Lemma 2.3. *Assume that $\mathfrak{A}$ is a Boolean algebra which does not satisfy the condition (r_1). Then there is $b \in \mathfrak{A}$ such that*

(∗) *if $b' \in [0, b]$, $b' > 0$, then the interval $[0, b']$ does not satisfy (r_1).*

Proof. In view of the assumption there exists an $\mathfrak{m}$-indexed system $(a_{t,s})_{s\in T, s\in S}$ of elements of $\mathfrak{A}$ such that (1) and (2) are valid and, moreover, there does not exist any function $\varphi \in S^T$ satisfying (3). Hence for each $\varphi \in S^T$ there exists a finite subset T' of T such that

$$(3') \qquad \bigwedge_{t\in T'} a_{t,\varphi(t)} = 0.$$

We put $b = \bigwedge_{t\in T} \bigvee_{s\in S} a_{t,s}$ and

$$a'_{t,s} = b' \wedge a_{t,s} \quad \text{for each } t \in T, s \in S,$$

where $b' \in [0, b]$, $b' > 0$.

Since $\mathfrak{A}$ is infinitely distributive, for each $t \in T$ we have

$$b' \wedge \left(\bigvee_{s\in S} a_{t,s}\right) = \bigvee_{s\in s} (b' \wedge a_{t,s}) = \bigvee_{s\in S} a'_{t,s},$$

$$b' = b' \wedge b = b' \wedge \left(\bigwedge_{t\in T} \bigvee_{s\in S} a_{t,s}\right) = \bigwedge_{t\in T} \bigvee_{s\in S} (b' \wedge a_{t,s}) = \bigwedge_{t\in t} \bigvee_{s\in S} a'_{t,s}.$$

Moreover, according to (3′) for each $\varphi \in S^T$,

$$0 = b' \wedge (\wedge_{t\in T'} a_{t,\varphi(t)}) = \bigwedge_{t\in T'} (b' \wedge a_{t,\varphi(t)}) = \bigwedge_{t\in T'} \bigvee_{s\in S} a'_{t,\varphi(t)}.$$

Thus the interval $[0, b']$ does not satisfy the condition (r_1). □

3. Generalized Boolean algebras

A lattice L with the least element 0 is a *generalized Boolean algebra* if for each $x \in L$, the interval $[0, x]$ of L is a Boolean algebra.

For a generalized Boolean algebra L we denote by $\mathscr{I}(L)$ the system of all ideals of L. This system is partially ordered by the set-theoretical inclusion. Then $\mathscr{I}(L)$ is a complete lattice.

If $\{L_i\}_{i\in I}$ is a nonempty system of elements of $\mathscr{I}(L)$, then

a) $\bigwedge_{i\in I} L_i = \bigcap_{i\in I} L_i$

and

b) $\bigvee_{i\in I} L_i$ is the set of all $x \in L$ such that there exist $i(1), \ldots, i(n) \in I$ and $x_{i(1)} \in L_{i(1)}, \ldots, x_{i(n)} \in L_{i(n)}$ with $x = x_{i(1)} \vee \ldots \vee x_{i(n)}$.

Definition 3.1. A nonempty collection C of generalized Boolean algebras is a *radical class* of generalized Boolean algebras if it satisfies the following conditions:

(i) C is closed under isomorphisms;
(ii) if $L \in C$ and L_1 is an ideal of L, then $L_1 \in C$;
(iii) if L is a generalized Boolean algebra and $\{L_i\}_{i\in I} \subseteq \mathscr{I}(L) \cap C$, then $\bigvee_{i\in I} L_i$ belongs to C.

Let $\mathfrak{m}$ be an infinite cardinal and let $C^0_{\mathfrak{m}}$ be as in Section 1.

Proposition 3.2. *$C^0_{\mathfrak{m}}$ is a radical class of generalized Boolean algebras.*

P r o o f. It is obvious that $C^0_{\mathfrak{m}}$ is closed with respect to isomorphisms. If $L \in C^0_{\mathfrak{m}}$ and if $L_1 \in \mathscr{I}(L)$, then in view of 2.2 we obtain $L_1 \in C^0_{\mathfrak{m}}$. Hence the conditions (i) and (ii) from 3.1 are satisfied.

By way of contradiction, assume that the condition (iii) from 3.1 fails to be valid. Hence there exists a generalized Boolean algebra L and a system $\{L_i\}_{i\in I} \subseteq \mathscr{I}(L) \cap C^0_{\mathfrak{m}}$ such that $\bigvee_{i\in I} L_i$ does not belong to $C^0_{\mathfrak{m}}$. Then there exixts $0 < x \in \bigvee_{i\in I} L_i$ with $[0, x] \notin C_{\mathfrak{m}}$. Thus there is $0 < b \in [0, x]$ satisfying the condition $(*)$ from 2.3.

In view of the relation $b \in \bigvee_{i\in I} L_i$ there are $i(1), \ldots, i(n) \in I$ and $x_{i(1)} \in L_{i(1)}, \ldots, x_{i(n)} \in L_{i(n)}$ such that $b = x_{i(1)} \vee \ldots \vee x_{i(n)}$. Without loss of generality we can suppose that $x_{i(1)} > 0, \ldots, x_{i(n)} > 0$.

Since $x_{i(1)} \in L_{i(1)}$, by applying 2.2 we obtain that $[0, x_{i(1)}]$ satisfies (r_1). Since $(*)$ is valid for b, we arrived at a contradiction. □

4. Lattice ordered groups

The group operation in a lattice ordered group will be written additively, though we do not assume the commutativity of this operation. Let G be a lattice ordered group. As usual, we set $G^+ = \{g \in G\colon\ g \geqslant 0\}$.

Let $\mathfrak{m}$ be an infinite cardinal.

Definition 4.1. We say that a lattice ordered group G satisfies (r_1') if it fulfils the same conditions as in (r_1) with the distinction that instead of $\mathfrak{A}$ we have now the set G^+.

Let $u \in G^+$. The element u is a *weak unit* of G if for each $g \in G$ with $g > 0$ the relation $g \wedge u > 0$ is valid. If for each $g \in G$ there is a positive integer n such that $g \leqslant nu$, then u is a *strong unit* of G. Every strong unit is a weak unit.

Definition 4.2. Assume that u is a weak unit of a lattice ordered group G. We say that G satisfies (r_2') if it fulfils the same conditions as in (r_2) with the distinctions that

1) instead of $\mathfrak{A}$ we have the set G^+;
2) the condition (4) is replaced by

   (4′) the element $\bigwedge_{t\in T}\bigvee_{s\in S} a_{t,s}$ is a weak unit of G.

Proposition 4.3. *Assume that G is a lattice ordered group having a weak unit. Then $(\mathrm{r}_1') \Rightarrow (\mathrm{r}_2')$.*

P r o o f. Suppose that (r_1') is valid. By way of contradiction, assume that (r_2') fails to hold. Then there exists an $\mathfrak{m}$-indexed set $(a_{t,s})_{t\in T, s\in S}$ of elements of G^+ such that (4′) is valid; moreover, there exists $0 < a \in G$ having the property that for each $\varphi \in S^T$,

(5′) $a \wedge \bigwedge_{t\in T'} a_{t,\varphi(t)} = 0$ for some finite subset T' of T.

Put $a_0 = a \wedge u$. Then $a_0 > 0$. Further, for each $t \in T$ and $s \in S$ we set $a'_{t,s} = a_0 \wedge a_{t,s}$. Consider the $\mathfrak{m}$-indexed set $(a'_{t,s})_{t\in T,s\in S}$.

All elements $a'_{t,s}$ belong to G^+. In view of the infinite distributivity of G and according to the condition (1) in (r_1) we conclude that all the joins $\bigvee_{s\in S} a'_{t,s}$ and the meet $\bigwedge_{t\in T}\bigvee_{s\in S} a'_{t,s}$ exist in G. Put $u = \bigwedge_{t\in T}\bigvee_{s\in S} a_{t,s}$. From (4′) we obtain

$$0 < u \wedge a_0 = \bigwedge_{t\in T}\bigvee_{s\in S}(a_{t,s}\wedge a_0) = \bigwedge_{t\in T}\bigvee_{s\in S} a'_{t,s}.$$

Applying (r_1') to the $\mathfrak{m}$-indexed system $(a'_{t,s})_{t\in T,s\in S}$ we conclude that there exists $\varphi \in S^T$ with

(+) $\bigwedge_{t\in T'} a'_{t,\varphi(t)} \neq 0$ for each finite subset T' of T.

But according to (5′) we have

$$u \wedge a \wedge \bigwedge_{t\in T'} a_{t,\varphi(t)} = 0 \quad \text{for some finite } T' \subseteq T,$$

whence

$$\bigwedge_{t\in T'} a'_{t,\varphi(t)} = 0 \quad \text{for some finite } T' \subseteq T.$$

In view of (+), we have arrived at a contradiction. □

For $X \subseteq G$ we put

$$X^\delta = \{y \in G\colon\ |x| \wedge |y| = 0 \quad \text{for each } x \in X\};$$

X^δ is a *polar* of G. The lattice ordered group G is *projectable* if for each $x \in G$ we have a direct product decomposition

$$G = \{x\}^{\delta\delta} \times \{x\}^\delta.$$

In such a case, x is a weak unit of the lattice ordered group $\{x\}^{\delta\delta}$.

Proposition 4.4. *Assume that G is a projectable lattice ordered group having a weak unit. Then* $(\mathrm{r}_2') \Rightarrow (\mathrm{r}_1')$.

P r o o f. Suppose that (r_2') is valid. By way of contradiction, assume that (r_1') does not hold.

Thus there exists an $\mathfrak{m}$-indexed set $(a_{t,s})_{t\in T, s\in S}$ of elements of G^+ such that

(i) all the joins $\bigvee\limits_{s\in S} a_{t,s}$ exist in G;

(ii) there is $v \in G$ such that $v > 0$ and $v = \bigwedge\limits_{t\in T} \bigvee\limits_{s\in S} a_{t,s}$;

(iii) for each $\varphi \in S^T$ and for each finite subset T' of T we have

$$\bigwedge_{t\in T'} u_{t,\varphi(t)} = 0.$$

Since G is projectable, the relation

$$G = \{v\}^{\delta\delta} \times \{v\}^\delta \tag{6}$$

is valid; moreover, v is a weak unit of the lattice ordered group $\{v\}^{\delta\delta}$.

For each $z \in G$ we denote by $z(\{v\}^{\delta\delta})$ the component of z in the direct factor $\{v\}^{\delta\delta}$ of G; the meaning of $z(\{v\}^\delta)$ is analogous.

In particular, for $z = v$ we have

$$v(\{v\}^{\delta\delta}) = v, \quad v(\{v\}^\delta) = 0.$$

For each $t \in T$ and $x \in S$ we put $a'_{t,s} = v \wedge a_{t,s}$. Then, in view of the infinite distributivity of G, we obtain

$$v = \bigwedge_{t\in T} \bigvee_{s\in S} a'_{t,s}.$$

Since $0 \leqslant a'_{t,s} \leqslant v$, we get

$$a'_{t,s}(\{v\}^{\delta\delta}) = a'_{t,s}, \quad a'_{t,s}(\{v\}^\delta) = 0.$$

For each $t \in T$ and $s \in S$ we denote

$$a''_{t,s} = a'_{t,s} + u(\{v\}^{\delta}).$$

Then we have

$$a''_{t,s}(\{v\}^{\delta\delta}) = a'_{t,s}, \quad a''_{t,s}(\{v\}^{\delta}) = u(\{v\}^{\delta}).$$

This yields that there exist all joins $\bigvee_{s\in S} a''_{t,s}$ and

$$\bigwedge_{t\in T}\bigvee_{s\in S} a''_{t,s} = v + u(\{v\}^{\delta}). \tag{7}$$

Let u be a weak unit in G; we infer that $u(\{v\}^{\delta})$ is a weak unit in the direct factor $\{v\}^{\delta}$. Moreover, v is a weak unit in the direct factor $\{v\}^{\delta\delta}$. Therefore in view of (6) and (7), $\bigwedge_{t\in T}\bigvee_{s\in S} a''_{t,s}$ is a weak unit of the lattice ordered group G.

Looking at the $\mathfrak{m}$-indexed system $(a'_{t,s})_{t\in T, s\in S}$ and at the definition of (r'_2) we conclude that for each $a \in G$ with $a > 0$ there exists $\varphi \in S^T$ such that

$$a \wedge \bigwedge_{t\in T'} a''_{t,\varphi(t)} \neq 0 \quad \text{for each finite subset } T' \text{ of } T. \tag{8}$$

From the definition of $a'_{t,s}$ and from (iii) we infer that for each $\varphi \in S^T$ and for each finite subset T' of T the relation

$$\bigwedge_{t\in T'} a'_{t,\varphi(t)} = 0 \tag{9}$$

is valid.

Since $a'_{t,s} \in \{v\}^{\delta\delta}$ and $u(\{v\}^{\delta}) \in \{v\}^{\delta}$, we get

$$a'_{t,s} \wedge u(\{v\}^{\delta}) = 0,$$

hence

$$a''_{t,s} = a'_{t,s} \vee (u(\{v\}^{\delta})).$$

Also, $v \wedge (u(\{v\})) = 0$.

In view of $v > 0$ we can take $a = v$ in (8) and we obtain for each finite $T' \subseteq T$,

$$\begin{aligned} a \wedge \bigwedge_{t\in T'} a''_{t,\varphi(t)} &= v \wedge \bigwedge_{t\in T'} (a'_{t,\varphi(t)} \vee (u(\{v\}^{\delta})) \\ &= \bigwedge_{t\in T'} ((v \wedge a'_{t,\varphi(t)}) \vee (v \wedge (u(\{v\}^{\delta}))) = \bigwedge_{t\in T'} (v \wedge a'_{t,\varphi(t)}) \\ &= v \wedge \bigwedge_{t\in T'} a'_{t,\varphi(t)} = 0 \end{aligned}$$

for each finite $T' \subseteq T$ (according to (9)). In view of (8), we have arrived at a contradiction. □

From 4.3 and 4.4 we obtain

Theorem 4.5. *Assume that G is a projectable lattice ordered group having a weak unit. Then* $(\mathrm{r}'_1) \Leftrightarrow (\mathrm{r}'_2)$.

5. The radical class $C^1_{\mathfrak m}$

For a lattice ordered group G we denote by $c(G)$ the system of all convex ℓ-subgroups of G; the system $c(G)$ is partially ordered by the set-theoretical inclusion. Then $c(G)$ is a complete lattice. If $\{G_i\}_{i\in I}$ is a nonempty subset of $c(G)$, then $\bigwedge\limits_{i\in I} G_i = \bigcap\limits_{i\in I} G_i$. Further, $\bigvee\limits_{i\in I} G_i$ is the set of all $g \in G$ such that there exist $x_1, \ldots, x_n \in \bigcup\limits_{i\in I} G_i$ with $g = x_1 + \ldots + x_n$. If, moreover, $g > 0$, then without loss of generality we can suppose that $x_1 > 0, \ldots, x_n > 0$.

Definition 5.1. A nonempty collection X of lattice ordered groups is a radical class if it satisfies the following conditions:

(i) X is closed with respect to isomorphisms;
(ii) if $G_1 \in X$ and $G_2 \in c(X)$, then $G_2 \in X$;
(iii) if G is a lattice ordered group and $\emptyset \neq \{G_i\}_{i\in I} \subseteq c(G) \cap X$, then $\bigvee\limits_{i\in I} G_i \in X$.

Let $\mathfrak m$ be an infinite cardinal. We denote by $C^1_{\mathfrak m}$ the class of all lattice ordered groups which satisfy the condition (r'_1).

The following assertion is easy to verify; the proof will be omitted.

Lemma 5.2. *Let G be a lattice ordered group and let $G_1 \in c(G)$.*

(i) *Let $\{x_i\}_{i\in I} \subseteq G_1$ and $p, q \in G_1$. If the relation $p = \bigwedge\limits_{i\in I} x_i$ is valid in G_1, then this relation holds also in G. Similarly, if $q = \bigvee\limits_{i\in i} x_i$ is valid in G_1, then the same holds in G.*
(ii) *Let $\{y_i\}_{i\in I} \subseteq G_1^+$, $y \in G$. If the relation $\bigwedge\limits_{i\in I} y_i = y$ is valid in G, then $y \in G_1^+$ and the mentioned relation holds also in G_1.*

Lemma 5.3. *Let G and G_1 be as in 5.2. If G satisfies the condition (r'_1), then G_1 satisfies the condition as well.*

Proof. This is a consequence of the definition of (r'_1) and of 5.2. □

Definition 5.4. Let G be a lattice ordered group and $g \in G^+$. The interval $[0, g]$ of G is said to be regular with respect to (r'_1) if the assertion of (r'_1) is valid whenever $(a_{t,s})_{t\in T, s\in S}$ is an $\mathfrak m$-indexed set of elements of the interval $[0, g]$.

Lemma 5.5. *Let G be a lattice ordered group which fails to satisfy the condition* (r_1'). *Then there is $0 < b \in G^+$ such that*

($*$) *if $0 < b' \in [0, b]$, then the interval $[0, b']$ is not regular with respect to* (r_1').

Proof. It suffices to apply analogous steps as in the proof of 2.3. □

Let $\mathfrak{m}$ be an infinite cardinal. We denote by $C^1_{\mathfrak{m}}$ the class of all lattice ordered groups which satisfy the condition (r_1').

Proposition 5.6. *$C^1_{\mathfrak{m}}$ is a radical class of lattice ordered groups.*

Proof. Consider the conditions (i), (ii) and (iii) from 5.1. The validity of (i) is obvious. The validity of (ii) follows from 5.3.

By way of contradiction, suppose that (iii) fails to be valid. Then there exists a nonempty system $\{G_i\}_{i\in I} \subseteq c(G) \cap C^1_{\mathfrak{m}}$ such that the lattice ordered group $\bigvee_{i\in I} G_i$ does not satisfy the condition (r_1'). Then according to 5.5 there exists $0 < b \in \bigvee_{i\in I} G_i$ such that the condition ($*$) from 5.5 is satisfied.

The element b can be expressed in the form $b = x_1 + \ldots + x_n$, where $x_1, \ldots, x_n$ are elements of the set $\bigcup_{i\in I} G_i^+$. Therefore all intervals $[0, x_1], \ldots, [0, x_n]$ are regular with respect to (r_1'). Without loss of generality we can suppose that $x_1 > 0$. Then in view of ($*$), this interval fails to be regular with respect to (r_1'); we have arrived at a contradiction. □

Let us denote by $C^2_{\mathfrak{m}}$ the class of all lattice ordered groups which satisfy the condition (r_2').

$C^2_{\mathfrak{m}}$ fails to be a radical class since it does not fulfil the condition (iii) from 5.1. Example: Let R be the set of all reals with the operation $+$ and with the natural linear order. Let I be an infinite set of indices and for each $i \in I$ let $R_i = R$; put $G = \prod_{i\in I} R_i$. For $i_0 \in I$ let $\overline{R}_{i_0}$ be the set of all $g \in G$ such that $g_i = 0$ whenever $i \in I$ and $i \neq i_0$. Then $\overline{R}_{i_0} \in c(G)$ and $\overline{R}_{i_0} \in C^2_{\mathfrak{m}}$ for any infinite cardinal $\mathfrak{m}$. But $\bigvee_{i\in I} \overline{R}_i$ has no weak unit, whence it does not belong to $C^2_{\mathfrak{m}}$. Therefore the condition (iii) from 5.1 does not hold for $C^2_{\mathfrak{m}}$.

Let $C^{02}_{\mathfrak{m}}$ be the class of all lattice ordered groups G such that for each $0 < g \in G$, the convex ℓ-subgroup G_g of G generated by the element g belongs to $C^2_{\mathfrak{m}}$. Without a proof, we present the following result

($*$) $C^{02}_{\mathfrak{m}}$ is a radical class of lattice ordered groups.

6. Generalized MV-algebras

The notion of generalized MV-algebra was introduced independently by Georgescu and Iorgulescu [6], [7] and by Rachůnek [15] (in [6] and [7], the term 'pseudo MV-algebra' was used).

An MV-algebra $\mathscr{A} = (A; \oplus, ^{\neg}, ^{\sim}, 0, 1)$ is an algebraic system of type (2,1,1,0,0) such the axioms (A1)–(A8) from [15] are satisfied.

Let G be a lattice ordered group with a strong unit u. The pair (G, u) is called a unital lattice ordered group. If no misunderstanding can occur we speak about G instead of (G, u).

Given (G, u), we put $A = [0, u]$ and for $x, y \in A$ we define

$$x \oplus y = (x + y) \wedge u, \quad x^{\neg} = u - x, \quad x^{\sim} = -x + u, \quad 1 = u.$$

Then $\mathscr{A} = (A; \oplus, ^{\neg}, ^{\sim}, 0, u)$ is a generalized MV-algebra; we set $\mathscr{A} = \Gamma(G, u)$.

Dvurečenskij [5] proved that for each MV-algebra $\mathscr{A}$ there exists a unital lattice ordered group (G, u) such that the relation

$$\mathscr{A} = \Gamma(G, u) \tag{1}$$

is valid.

Below, when speaking about a generalized MV-algebra $\mathscr{A}$ we always assume that (1) holds.

We consider the partial order $\leqslant$ on the set A which is induced by the partial order on G.

Now we can apply for a generalized MV-algebra $\mathscr{A}$ the conditions (r_1) and (r_2) from Section 2 with the distinction that we take $\mathscr{A}$ instead of the Boolean algebra $\mathfrak{A}$.

Thus we obviously have

Lemma 6.1. *A generalized MV-algebra $\mathscr{A}$ satisfies (r_1) if and only if the interval $[0, u]$ of G is regular with respect to (r_1').*

Lemma 6.2. *Let (G, u) be a unital lattice ordered group. The following conditions are equivalent:*

(i) *G satisfies (r_1');*
(ii) *The interval $[0, u]$ of G is regular with respect to (r_1').*

Proof. The validity of the implication (i) $\Rightarrow$ (ii) is obvious. Assume that (ii) holds. By way of contradiction, suppose that (i) fails to be valid.

In view of 5.5, there is $0 < b \in G$ such that, whenever $b' \in G$, $0 < b' \leqslant b$, then the interval $[0, b']$ is not regular with respect to (r_1'). Put $b' = b \wedge u$. We have $0 < b'$, hence the mentioned assertion for b' holds. Therefore (ii) is not valid, which is a contradiction. □

Proposition 6.3. *Let $\mathscr{A}$ and (G,u) be as above. Then the following conditions are equivalent:*

(i) *G satisfies* (r_1');

(ii) *$\mathscr{A}$ satisfies* (r_1).

P r o o f. This is a consequence of 6.1 and 6.2. □

Proposition 6.4. *Let $\mathscr{A}$ be a generalized MV-algebra. Then* $(\mathrm{r}_1) \Rightarrow (\mathrm{r}_2)$.

P r o o f. It suffices to apply the same idea as in the proof of 6.4; we also use the fact that u is the greatest element of $\mathscr{A}$. □

As above, let A be the underlying set of $\mathscr{A}$. For $a \in A$ we put $A_a = [0,a]$ and for each $x_1, x_2 \in A_1$ we set

$$x_1 \oplus_a x_2 = (x_1 + x_2) \wedge a \quad x_1^{\neg_a} = a - x_1, \quad x_1^{\sim_a} = -x_1 + a, \quad 1_a = a.$$

Then the algebraic structure $\mathscr{A}_a = (A_a, \oplus_a, {}^{\neg_a}, {}^{\sim_a}, 0, 1_a)$ is a generalized MV-algebra. It will be called an interval subalgebra of $\mathscr{A}$.

Let a be as above and let $b \in A$ such that $a \wedge b = 0$, $a \vee b = 1$. For each $x \in A$ we put $\varphi(x) = (x \wedge a, x \wedge b)$. Then φ is an isomorphism of $\mathscr{A}$ onto the direct product $\mathscr{A}_a \times \mathscr{A}_b$ (cf. [11]). In this situation we will write $\mathscr{A} = \mathscr{A}_a \times \mathscr{A}_b$.

For $\emptyset \neq X \subseteq A$ we put

$$X^{\delta_1} = \{y \in A\colon\ y \wedge x = 0 \text{ for each } x \in X\}.$$

The generalized MV-algebra $\mathscr{A}$ is *projectable* it for each $x \in A$, the relation

$$\mathscr{A} = \{x\}^{\delta_1\delta_1} \times \{x\}^{\delta_1}$$

is valid (meaning that $\{x\}^{\delta_1\delta_1}$ has a greatest element a, $\{x\}^{\delta_1}$ has a greatest element b and $\mathscr{A} = \mathscr{A}_a \times \mathscr{A}_b$).

It is easy to verify that there exists a one-to-one correspondence between polars of $\mathscr{A}$ and polars of G. From this and from Theorem 6.4 in [11] we can deduce by a simple argument

Lemma 6.5. *Let $\mathscr{A}$ and (G,u) be as above. Then $\mathscr{A}$ is projectable if and only if G is projectable.*

Lemma 6.6. *Let us apply the notation as above. The following conditions are equivalent:*

(i) *G satisfies* (r_2')*;*

(ii) *$\mathscr{A}$ satisfies* (r_2).

P r o o f. The case $G=\{0\}$ being trivial we assume that $G\neq\{0\}$.

a) The validity of the implication (i) $\Rightarrow$ (ii) is obvious (since $u=1$ is a strong unit of G).

b) Assume that (ii) is valid. By way of contradiction, suppose that (i) fails to hold. Hence there is an $\mathfrak{m}$-indexed set $(a_{t,s})_{t,\in T,s\in S}$ of elements of G^+ such that

α) the element

$$\bigwedge_{t\in T}\bigvee_{s\in S} a_{t,s}=v$$

exists in G;

β) v is a weak unit in G;

γ) there exists $0<a\in G$ such that for each $\varphi\in S^T$,

$$a\wedge\bigwedge_{t\in T'} a_{t,\varphi(t)}=0\quad\text{for each finite subset } T' \text{ of } T.$$

We denote by G_1 the convex ℓ-subgroup of G generated by the element v. We have $v>0$, thus $G_1\neq\{0\}$. For each $t\in T$ and $x\in S$ we put $a'_{t,s}=a_{t,s}\wedge v$. Then $(a'_{t,s})_{t\in T,s\in S}$ is an $\mathfrak{m}$-indexed set of elements of the interval $[0,v]$ such that

α_1) $\bigwedge_{t\in T}\bigvee_{s\in S} a'_{t,s}=v$,

β_1) v is the greatest element of the generalized MV-algebra $\Gamma(G,v)=\mathscr{A}_1$.

The underlying set A_1 of $\mathscr{A}_1$ is an interval of the underlying lattice of $\mathscr{A}$. From this and from the fact that $\mathscr{A}$ satisfies (r_2') we conclude that $\mathscr{A}_1$ satisfies (r_2') as well.

Let $0<a\in A$. Then $0<a\wedge v$ and $a\wedge v\in G_1$. Put $a\wedge v=a_1$. Since G_1 satisfies (r_2'), there exists $\varphi\in S^T$ such that

$$a_1\wedge\bigwedge_{t\in T'} a'_{t,\varphi(t)}>0\quad\text{for each finite subset } T' \text{ of } T. \tag{1}$$

Let the element a be as in γ). We have

$$a\wedge\bigwedge_{t\in T'} a_{t,\varphi(t)}\geqslant a_1\wedge\bigwedge_{t\in T'} a'_{t,\varphi(t)}>0,$$

which contradicts γ_1). Therefore (i) must hold. □

Proposition 6.7. *Let $\mathscr{A}$ be a generalized MV-algebra. Suppose that $\mathscr{A}$ is projectable. Then the relation* $(\mathrm{r}_1) \Leftrightarrow (\mathrm{r}_2)$ *holds for* $\mathscr{A}$.

Proof. In view of 6.5, G is projectable. Hence according to 4.5 we have $(\mathrm{r}_1') \Leftrightarrow (\mathrm{r}_2')$. Further, 6.3 yields $(\mathrm{r}_1) \Leftrightarrow (\mathrm{r}_1')$. In view of 6.6, $(\mathrm{r}_2') \Leftrightarrow (\mathrm{r}_2)$. Hence $(\mathrm{r}_1) \Leftrightarrow (\mathrm{r}_2)$. □

Specker groups were investigated by Conrad and Darnel [3] and by the author [12]. For each Boolean algebra B there exists a Specker group G_0 such that G_0 is a lattice ordered group having the property that there exists an ideal B^1 of the underlying lattice of G_0^+ with $B \simeq B^1$.

We denote by u the greatest element of B^1. Further, let G be the convex ℓ-subgroup of G_0 generated by the element u. Put $\mathscr{A} = \Gamma(G, u)$. Then without loss of generality we can assume that B is the underlying lattice of $\mathscr{A}$.

Each Specker group is projectable and abelian; this yields that G is projectable as well. The condition (r_1) for the Boolean algebra B is, in fact, identical to the condition (r_1) for the MV-algebra $\mathscr{A}$; the situation for (r_2) is analogous. Therefore the equivalence $(\mathrm{r}_1) \Leftrightarrow (\mathrm{r}_2)$ for Boolean algebras (cf. Section 1) is a consequence of 6.7.

We remark that in the proof of the implication $(\mathrm{r}_2) \Rightarrow (\mathrm{r}_1)$ for Boolean algebras in the proof of 19.3 in [20], Stone's theorem and arguments of topological character were applied. In our method, all the proofs are purely algebraic.

7. Radical classes of generalized MV-algebras

The collection of all MV-algebras or of all generalized MV-algebras will be denoted by $\mathscr{M}$ or by $\mathscr{M}_g$, respectively.

The notion of radical class of MV-algebras was introduced in [10]. If we modify the definition from [10] in such a way that instead of elements of $\mathscr{M}$, the elements of $\mathscr{M}_g$ are taken into account, then we obtain the definition of *a radical class of generalized MV-algebras.*

For the sake of completness, we present the definition in detail.

Let X be a nonempty subclass of $\mathscr{M}_g$ such that the following conditions are satisfied:

(i) X is closed with respect to isomorphisms;
(ii) if $\mathscr{A} \in X$, then each interval subalgebra of $\mathscr{A}$ belongs to X;
(iii) if $\mathscr{A} \in \mathscr{M}_g$ and if $a_1, \ldots, a_n \in A$ such that all interval subalgebras $\mathscr{A}_{a_1}, \ldots, \mathscr{A}_{a_n}$ of $\mathscr{A}$ belong to X, then the interval subalgebra $\mathscr{A}_a$ also belongs to X, where $a = a_1 \vee \ldots \vee a_n$.

Under these assumptions, X is called a radical class of generalized MV-algebras.

We denote by $\mathscr{R}_\ell$ and $\mathscr{R}_g$ the collection of all radical classes of lattice ordered groups or the collection of all radical classes of generalized MV-algebras, respectively.

Let $X \in \mathscr{R}_\ell$. We denote by $\varphi_1(X)$ the class of all generalized MV-algebras $\mathscr{A}$ such that there exists a unital lattice ordered group (G, u) with $(G, u) \in X$, $\Gamma(G, u) = \mathscr{A}$.

Further, let $Y \in R_g$. The symbol $\varphi_2(Y)$ will denote the class of all lattice ordered groups G such that for each $0 \leqslant u \in G$ we have $\Gamma(G_u, u) \in Y$, where G_u is the convex ℓ-subgroup of G generated by the element u.

Lemma 7.1. *For each $X \in \mathscr{R}_\ell$ and $Y \in R_g$ we have $\varphi_1(X) \in \mathscr{R}_g$ and $\varphi_2(Y) \in \mathscr{R}_\ell$. The mapping φ_1 is a bijection of $\mathscr{R}_\ell$ onto $\mathscr{R}_g$; further, φ_2 is a bijection of $\mathscr{R}_g$ onto $\mathscr{R}_\ell$ and $\varphi_2 = \varphi_1^{-1}$.*

Proof. It suffices to apply the same argument as in [10] by investigating the relation between radical classes of abelian lattice ordered groups and radical classes of MV-algebras. □

Let $\mathfrak{m}$ be an infinite cardinal and $i \in \{1, 2\}$. We denote by $D^i_{\mathfrak{m}}$ the class of all generalized MV-algebras which satisfy the condition ($\mathrm{r_i}$).

Proposition 7.2. *$D^1_{\mathfrak{m}}$ is a radical class of generalized MV-algebras.*

Proof. This is a consequence of 5.6 and 7.1. □

On the other hand, $D^2_{\mathfrak{m}}$ fails to be a radical class. We verify this as follows. Put $D^2_{\mathfrak{m}} = Y$ and let $\varphi_2(Y)$ be as above. Further, let $C^2_{\mathfrak{m}}$ be as in Section 5. Then in view of 6.6 we have $\varphi_2(Y) = C^2_{\mathfrak{m}}$. If $Y \in \mathscr{R}_g$, then in view of 7.1 we would have $C^2_{\mathfrak{m}} \in \mathscr{R}_\ell$. But in Section 5 we observed that $C^2_{\mathfrak{m}}$ fails to be a radical class; hence we have arrived at a contradiction.

References

[1] *C. C. Chang*: On the representation of α-complete Boolean algebras. Trans. Amer. Math. Soc. *85* (1957), 208–218.

[2] *P. Conrad*: K-radical classes of lattice ordered groups. In: Proc. Conf. Carbondale, Lecture Notes Math. vol. 848, Springer Verlag, New York, 1981, pp. 186–207.

[3] *P. Conrad and M. R. Darnel*: Subgroups and hulls of Specker lattice ordered groups. Czech. Math. J. *51* (2001), 395–413.

[4] *M. Darnel*: Closure operators on radicals of lattice ordered groups. Czech. Math. J. *37* (1987), 51–64.

[5] *A. Dvurečenskij*: Pseudo MV-algebras are intervals in ℓ-groups. J. Austral. Math. Soc. *72* (2002), 427–445.

[6] *G. Georgescu and A. Iorgulescu*: Pseudo MV-algebras: a noncommutative extension of MV-algebras. Proc. Fourth Int. Symp. Econ. Inf., Bucharest. 1999, pp. 961–968.

[7] *G. Georgescu and A. Iorgulescu*: Pseudo MV-algebras. Multiple Valued Logic *6* (2001), 95–135.

[8] *J. Jakubík*: Radical mappings and radical classes of lattice ordered groups. Symposia Math. 21. Academic Press, New York-London, 1977, pp. 451–477.

[9] *J. Jakubík*: Radical classes of generalized Boolean algebras. Czech. Math. J. *48* (1998), 253–268.

[10] *J. Jakubík*: Radical classes of MV-algebras. Czech. Math. J. *49* (1999), 191–211.

[11] *J. Jakubík*: Direct product decompositions of pseudo MV-algebras. Archivum Math. *37* (2001), 131–142.

[12] *J. Jakubík*: Torsion classes of Specker lattice ordered groups. Czech. Math. J. *52* (2002), 469–482.

[13] *L. H. Loomis*: On the representation of σ-complete Boolean algebras. Bull. Amer. Math. Soc. *53* (1947), 757–760.

[14] *R. S. Pierce*: Representation theorems for certain Boolean algebras. Proc. Amer. Math. Soc. *10* (1959), 42–50.

[15] *J. Rachůnek*: A non-commutative generalization of MV-algebras. Czech. Math. J. *52* (2002), 255–273.

[16] *D. Scott*: A new characterization of α-representable Boolean algebras. Bull. Amer. Math. Soc. *61* (1955), 522–523.

[17] *E. C. Smith, Jr.*: A distributivity condition for Boolean algebras. Ann. Math. *64* (1956), 551–561.

[18] *R. Sikorski*: On the representation of Boolean algebras as fields of set. Fund. Math. *35* (1958), 247–258.

[19] *R. Sikorski*: Distributivity and representability. Fund. Math. *48* (1959), 95–103.

[20] *R. Sikorski*: Boolean Algebras. Second Edition, Springer Verlag, Berlin-Göttingen-Heidelberg-New York, 1964.

[21] *Dao Rong Ton*: Product radical classes of ℓ-groups. Czech. Math. J. *42* (1992), 129–142.

Author's address: Ján Jakubík, Matematický ústav SAV, Grešákova 6, 040 01 Košice, Slovakia, e-mail: `kstefan@saske.sk`.