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PROJECTION REPRESENTABLE RELATIONS ON MENGER $(2,n)$-SEMIGROUPS

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Abstract. Abstract characterizations of relations of nonempty intersection, inclusion end equality of domains for partial n-place functions are presented. Representations of Menger $(2,n)$-semigroups by partial n-place functions closed with respect to these relations are investigated.

Keywords: n-place function, algebra of functions, Menger algebra, $(2,n)$-semigroup

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1. Introduction

Investigation of partial multiplace functions by algebraic methods plays an important role in modern mathematics where we consider various operations on sets of functions which are naturally defined. The basic operation for n-place functions is a superposition (composition) O of $n+1$ such functions, but there are some other naturally defined operations which are also worth considering. In this paper we consider binary Mann's compositions $\underset{1}{\oplus},\ldots,\underset{n}{\oplus}$ for partial n-place functions introduced in [4], which have many important applications for the studies of binary and n-ary operations. Algebras of n-place functions closed with respect to these compositions were investigated, for example, in [11] and [16].

2. Preliminaries and notations

Let A^n be the n-th Cartesian product of a set A. Any partial mapping from A^n into A is called a *partial n-place function.* The set of all such mappings is denoted by $\mathscr{F}(A^n, A)$. On $\mathscr{F}(A^n, A)$ we define the *Menger superposition* (composition) of n-place functions $O\colon (f, g_1, \ldots, g_n) \mapsto f[g_1 \ldots g_n]$ and n binary compositions $\underset{1}{\oplus}, \ldots, \underset{n}{\oplus}$ putting

$$f[g_1 \ldots g_n](a_1, \ldots, a_n) = f(g_1(a_1, \ldots, a_n), \ldots, g_n(a_1, \ldots, a_n)), \tag{1}$$

$$(f \underset{i}{\oplus} g)(a_1, \ldots, a_n) = f(a_1, \ldots, a_{i-1}, g(a_1, \ldots, a_n), a_{i+1}, \ldots, a_n), \tag{2}$$

for all $f, g, g_1, \ldots, g_n \in \mathscr{F}(A^n, A)$ and $(a_1, \ldots, a_n) \in A^n$, where the left- and the right-hand sides of (1) and (2) are defined or not defined simultaneously. Since, as it is not difficult to verify, each composition $\underset{i}{\oplus}$ is an associative operation, algebras of the form $(\Phi; \underset{1}{\oplus}, \ldots, \underset{n}{\oplus})$ and $(\Phi; O, \underset{1}{\oplus}, \ldots, \underset{n}{\oplus})$, where $\Phi \subset \mathscr{F}(A^n, A)$, are called respectively *$(2, n)$-semigroups* and *Menger $(2, n)$-semigroups of n-place functions.*

According to the general convention used in the theory of n-ary systems, the sequence $x_i, x_{i+1}, \ldots, x_j$, where $i \leqslant j$, can be written as x_i^j (for $i > j$ it is the empty symbol). With this convention (1) and (2) can be written as

$$\begin{aligned} f[g_1^n](a_1^n) &= f(g_1(a_1^n), \ldots, g_n(a_1^n)), \\ (f \underset{i}{\oplus} g)(a_1^n) &= f(a_1^{i-1}, g(a_1^n), a_{i+1}^n). \end{aligned}$$

An algebra $(G; o)$ with one $(n+1)$-ary operation o satisfying the identity

$$o(o(x_0^n), y_1^n) = o(x_0, o(x_1, y_1^n), \ldots, o(x_n, y_1^n))$$

is called a *Menger algebra of rank n* (cf. [1], [10]). Such operation is called *superassociative* and by many authors is written as $o(x_0^n) = x_0[x_1^n]$. Such notation is motivated by the fact that the composition O of n-place functions is, as it is not difficult to see, an $(n+1)$-ary superassociative operation. In this convention the above identity has the form

$$x_0[x_1^n][y_1^n] = x_0[x_1[y_1^n] \ldots x_n[y_1^n]\,], \tag{3}$$

where $x_0[x_1^n][y_1^n]$ must be read as $(x_0[x_1^n])[y_1^n]$.

It is clear that an arbitrary semigroup is a Menger algebra of rank 1. Some properties of Menger algebras can be characterized by its *diagonal semigroup* (see [10]), i.e., the semigroup $(G, \star)$, where $x \star y = o(x, y, \ldots, y)$.

Let $\{\underset{1}{\oplus},\ldots,\underset{n}{\oplus}\}$ be the collection of associative binary operations defined on G. According to [11] and [16], an algebra $(G;\underset{1}{\oplus},\ldots,\underset{n}{\oplus})$ is called a $(2,n)$-*semigroup*. By a *Menger* $(2,n)$-*semigroup* we mean an algebra $(G;o,\underset{1}{\oplus},\ldots,\underset{n}{\oplus})$, where $(G;o)$ is a Menger algebra of rank n and $(G;\underset{1}{\oplus},\ldots,\underset{n}{\oplus})$ is a $(2,n)$-semigroup. Any homomorphism of a (Menger) $(2,n)$-semigroup onto some (Menger) $(2,n)$-semigroup of n-place functions is called a *representation by n-place functions*. A representation is *faithful* if it is an isomorphism (cf. [10]).

All expressions of the form $(\ldots((x\underset{i_1}{\oplus}y_1)\underset{i_2}{\oplus}y_2)\ldots)\underset{i_s}{\oplus}y_s$, where $\underset{i_k}{\oplus}$ are operations from the collection $\{\underset{1}{\oplus},\ldots,\underset{n}{\oplus}\}$ and $x,y_1,\ldots,x_s\in G$, are written as $x\underset{i_1}{\oplus}y_1\underset{i_2}{\oplus}\ldots\underset{i_s}{\oplus}y_s$ or, in the abbreviated form, as $x\overset{i_s}{\underset{i_1}{\oplus}}y_1^s$. The symbol $\mu_i(\overset{i_s}{\underset{i_1}{\oplus}}x_1^s)$, in the case $i=i_k$ and $i\neq i_p$ for all $p<k\leqslant s$, denotes the element $x_{i_k}\overset{i_s}{\underset{i_{k+1}}{\oplus}}x_{k+1}^s$. In any other case it is the empty symbol. For example, $\mu_1(\underset{2}{\oplus}x\underset{2}{\oplus}y\underset{3}{\oplus}z\underset{1}{\oplus}u\underset{3}{\oplus}v)=u\underset{3}{\oplus}v$, $\mu_2(\underset{2}{\oplus}x\underset{2}{\oplus}y\underset{3}{\oplus}z\underset{1}{\oplus}u\underset{3}{\oplus}v)=x\underset{2}{\oplus}y\underset{3}{\oplus}z\underset{1}{\oplus}u\underset{3}{\oplus}v$, $\mu_3(\underset{2}{\oplus}x\underset{2}{\oplus}y\underset{3}{\oplus}z\underset{1}{\oplus}u\underset{3}{\oplus}v)=z\underset{1}{\oplus}u\underset{3}{\oplus}v$. The symbol $\mu_4(\underset{2}{\oplus}x\underset{2}{\oplus}y\underset{3}{\oplus}z\underset{1}{\oplus}u\underset{3}{\oplus}v)$ is empty.

In [11] it is proved that a $(2,n)$-semigroup $(G;\underset{1}{\oplus},\ldots,\underset{n}{\oplus})$ has a faithful representation by n-place functions if and only if it satisfies the implication[1]

$$\bigwedge_{i=1}^{n}\Big(\mu_i(\overset{i_s}{\underset{i_1}{\oplus}}x_1^s)=\mu_i(\overset{j_k}{\underset{j_1}{\oplus}}y_1^k)\Big)\longrightarrow g\overset{i_s}{\underset{i_1}{\oplus}}x_1^s=g\overset{j_k}{\underset{j_1}{\oplus}}y_1^k \tag{4}$$

for all $g,x_1,\ldots,x_s,y_1,\ldots,y_k\in G$. A Menger $(2,n)$-semigroup has a faithful representation if and only if it satisfies (4) and

$$(x\underset{i}{\oplus}y)[z_1^n]=x[z_1^{i-1}y[z_1^n]\,z_{i+1}^n], \tag{5}$$

$$x[y_1^n]\underset{i}{\oplus}z=x[(y_1\underset{i}{\oplus}z)\ldots(y_n\underset{i}{\oplus}z)], \tag{6}$$

$$x\overset{i_s}{\underset{i_1}{\oplus}}y_1^s=x[\mu_1(\overset{i_s}{\underset{i_1}{\oplus}}y_1^s)\ldots\mu_n(\overset{i_s}{\underset{i_1}{\oplus}}y_1^s)], \tag{7}$$

where $\{i_1,\ldots,i_s\}=\{1,\ldots,n\}$ and $i=1,\ldots,n$. In the sequel, any (Menger) $(2,n)$-semigroup satisfying the condition (4) (respectively, (4), (5), (6) and (7)) will be called *representable*.

[1] We use the following notation: $\sim$-negation, $\wedge$-conjunction, $\vee$-disjunction, $\rightarrow$-implication, $\leftrightarrow$-equivalence, $\forall$-universal quantifier, $\exists$-existential quantifier.

Let Φ be some set of n-place functions, i.e., $\Phi \subset \mathscr{F}(A^n, A)$. Consider the following three binary relations on Φ:

$$\chi_\Phi = \{(f,g) \in \Phi \times \Phi \mid \mathrm{pr}_1 f \subset \mathrm{pr}_1 g\},$$
$$\gamma_\Phi = \{(f,g) \in \Phi \times \Phi \mid \mathrm{pr}_1 f \cap \mathrm{pr}_1 g \neq \emptyset\},$$
$$\pi_\Phi = \{(f,g) \in \Phi \times \Phi \mid \mathrm{pr}_1 f = \mathrm{pr}_1 g\},$$

where $\mathrm{pr}_1 f$ is the domain of f, called respectively: *inclusion of domains*, *co-definability* and *equality of domains*.

Abstract characterizations of such relations for semigroups of transformations were studied in [7], [8], [9] and for Menger algebras of n-place functions in [12], [13], [14]. We characterize these relations in $(2,n)$-semigroups and in Menger $(2,n)$-semigroups of n-place functions.

Consider a representable (Menger) $(2,n)$-semigroup $(G; \underset{1}{\oplus}, \ldots, \underset{n}{\oplus})$ (respectively, $(G; o, \underset{1}{\oplus}, \ldots, \underset{n}{\oplus})$) and its representation P by n-place functions. On the set G we define three binary relations:

$$\chi_P = \{(g_1,g_2) \mid \mathrm{pr}_1 P(g_1) \subset \mathrm{pr}_1 P(g_2)\},$$
$$\gamma_P = \{(g_1,g_2) \mid \mathrm{pr}_1 P(g_1) \cap \mathrm{pr}_1 P(g_2) \neq \emptyset\},$$
$$\pi_P = \{(g_1,g_2) \mid \mathrm{pr}_1 P(g_1) = \mathrm{pr}_1 P(g_2)\}.$$

It is not difficult to see that χ_P is a quasi-order, i.e., χ_P is reflexive and transitive relation, and π_P is an equivalence such that $\pi_P = \chi_P \cap \chi_P^{-1}$, where $\chi_P^{-1} = \{(b,a) \mid (a,b) \in \chi_P\}$.

Let $(P_i)_{i \in I}$ be a family of representations of a representable $(2,n)$-semigroup $(G; \underset{1}{\oplus}, \ldots, \underset{n}{\oplus})$ (respectively, representable Menger $(2,n)$-semigroup $(G; o, \underset{1}{\oplus}, \ldots, \underset{n}{\oplus})$) by n-place functions defined on sets $(A_i)_{i \in I}$ respectively, where the sets A_i are pair-wise disjoint. The *sum* of $(P_i)_{i \in I}$ is the mapping $P\colon\ g \mapsto P(g)$, denoted by $\sum\limits_{i \in I} P_i$, where $P(g)$ is an n-place function on $A = \bigcup\limits_{i \in I} A_i$, such that $P(g) = \bigcup\limits_{i \in I} P_i(g)$ for every $g \in G$. The sum of any family of representations by n-place functions is also a representation by n-place functions and

$$\chi_P = \bigcap_{i \in I} \chi_{P_i}, \quad \gamma_P = \bigcup_{i \in I} \gamma_{P_i}, \quad \pi_P = \bigcap_{i \in I} \pi_{P_i}. \tag{8}$$

Let 0 be a zero of a $(2,n)$-semigroup $(G; \underset{1}{\oplus}, \ldots, \underset{n}{\oplus})$ (respectively, Menger $(2,n)$-semigroup $(G; o, \underset{1}{\oplus}, \ldots, \underset{n}{\oplus})$), i.e., $0 \underset{i}{\oplus} g = g \underset{i}{\oplus} 0 = 0$ (respectively, $0 \underset{i}{\oplus} g = g \underset{i}{\oplus} 0 = 0$ and $0[g_1^n] = g[g_1^{i-1}\, 0\, g_{i+1}^n] = 0$) for all $i = 1, \ldots, n$ and $g, g_1, \ldots, g_n \in G$. We say that a binary relation $\varrho \subset G \times G$ is 0-*reflexive*, if $(g,g) \in \varrho$ for all $g \in G \setminus \{0\}$. A symmetric relation ϱ which is reflexive if $0 \in \mathrm{pr}_1\, \varrho$, and 0-reflexive if $0 \notin \mathrm{pr}_1\, \varrho$, is called a 0-*quasi-equivalence*. If G does not contains a zero, then by a 0-quasi-equivalence relation we understand a reflexive and symmetric binary relation.

A binary relation Δ on a Menger $(2,n)$-semigroup $(G; o, \underset{1}{\oplus}, \ldots, \underset{n}{\oplus})$ is called:

- *l-regular*, if

$$x \,\Delta\, y \longrightarrow x[z_1^n]\Delta y[z_1^n], \tag{9}$$

$$x \,\Delta\, y \longrightarrow x \underset{i}{\oplus} z \Delta y \underset{i}{\oplus} z \tag{10}$$

  for all $i = 1, \ldots, n$ and $x, y, z, z_1, \ldots, z_n \in G$,

- *l-cancellative*, if

$$x[z_1^n]\Delta y[z_1^n] \longrightarrow x \,\Delta\, y, \tag{11}$$

$$x \underset{i}{\oplus} z \Delta y \underset{i}{\oplus} z \longrightarrow x \,\Delta\, y \tag{12}$$

  for all $i = 1, \ldots, n$ and $x, y, z, z_1, \ldots, z_n \in G$,

- *v-negative*, if

$$x[y_1^n]\Delta y_i, \;\; i = 1, \ldots, n, \tag{13}$$

$$x \overset{i_s}{\underset{i_1}{\oplus}} z_1^s \Delta \mu_j (\overset{i_s}{\underset{i_1}{\oplus}} z_1^s) \tag{14}$$

  for all $x, y_1, \ldots, y_n, z_1, \ldots, z_s \in G$ and $j \in \{i_1, \ldots, i_s\}$.

In the case of $(2,n)$-semigroups these relations are defined by (10), (12) and (14), respectively.[2]

3. Projection representable relations on Menger $(2,n)$-semigroups

Let $\mathscr{G} = (G; o, \underset{1}{\oplus}, \ldots, \underset{n}{\oplus})$ be a representable Menger $(2,n)$-semigroup and let χ, γ and π be binary relations on G. We say that the triplet (χ, γ, π) is (*faithful*) *projection representable for* $\mathscr{G}$ if there exists a (faithful) representation P of $\mathscr{G}$ by n-place functions for which $\chi = \chi_P$, $\gamma = \gamma_P$ and $\pi = \pi_P$. Analogously we define projection representable pairs and separate relations.

In the sequel, instead of $(g_1, g_2) \in \chi$, $(g_1, g_2) \in \gamma$ and $(g_1, g_2) \in \pi$ we will write $g_1 \sqsubset g_2$, $g_1 \top g_2$ and $g_1 \equiv g_2$, respectively.

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[2] If Δ is a quasi-order relation, then the condition (13) is equivalent to condition

$$(\forall x)(\forall y)(\forall u)(\forall \overline{w})(\forall i)((x, u[\overline{w}|_i y]) \in \Delta \longrightarrow (x,y) \in \Delta),$$

where $u, x, y \in G$, $\overline{w} \in G^n$, $i \in \{1, \ldots, n\}$ and $u[\overline{w}|_i y] = u[w_1^{i-1} y w_{i+1}^n]$ (see [10]).

Theorem 1. *A triplet (χ,γ,π) of binary relations on G is projection representable for a representable Menger $(2,n)$-semigroup $\mathscr{G}$ if and only if the following conditions are satisfied:*

(a) *χ is an l-regular and v-negative quasi-order,*

(b) *γ is an l-cancellative 0-quasi-equivalence,*

(c) *$\pi=\chi\cap\chi^{-1}$ and*

$$h_1 \top h_2 \ \wedge\ h_1 \sqsubset g_1 \ \wedge\ h_2 \sqsubset g_2 \longrightarrow g_1 \top g_2 \tag{15}$$

for all $h_1,h_2,g_1,g_2\in G$.

P r o o f. *Necessity.* Let $(\Phi;O,\underset{1}{\oplus},\ldots,\underset{n}{\oplus})$ be a Menger $(2,n)$-semigroup of n-place functions determined on the set A. Let us show that the triplet $(\chi_\Phi,\gamma_\Phi,\pi_\Phi)$ satisfies all the conditions of the theorem.

At first we prove the condition (a). The relation χ_Φ is obviously a quasi-order. Let $f,g,h_1,\ldots,h_n\in\Phi$ and $(f,g)\in\chi_\Phi$, i.e., $\mathrm{pr}_1 f\subset\mathrm{pr}_1 g$. Suppose that $\bar a\in\mathrm{pr}_1 f[h_1^n]$ for some $\bar a\in A^n$. Then $\{f[h_1^n](\bar a)\}\neq\emptyset$, i.e., $\{f(h_1(\bar a),\ldots,h_n(\bar a))\}\neq\emptyset$. Thus $(h_1(\bar a),\ldots,h_n(\bar a))\in\mathrm{pr}_1 f$ and, consequently, $(h_1(\bar a),\ldots,h_n(\bar a))\in\mathrm{pr}_1 g$. Therefore $\{g(h_1(\bar a),\ldots,h_n(\bar a))\}\neq\emptyset$, whence $\{g[h_1^n](\bar a)\}\neq\emptyset$, i.e., $\bar a\in\mathrm{pr}_1 g[h_1^n]$. So, $\mathrm{pr}_1 f[h_1^n]\subset\mathrm{pr}_1 g[h_1^n]$, which implies $(f[h_1^n],g[h_1^n])\in\chi_\Phi$. Similarly we can prove that for all $f,g,h\in\Phi$ and $i=1,\ldots,n$, from $(f,g)\in\chi_\Phi$ it follows that $(f\underset{i}{\oplus}h,g\underset{i}{\oplus}h)\in\chi_\Phi$. This means that the relation χ_Φ is l-regular. The proof of the v-negativity is analogous.

To prove (b) let Θ be a zero of a Menger $(2,n)$-semigroup $(\Phi;O,\underset{1}{\oplus},\ldots,\underset{n}{\oplus})$. If $\Theta\neq\emptyset$, then $\mathrm{pr}_1\Theta\neq\emptyset$, whence $(\Theta,\Theta)\in\gamma_\Phi$. Thus $\Theta\in\mathrm{pr}_1\gamma_\Phi$. So, in this case γ_Φ is reflexive. For $\Theta=\emptyset$ we have $\mathrm{pr}_1\Theta=\emptyset$. Therefore $\Theta\notin\mathrm{pr}_1\gamma_\Phi$, i.e., $(f,f)\in\gamma_\Phi$ for every $f\neq\Theta$. Hence γ_Φ is Θ-reflexive. Since γ_Φ is symmetric, the above means that γ_Φ is a Θ-quasi-equivalence. If Φ does not contain a zero, then γ_Φ is a reflexive and symmetric binary relation.

Suppose now that $(f[h_1^n],g[h_1^n])\in\gamma_\Phi$ for some $f,g\in\Phi$, $h_1,\ldots,h_n\in\Phi$. Then $\mathrm{pr}_1 f[h_1^n]\cap\mathrm{pr}_1 g[h_1^n]\neq\emptyset$, i.e., there exists $\bar a\in A^n$ such that $\bar a\in\mathrm{pr}_1 f[h_1^n]$ and $\bar a\in\mathrm{pr}_1 g[h_1^n]$. Therefore $\{f[h_1^n](\bar a)\}\neq\emptyset$ and $\{g[h_1^n](\bar a)\}\neq\emptyset$. Thus $\{f(h_1(\bar a),\ldots,h_n(\bar a))\}\neq\emptyset$ and $\{g(h_1(\bar a),\ldots,h_n(\bar a))\}\neq\emptyset$, which shows that $(h_1(\bar a),\ldots,h_n(\bar a))\in\mathrm{pr}_1 f\cap\mathrm{pr}_1 g$. So, $(f,g)\in\gamma_\Phi$. Analogously, for $f,g,h\in\Phi$, $i=1,\ldots,n$, from $(f\underset{i}{\oplus}h,g\underset{i}{\oplus}h)\in\gamma_\Phi$ it follows that $(f,g)\in\gamma_\Phi$. So, γ_Φ is l-cancellative.

Since in (c) the first condition is obvious, we prove (15) only. For this let $(h_1,h_2)\in\gamma_\Phi$, $(h_1,g_1)\in\chi_\Phi$ and $(h_2,g_2)\in\chi_\Phi$ for some $h_1,h_2,g_1,g_2\in\Phi$. Then $\mathrm{pr}_1 h_1\cap\mathrm{pr}_1 h_2\neq\emptyset$, $\mathrm{pr}_1 h_1\subset\mathrm{pr}_1 g_1$ and $\mathrm{pr}_1 h_2\subset\mathrm{pr}_1 g_2$, whence $\emptyset\neq\mathrm{pr}_1 h_1\cap\mathrm{pr}_1 h_2\subset\mathrm{pr}_1 g_1\cap\mathrm{pr}_1 g_2$. Thus $\mathrm{pr}_1 g_1\cap\mathrm{pr}_1 g_2\neq\emptyset$, i.e., $(g_1,g_2)\in\gamma_\Phi$, which proves (15) and completes the proof of the necessity of the conditions formulated in the theorem. $\square$

To prove the sufficiency of these conditions we must introduce some additional constructions. Consider the triplet (χ,γ,π) of binary relations on a representable Menger $(2,n)$-semigroup $\mathscr{G} = (G; o, \underset{1}{\oplus}, \ldots, \underset{n}{\oplus})$ satisfying all the conditions of the theorem. Let $e_1,\ldots,e_n$ be pairwise different elements not belonging to G. For all $x_1,\ldots,x_s \in G$, $i=1,\ldots,n$, and operations $\underset{i_1}{\oplus},\ldots,\underset{i_s}{\oplus}$ defined on G, we denote by $\mu_i^*(\overset{i_s}{\underset{i_1}{\oplus}} x_1^s)$ an element of $G^* = G \cup \{e_1,\ldots,e_n\}$ such that

$$\mu_i^*(\overset{i_s}{\underset{i_1}{\oplus}} x_1^s) = \begin{cases} \mu_i(\overset{i_s}{\underset{i_1}{\oplus}} x_1^s) & \text{if } i \in \{i_1,\ldots,i_s\}, \\ e_i & \text{if } \ i \notin \{i_1,\ldots,i_s\}. \end{cases}$$

Consider the set $\mathfrak{A}^* = G^n \cup \mathfrak{A}_0 \cup \{(e_1,\ldots,e_n)\}$, where $\mathfrak{A}_0$is the collection of all n-tuples $(x_1,\ldots,x_n) \in (G^*)^n$ for which there exists $y_1,\ldots,y_s \in G$ and $i_1,\ldots,i_n \in \{1,\ldots,n\}$ such that $x_i = \mu_i^*(\overset{i_s}{\underset{i_1}{\oplus}} y_1^s)$. Let $(h_1,h_2) \in G^2$ be fixed. For each $g \in G$ we define a partial n-place function $P_{(h_1,h_2)}(g)\colon\ \mathfrak{A}^* \to G$ such that

$$x_1^n \in \mathrm{pr}_1\, P_{(h_1,h_2)}(g) \longleftrightarrow \begin{cases} h_1 \sqsubset g[x_1^n] \,\vee\, h_2 \sqsubset g[x_1^n] & \text{if } x_1^n \in G^n, \\ h_1 \sqsubset g \,\vee\, h_2 \sqsubset g & \text{if } x_1^n = e_1^n, \\ h_1 \sqsubset g \overset{i_s}{\underset{i_1}{\oplus}} y_1^s \,\vee\, h_2 \sqsubset g \overset{i_s}{\underset{i_1}{\oplus}} y_1^s & \text{if } \ x_i = \mu_i^*(\overset{i_s}{\underset{i_1}{\oplus}} y_1^s), \\ & i=1,\ldots,n, \ \text{for} \\ & \text{some } y_1^s \in G^s \text{and} \\ & i_1\ldots,i_s \in \{1,\ldots,n\}. \end{cases}$$

For $x_1^n \in \mathrm{pr}_1\, P_{(h_1,h_2)}(g)$ we put

$$P_{(h_1,h_2)}(g)(x_1^n) = \begin{cases} g[x_1^n] & \text{if } x_1^n \in G^n, \\ g & \text{if } x_1^n = e_1^n, \\ g \overset{i_s}{\underset{i_1}{\oplus}} y_1^s & \text{if } \ x_i = \mu_i^*(\overset{i_s}{\underset{i_1}{\oplus}} y_1^s), \\ & i=1,\ldots,n, \ \text{for} \\ & \text{some } y_1^s \in G^s \text{ and} \\ & i_1\ldots,i_s \in \{1,\ldots,n\}. \end{cases} \tag{16}$$

Let us show that $P_{(h_1,h_2)}$ is a representation of $\mathscr{G}$ by n-place functions.

Proposition 1. *The function $P_{(h_1,h_2)}(g)$ is single-valued.*

P r o o f. Let $x_1^n \in \mathrm{pr}_1\, P_{(h_1,h_2)}(g)$, where $g, h_1, h_2 \in G$ are fixed. Since for $x_1^n \in G^n$ and $x_1^n = e_1^n$ the value of $P_{(h_1,h_2)}(g)(x_1^n)$ is uniquely determined, we verify only the case when $x_i = \mu_i^*(\overset{i_s}{\underset{i_1}{\oplus}} y_1^s)$, $i = 1, \ldots, n$, for some $y_1^s \in G^s$. If for some $z_1^k \in G^k$ and $j_1, \ldots, j_k \in \{1, \ldots, n\}$ we have also $x_i = \mu_i^*(\overset{j_k}{\underset{j_1}{\oplus}} z_1^k)$, $i = 1, \ldots, n$, then $\mu_i(\overset{i_s}{\underset{i_1}{\oplus}} y_1^s) = \mu_i(\overset{j_k}{\underset{j_1}{\oplus}} z_1^k)$ for every $i = 1, \ldots, n$, which, according to (4), implies $g \overset{i_s}{\underset{i_1}{\oplus}} y_1^s = g \overset{j_k}{\underset{j_1}{\oplus}} z_1^k$. This means that also in this case $P_{(h_1,h_2)}(g)(x_1^n)$ is uniquely determined. Thus, the function $P_{(h_1,h_2)}(g)$ is single-valued. □

Proposition 2. *For all $g, g_1, \ldots, g_n, h_1, h_2 \in G$ we have*

$$P_{(h_1,h_2)}(g[g_1^n]) = P_{(h_1,h_2)}(g)[P_{(h_1,h_2)}(g_1) \ldots P_{(h_1,h_2)}(g_n)].$$

P r o o f. Let $g, g_1, \ldots, g_n \in G$ and $x_1^n \in \mathrm{pr}_1\, P_{(h_1,h_2)}(g[g_1^n])$. If $x_1^n \in G^n$, then

$$h_1 \sqsubset g[g_1^n][x_1^n] \vee h_2 \sqsubset g[g_1^n][x_1^n],$$

whence, applying the superassociativity, we obtain

$$h_1 \sqsubset g[g_1[x_1^n] \ldots g_n[x_1^n]] \vee h_2 \sqsubset g[g_1[x_1^n] \ldots g_n[x_1^n]]. \tag{17}$$

This together with the v-negativity of χ implies

$$h_1 \sqsubset g_i[x_1^n] \vee h_2 \sqsubset g_i[x_1^n],\ i = 1, \ldots, n. \tag{18}$$

¿From (17) it follows that $(g_1[x_1^n], \ldots, g_n[x_1^n]) \in \mathrm{pr}_1\, P_{(h_1,h_2)}(g)$, from (18) that $x_1^n \in \mathrm{pr}_1\, P_{(h_1,h_2)}(g_i)$, $i = 1, \ldots, n$. So, if $x_1^n \in G^n$, then

$$x_1^n \in \mathrm{pr}_1\, P_{(h_1,h_2)}(g[g_1^n]) \longleftrightarrow \begin{cases} (g_1[x_1^n], \ldots, g_n[x_1^n]) \in \mathrm{pr}_1\, P_{(h_1,h_2)}(g), \\ \bigwedge\limits_{i=1}^{n} x_1^n \in \mathrm{pr}_1\, P_{(h_1,h_2)}(g_i). \end{cases} \tag{19}$$

Analogously we can verify that

$$e_1^n \in \mathrm{pr}_1\, P_{(h_1,h_2)}(g[g_1^n]) \longleftrightarrow \begin{cases} (g_1, \ldots, g_n) \in \mathrm{pr}_1\, P_{(h_1,h_2)}(g), \\ \bigwedge\limits_{i=1}^{n} e_1^n \in \mathrm{pr}_1\, P_{(h_1,h_2)}(g_i). \end{cases} \tag{20}$$

Now let $x_i = \mu_i^*(\overset{i_s}{\underset{i_1}{\oplus}} y_1^s)$, $i = 1, \ldots, n$, for some $i_1, \ldots, i_s \in \{1, \ldots, n\}$ and $y_1^s \in G^s$. Then $x_1^n \in \mathrm{pr}_1 P_{(h_1,h_2)}(g[g_1^n])$ implies

$$h_1 \sqsubset g[g_1^n] \overset{i_s}{\underset{i_1}{\oplus}} y_1^s \vee h_2 \sqsubset g[g_1^n] \overset{i_s}{\underset{i_1}{\oplus}} y_1^s,$$

which, by (6), is equivalent to

$$h_1 \sqsubset g[(g_1 \overset{i_s}{\underset{i_1}{\oplus}} y_1^s) \ldots (g_n \overset{i_s}{\underset{i_1}{\oplus}} y_1^s)] \vee h_2 \sqsubset g[(g_1 \overset{i_s}{\underset{i_1}{\oplus}} y_1^s) \ldots (g_n \overset{i_s}{\underset{i_1}{\oplus}} y_1^s)]. \tag{21}$$

¿From this, applying the v-negativity of χ, we obtain

$$h_1 \sqsubset g_i \overset{i_s}{\underset{i_1}{\oplus}} y_1^s \vee h_2 \sqsubset g_i \overset{i_s}{\underset{i_1}{\oplus}} y_1^s \tag{22}$$

for every $i = 1, \ldots, n$.

The condition (21) is equivalent to $(g_1 \overset{i_s}{\underset{i_1}{\oplus}} y_1^s, \ldots, g_n \overset{i_s}{\underset{i_1}{\oplus}} y_1^s) \in \mathrm{pr}_1 P_{(h_1,h_2)}(g)$. The condition (22) shows that $x_1^n \in \mathrm{pr}_1 P_{(h_1,h_2)}(g_i)$ for every $i = 1, \ldots, n$, where $x_i = \mu_i^*(\overset{i_s}{\underset{i_1}{\oplus}} y_1^s)$, $i = 1, \ldots, n$. So,

$$x_1^n \in \mathrm{pr}_1 P_{(h_1,h_2)}(g[g_1^n]) \longleftrightarrow \begin{cases} (g_1 \overset{i_s}{\underset{i_1}{\oplus}} y_1^s, \ldots, g_n \overset{i_s}{\underset{i_1}{\oplus}} y_1^s) \in \mathrm{pr}_1 P_{(h_1,h_2)}(g), \\ \bigwedge\limits_{i=1}^{n} x_1^n \in \mathrm{pr}_1 P_{(h_1,h_2)}(g_i), \end{cases} \tag{23}$$

where $x_i = \mu_i^*(\overset{i_s}{\underset{i_1}{\oplus}} y_1^s)$, $i = 1, \ldots, n$.

Let $x_1^n \in \mathrm{pr}_1 P_{(h_1,h_2)}(g[g_1^n])$. If $x_1^n \in G^n$, then, according to (16) and (19), we have

$$\begin{aligned} P_{(h_1,h_2)}(g[g_1^n])(x_1^n) &= g[g_1^n][x_1^n] = g[g_1[x_1^n] \ldots g_n[x_1^n]] \\ &= P_{(h_1,h_2)}(g)(g_1[x_1^n], \ldots, g_n[x_1^n]) \\ &= P_{(h_1,h_2)}(g) \left(P_{(h_1,h_2)}(g_1)(x_1^n), \ldots, P_{(h_1,h_2)}(g_n)(x_1^n)\right) \\ &= P_{(h_1,h_2)}(g) \left[P_{(h_1,h_2)}(g_1) \ldots P_{(h_1,h_2)}(g_n)\right] (x_1^n). \end{aligned}$$

Similarly, we can prove that

$$P_{(h_1,h_2)}(g[g_1^n])(e_1^n) = P_{(h_1,h_2)}(g) \left[P_{(h_1,h_2)}(g_1) \ldots P_{(h_1,h_2)}(g_n)\right] (e_1^n)$$

for $e_1^n \in \mathrm{pr}_1 P_{(h_1,h_2)}(g[g_1^n])$.

If $x_1^n \in \mathrm{pr}_1\, P_{(h_1,h_2)}(g[g_1^n])$, where $x_i = \mu_i^*(\overset{i_s}{\underset{i_1}{\oplus}} y_1^s)$, $i = 1, \ldots, n$, for some $y_1^s \in G^s$, $i_1, \ldots, i_s \in \{1, \ldots, n\}$, then, according to (16) and (23), we obtain

$$\begin{aligned} P_{(h_1,h_2)}(g[g_1^n])(x_1^n) &= g[g_1^n] \overset{i_s}{\underset{i_1}{\oplus}} y_1^s = g[(g_1 \overset{i_s}{\underset{i_1}{\oplus}} y_1^s) \ldots (g_n \overset{i_s}{\underset{i_1}{\oplus}} y_1^s)] \\ &= P_{(h_1,h_2)}(g)(g_1 \overset{i_s}{\underset{i_1}{\oplus}} y_1^s, \ldots, g_n \overset{i_s}{\underset{i_1}{\oplus}} y_1^s) \\ &= P_{(h_1,h_2)}(g)\left(P_{(h_1,h_2)}(g_1)(x_1^n), \ldots, P_{(h_1,h_2)}(g_n)(x_1^n)\right) \\ &= P_{(h_1,h_2)}(g)\left[P_{(h_1,h_2)}(g_1) \ldots P_{(h_1,h_2)}(g_n)\right](x_1^n). \end{aligned}$$

The proof of Proposition 2 is complete. $\square$

Proposition 3. *For all $g_1, g_2, h_1, h_2 \in G$ and $i = 1, \ldots, n$ we have*

$$P_{(h_1,h_2)}(g_1 \underset{i}{\oplus} g_2) = P_{(h_1,h_2)}(g_1) \underset{i}{\oplus} P_{(h_1,h_2)}(g_2).$$

P r o o f. Let $x_1^n \in \mathrm{pr}_1\, P_{(h_1,h_2)}(g_1 \underset{i}{\oplus} g_2)$. If $x_1^n \in G^n$, then

$$h_1 \sqsubset (g_1 \underset{i}{\oplus} g_2)[x_1^n] \vee h_2 \sqsubset (g_1 \underset{i}{\oplus} g_2)[x_1^n],$$

which, by (5), is equivalent to

$$h_1 \sqsubset g_1[x_1^{i-1} g_2[x_1^n] x_{i+1}^n] \vee h_2 \sqsubset g_1[x_1^{i-1} g_2[x_1^n] x_{i+1}^n]. \tag{24}$$

This, according to the v-negativity of χ, implies

$$h_1 \sqsubset g_2[x_1^n] \vee h_2 \sqsubset g_2[x_1^n]. \tag{25}$$

The condition (24) means that $(x_1^{i-1}, g_2[x_1^n], x_{i+1}^n) \in \mathrm{pr}_1\, P_{(h_1,h_2)}(g_1)$. From (25) we obtain $x_1^n \in \mathrm{pr}_1\, P_{(h_1,h_2)}(g_2)$. So, for $x_1^n \in G^n$ we have

$$x_1^n \in \mathrm{pr}_1\, P_{(h_1,h_2)}(g_1 \underset{i}{\oplus} g_2) \longleftrightarrow \begin{cases} (x_1^{i-1}, g_2[x_1^n], x_{i+1}^n) \in \mathrm{pr}_1\, P_{(h_1,h_2)}(g_1), \\ x_1^n \in \mathrm{pr}_1\, P_{(h_1,h_2)}(g_2). \end{cases} \tag{26}$$

Consider now the case when $x_1^n = e_1^n$. In this case $e_1^n \in \mathrm{pr}_1\, P_{(h_1,h_2)}(g_1 \underset{i}{\oplus} g_2)$ means, by (17), that

$$h_1 \sqsubset g_1 \underset{i}{\oplus} g_2 \vee h_2 \sqsubset g_1 \underset{i}{\oplus} g_2. \tag{27}$$

Because $g_1 \underset{i}{\oplus} g_2 \sqsubset \mu_i(\underset{i}{\oplus} g_2) = g_2$, by the v-negativity of χ, the above condition gives

$$h_1 \sqsubset g_2 \vee h_2 \sqsubset g_2. \tag{28}$$

But $\mu_i^*(\underset{i}{\oplus} g_2) = \mu_i(\underset{i}{\oplus} g_2) = g_2$ and $\mu_k^*(\underset{i}{\oplus} g_2) = e_k$ for $k \in \{1,\dots,n\} \setminus \{i\}$, so, (27) implies $(e_1^{i-1}, g_2, e_{i+1}^n) \in \operatorname{pr}_1 P_{(h_1,h_2)}(g_1)$. On the other hand, from (28) it follows that $e_1^n \in \operatorname{pr}_1 P_{(h_1,h_2)}(g_2)$. Therefore

$$e_1^n \in \operatorname{pr}_1 P_{(h_1,h_2)}(g_1 \underset{i}{\oplus} g_2) \longleftrightarrow \begin{cases} (e_1^{i-1}, g_2, e_{i+1}^n) \in \operatorname{pr}_1 P_{(h_1,h_2)}(g_1), \\ e_1^n \in \operatorname{pr}_1 P_{(h_1,h_2)}(g_2). \end{cases} \tag{29}$$

In the third case when $x_i = \mu_i^*(\overset{i_s}{\underset{i_1}{\oplus}} y_1^s)$, $i = 1,\dots,n$, for some $y_1^s \in G^s$, $i_1,\dots,i_s \in \{1,\dots,n\}$, from $x_1^n \in \operatorname{pr}_1 P_{(h_1,h_2)}(g_1 \underset{i}{\oplus} g_2)$ we conclude that

$$h_1 \sqsubset (g_1 \underset{i}{\oplus} g_2) \overset{i_s}{\underset{i_1}{\oplus}} y_1^s \vee h_2 \sqsubset (g_1 \underset{i}{\oplus} g_2) \overset{i_s}{\underset{i_1}{\oplus}} y_1^s. \tag{30}$$

Since χ is v-negative, we have $(g_1 \underset{i}{\oplus} g_2) \overset{i_s}{\underset{i_1}{\oplus}} y_1^s \sqsubset \mu_i(\underset{i}{\oplus} g_2 \overset{i_s}{\underset{i_1}{\oplus}} y_1^s) = g_2 \overset{i_s}{\underset{i_1}{\oplus}} y_1^s$, which means that (30) can be written in the form

$$h_1 \sqsubset g_2 \overset{i_s}{\underset{i_1}{\oplus}} y_1^s \vee h_2 \sqsubset g_2 \overset{i_s}{\underset{i_1}{\oplus}} y_1^s. \tag{31}$$

But $\mu_i^*(\underset{i}{\oplus} g_2 \overset{i_s}{\underset{i_1}{\oplus}} y_1^s) = \mu_i(\underset{i}{\oplus} g_2 \overset{i_s}{\underset{i_1}{\oplus}} y_1^s) = g_2 \overset{i_s}{\underset{i_1}{\oplus}} y_1^s$ and $\mu_k^*(\underset{i}{\oplus} g_2 \overset{i_s}{\underset{i_1}{\oplus}} y_1^s) = \mu_k^*(\overset{i_s}{\underset{i_1}{\oplus}} y_1^s)$ for $k \in \{1,\dots,n\} \setminus \{i\}$. This, together with the condition (30), proves $(x_1^{i-1}, g_2 \overset{i_s}{\underset{i_1}{\oplus}} y_1^s, x_{i+1}^n) \in \operatorname{pr}_1 P_{(h_1,h_2)}(g_1)$. Similarly, from (31) we can deduce $x_1^n \in \operatorname{pr}_1 P_{(h_1,h_2)}(g_2)$. Therefore

$$x_1^n \in \operatorname{pr}_1 P_{(h_1,h_2)}(g_1 \underset{i}{\oplus} g_2) \longleftrightarrow \begin{cases} (x_1^{i-1}, g_2 \overset{i_s}{\underset{i_1}{\oplus}} y_1^s, x_{i+1}^n) \in \operatorname{pr}_1 P_{(h_1,h_2)}(g_1) \\ x_1^n \in \operatorname{pr}_1 P_{(h_1,h_2)}(g_2), \end{cases}$$

where $x_i = \mu_i^*(\overset{i_s}{\underset{i_1}{\oplus}} y_1^s)$, $i = 1,\dots,n$.

Let $x_1^n \in \operatorname{pr}_1 P_{(h_1,h_2)}(g_1 \underset{i}{\oplus} g_2)$. If $x_1^n \in G^n$, then, according to (16) and (26), we have

$$\begin{aligned} P_{(h_1,h_2)}(g_1 \underset{i}{\oplus} g_2)(x_1^n) &= (g_1 \underset{i}{\oplus} g_2)[x_1^n] = g_1[x_1^{i-1} g_2[x_1^n] x_{i+1}^n] \\ &= P_{(h_1,h_2)}(g_1)\left(x_1^{i-1}, g_2[x_1^n], x_{i+1}^n\right) \\ &= P_{(h_1,h_2)}(g_1)\left(x_1^{i-1}, P_{(h_1,h_2)}(g_2)(x_1^n), x_{i+1}^n\right) \\ &= P_{(h_1,h_2)}(g_1) \underset{i}{\oplus} P_{(h_1,h_2)}(g_2)(x_1^n). \end{aligned}$$

If $x_1^n = e_1^n$, then, analogously as in the previous case, using (16) and (29) we obtain

$$P_{(h_1,h_2)}(g_1 \underset{i}{\oplus} g_2)(e_1^n) = P_{(h_1,h_2)}(g_1) \underset{i}{\oplus} P_{(h_1,h_2)}(g_2)(e_1^n).$$

Similarly, in the case when $x_i = \mu_i^*(\overset{i_s}{\underset{i_1}{\oplus}} y_1^s)$, $i = 1, \ldots, n$, for some $y_1^s \in G^s$, $i_1, \ldots, i_s \in \{1, \ldots, n\}$, we have

$$\begin{aligned} P_{(h_1,h_2)}(g_1 \underset{i}{\oplus} g_2)(x_1^n) &= (g_1 \underset{i}{\oplus} g_2) \overset{i_s}{\underset{i_1}{\oplus}} y_1^s \\ &= P_{(h_1,h_2)}(g_1)(x_1^{i-1}, g_2 \overset{i_s}{\underset{i_1}{\oplus}} y_1^s, x_{i+1}^n) \\ &= P_{(h_1,h_2)}(g_1)\left(x_1^{i-1}, P_{(h_1,h_2)}(g_2)(x_1^n), x_{i+1}^n\right) \\ &= P_{(h_1,h_2)}(g_1) \underset{i}{\oplus} P_{(h_1,h_2)}(g_2)(x_1^n). \end{aligned}$$

This completes our proof of Proposition 3. □

Basing on these propositions we are able to prove the sufficiency of the conditions of Theorem 1.

Sufficiency. Let the triplet (χ, γ, π) of binary relations on a representable Menger $(2,n)$-semigroup $\mathscr{G} = (G; o, \underset{1}{\oplus}, \ldots, \underset{n}{\oplus})$ satisfy all the conditions of the theorem. Then, as it follows from Propositions 1–3, for all $h_1, h_2 \in G$, the mapping $P_{(h_1,h_2)}$ is a representation of $\mathscr{G}$ by n-place functions. Consider the family of representations $P_{(h_1,h_2)}$ such that $(h_1, h_2) \in \gamma$. Let P be the sum of this family, i.e., $P = \sum\limits_{(h_1,h_2)\in\gamma} P_{(h_1,h_2)}$. Of course, P is a representation of $\mathscr{G}$ by n-place functions. Let us show that $\chi = \chi_P$, $\gamma = \gamma_P$ and $\pi = \pi_P$.

Let $(g_1, g_2) \in \chi_P$. Then, according to (8), we have[3] $(g_1, g_2) \in \chi_{(h_1,h_2)}$ for all $(h_1, h_2) \in \gamma$, i.e.,

$$(\forall (h_1, h_2) \in \gamma)\left(\mathrm{pr}_1\, P_{(h_1,h_2)}(g_1) \subset \mathrm{pr}_1\, P_{(h_1,h_2)}(g_2)\right),$$

which is equivalent to

$$(\forall (h_1, h_2) \in \gamma)(\forall x_1^n)\left(x_1^n \in \mathrm{pr}_1\, P_{(h_1,h_2)}(g_1) \longrightarrow x_1^n \in \mathrm{pr}_1\, P_{(h_1,h_2)}(g_2)\right).$$

¿From this, for $x_1^n = e_1^n$, we obtain

$$(\forall (h_1, h_2) \in \gamma)\left(e_1^n \in \mathrm{pr}_1\, P_{(h_1,h_2)}(g_1) \longrightarrow e_1^n \in \mathrm{pr}_1\, P_{(h_1,h_2)}(g_2)\right),$$

[3] $\chi_{(h_1,h_2)}$ denotes this quasi-order which corresponds to the representation $P_{(h_1,h_2)}$. Analogously are defined $\gamma_{(h_1,h_2)}$ and $\pi_{(h_1,h_2)}$.

which means that

$$(\forall (h_1,h_2)\in\gamma)\,(h_1\sqsubset g_1 \,\vee\, h_2\sqsubset g_1 \longrightarrow h_1\sqsubset g_2 \,\vee\, h_2\sqsubset g_2)\,.$$

Let $g_1\neq 0$. Then $g_1\top g_1$ and the above implication gives $g_1\sqsubset g_1\longrightarrow g_1\sqsubset g_2$. This proves $(g_1,g_2)\in\chi$ because χ is reflexive. If $g_1=0$, then $0=0[g_2\ldots g_2]\sqsubset g_2$, by the v-negativity of χ. Hence $(0,g_2)\in\chi$. So, $(g_1,g_2)\in\chi$, i.e., $\chi_P\subset\chi$.

Conversely, let $(g_1,g_2)\in\chi$, $(h_1,h_2)\in\gamma$ and $x_1^n\in \mathrm{pr}_1\,P_{(h_1,h_2)}(g_1)$. If $x_1^n\in G^n$, then $h_1\sqsubset g_1[x_1^n]\,\vee\, h_2\sqsubset g_1[x_1^n]$. Since the l-regularity of χ together with $g_1\sqsubset g_2$ implies $g_1[x_1^n]\sqsubset g_2[x_1^n]$, from the above we conclude that $h_1\sqsubset g_2[x_1^n]\vee h_2\sqsubset g_2[x_1^n]$, i.e., $x_1^n\in\mathrm{pr}_1\,P_{(h_1,h_2)}(g_2)$. Similarly, in the case $x_1^n=e_1^n$, from $e_1^n\in\mathrm{pr}_1\,P_{(h_1,h_2)}(g_1)$ it follows that $e_1^n\in\mathrm{pr}_1\,P_{(h_1,h_2)}(g_2)$. In the case when $x_i=\mu_i^*(\overset{i_s}{\underset{i_1}{\oplus}}y_1^s)$, $i=1,\ldots,n$, for some $y_1^s\in G^s$, $i_1,\ldots,i_s\in\{1,\ldots,n\}$, applying the l-regularity of χ to $g_1\sqsubset g_2$, we obtain $g_1\overset{i_s}{\underset{i_1}{\oplus}}y_1^s\sqsubset g_2\overset{i_s}{\underset{i_1}{\oplus}}y_1^s$, whence, in view of $h_1\sqsubset g_1\overset{i_s}{\underset{i_1}{\oplus}}y_1^s\,\vee\, h_2\sqsubset g_1\overset{i_s}{\underset{i_1}{\oplus}}y_1^s$, we obtain $h_1\sqsubset g_2\overset{i_s}{\underset{i_1}{\oplus}}y_1^s\,\vee\, h_2\sqsubset g_2\overset{i_s}{\underset{i_1}{\oplus}}y_1^s$. Therefore $x_1^n\in\mathrm{pr}_1\,P_{(h_1,h_2)}(g_2)$, which proves $\mathrm{pr}_1\,P_{(h_1,h_2)}(g_1)\subset\mathrm{pr}_1\,P_{(h_1,h_2)}(g_2)$ for all $(h_1,h_2)\in\gamma$. Thus $(g_1,g_2)\in\chi_P$, i.e., $\chi\subset\chi_P$. Consequently, $\chi=\chi_P$. This, together with the condition (c) formulated in the theorem, gives $\pi=\chi\cap\chi^{-1}=\chi_P\cap\chi_P^{-1}=\pi_P$. So, $\pi=\pi_P$.

Now let $(g_1,g_2)\in\gamma_P$. Then, according to (8), we have $(g_1,g_2)\in\gamma_{(h_1,h_2)}$ for some $(h_1,h_2)\in\gamma$, i.e.,

$$(\exists (h_1,h_2)\in\gamma)\left(\mathrm{pr}_1\,P_{(h_1,h_2)}(g_1)\cap\mathrm{pr}_1\,P_{(h_1,h_2)}(g_2)\neq\emptyset\right),$$

which is equivalent to

$$(\exists (h_1,h_2)\in\gamma)(\exists x_1^n)\left(x_1^n\in\mathrm{pr}_1\,P_{(h_1,h_2)}(g_1)\,\wedge\, x_1^n\in\mathrm{pr}_1\,P_{(h_1,h_2)}(g_2)\right).$$

This, for $x_1^n\in G^n$, implies $h_1\sqsubset g_1[x_1^n]\,\vee\, h_2\sqsubset g_1[x_1^n]$ and $h_1\sqsubset g_2[x_1^n]\,\vee\, h_2\sqsubset g_2[x_1^n]$. From the above, in view of $h_1\top h_2$ and (15), we obtain $g_1[x_1^n]\top g_2[x_1^n]$, whence, applying the l-cancellativity of γ, we get $g_1\top g_2$, i.e., $(g_1,g_2)\in\gamma$.

In the similar way, we can see that in the case of $x_1^n=e_1^n$ the condition $(g_1,g_2)\in\gamma$ also holds.

If $x_i=\mu_i^*(\overset{i_s}{\underset{i_1}{\oplus}}y_1^s)$, $i=1,\ldots,n$, for some $y_1^s\in G^s$, $i_1,\ldots,i_s\in\{1,\ldots,n\}$, then $h_1\sqsubset g_1\overset{i_s}{\underset{i_1}{\oplus}}y_1^s\,\vee\, h_2\sqsubset g_1\overset{i_s}{\underset{i_1}{\oplus}}y_1^s$ and $h_1\sqsubset g_2\overset{i_s}{\underset{i_1}{\oplus}}y_1^s\,\vee\, h_2\sqsubset g_2\overset{i_s}{\underset{i_1}{\oplus}}y_1^s$, whence, by $h_1\top h_2$ and (15), we obtain $g_1\overset{i_s}{\underset{i_1}{\oplus}}y_1^s\top g_2\overset{i_s}{\underset{i_1}{\oplus}}y_1^s$. This gives $g_1\top g_2$ because γ is l-cancellative. In this way we have proved that in any case $\gamma_P\subset\gamma$.

Conversely, let $(g_1, g_2) \in \gamma$. Since χ is reflexive, $g_1 \sqsubset g_1$ and $g_2 \sqsubset g_2$, whence $g_1 \sqsubset g_1 \vee g_2 \sqsubset g_1$ and $g_1 \sqsubset g_2 \vee g_2 \sqsubset g_2$. Consequently, $e_1^n \in \mathrm{pr}_1\, P_{(g_1,g_2)}(g_1)$ and $e_1^n \in \mathrm{pr}_1\, P_{(g_1,g_2)}(g_2)$. Thus $(g_1, g_2) \in \gamma_{(g_1,g_2)} \subset \gamma_P$, i.e., $\gamma \subset \gamma_P$. So, $\gamma = \gamma_P$.

This completes the proof of the theorem. □

Problem 1. *Find the necessary and sufficient conditions under which the triplet (χ, γ, π) of binary relations will be faithful projection representable for a representable Menger $(2, n)$-semigroup.*

Deleting from Theorem 1 the equality $\pi = \chi \cap \chi^{-1}$ we obtain the necessary and sufficient conditions under which the pair (χ, γ) of binary relations is projection representable for a representable Menger $(2, n)$-semigroup. Furthermore, all parts of the proof of this theorem connected with these two relations are valid. So, we have the following

Theorem 2. *A pair (χ, γ) of binary relations on G is projection representable for a representable Menger $(2, n)$-semigroup $\mathscr{G}$ if and only if χ is an l-regular and v-negative quasi-order, γ is an l-cancellative 0-quasi-equivalence and the implication (15) is satisfied.*

Problem 2. *Find the necessary and sufficient conditions under which the pair (χ, γ) of binary relations will be faithful projection representable for a representable Menger $(2, n)$-semigroup.*

Let $\mathscr{G} = (G; o, \underset{1}{\oplus}, \ldots, \underset{n}{\oplus})$ be a representable Menger $(2, n)$-semigroup. Let us consider on G the set $T_n(G)$ of mappings $t\colon\ x \mapsto t(x)$ defined as follows:

(a) $x \in T_n(G)$, i.e., $T_n(G)$ contains the identity transformation of G,
(b) if $i \in \{1, \ldots, n\}$, $a, b_1, \ldots, b_{i-1}, b_{i+1}, \ldots, b_n \in G$ and $t(x) \in T_n(G)$, then $a[b_1^{i-1} t(x) b_{i+1}^n] \in T_n(G)$,
(c) $T_n(G)$ contains those and only those mappings which are defined by (a) and (b).

Let us consider on G two binary relations δ_1 and δ_2 defined in the following way:

1. $(g_1, g_2) \in \delta_1 \longleftrightarrow g_1 = t(g_2)$ for some $t \in T_n(G)$,
2. $$(g_1, g_2) \in \delta_2 \longleftrightarrow \begin{cases} g_1 = (x \overset{i_s}{\underset{i_1}{\oplus}} y_1^s)[\overline{z}] \ \text{ and } \ g_2 = \mu_i(\overset{i_s}{\underset{i_1}{\oplus}} y_1^s)[\overline{z}] \ \text{ for some} \\ x \in G,\ y_1^s \in G^s,\ \overline{z} \in G^n,\ i, i_1, \ldots, i_s \in \{1, \ldots, n\}, \\ \text{where the symbol } [\overline{z}] \text{ can be empty.} \end{cases}$$

It is not difficult to see that δ_1 and δ_2 are l-regular relations, additionally δ_1 is a quasi-order. Moreover, *a binary relation $\varrho \subset G \times G$ is v-negative if and only if it contains δ_1 and δ_2.*

Let π be an l-regular equivalence on a representable Menger $(2,n)$-semigroup $\mathcal{G}$. Denote by $\chi(\pi)$ the binary relation $f_t(f_R(\delta_2)\circ\delta_1\circ\pi)$, where f_R and f_t are respectively the reflexive and the transitive closure operations (cf. [6]), and $\circ$ is the composition of relations,[4] i.e.,

$$\chi(\pi) = f_t(f_R(\delta_2)\circ\delta_1\circ\pi) = \bigcup_{n=1}^{\infty}((\delta_2\cup\triangle_G)\circ\delta_1\circ\pi)^n\,. \tag{32}$$

Since π, δ_1 and $f_R(\delta_2)$ are reflexive l-regular relations, $\chi(\pi)$ is an l-regular quasi-order containing π, δ_1 and δ_2. So, $\chi(\pi)$ is a v-negative quasi-order.

Proposition 4. *$\chi(\pi)$ is the smallest l-regular and v-negative quasi-order containing π, i.e., $\chi(\pi)\subset\chi$, where χ is any l-regular and v-negative quasi-order containing π.*

Proof. Let χ be an arbitrary l-regular and v-negative quasi-order containing π. Then $\delta_1\subset\chi$ and $\delta_2\subset\chi$, because χ is v-negative. Thus, $\pi\subset\chi$, $\delta_1\subset\chi$ and $f_R(\delta_2)\subset\chi$, whence $f_R(\delta_2)\circ\delta_1\circ\pi\subset\chi^3\subset\chi$. From this, applying the transitivity of χ, we obtain $(f_R(\delta_2)\circ\delta_1\circ\pi)^n\subset\chi^n\subset\chi$ for every natural n. Therefore $\bigcup\limits_{n=1}^{\infty}((\delta_2\cup\triangle_G)\circ\delta_1\circ\pi)^n\subset\chi$, i.e., $\chi(\pi)\subset\chi$. $\square$

Theorem 3. *A pair (γ,π) of binary relations on a representable Menger $(2,n)$-semigroup $\mathcal{G}$ is projection representable if and only if*

(a) *γ is an l-cancellative 0-quasi-equivalence,*
(b) *π is an l-regular equivalence such that $\chi(\pi)\cap(\chi(\pi))^{-1}\subset\pi$,*
(c) *the following condition*

$$h_1\top h_2\;\wedge\;h_1\sqsubset_\pi g_1\;\wedge\;h_2\sqsubset_\pi g_2\longrightarrow g_1\top g_2, \tag{33}$$

where $h\sqsubset_\pi g$ means $(h,g)\in\chi(\pi)$, is satisfied for all $g_1,g_2,h_1,h_2\in G$.

Proof. Let P be such representation on a representable Menger $(2,n)$-semigroup $\mathcal{G}$ for which $\gamma=\gamma_P$ and $\pi=\pi_P$. Then, by Proposition 3, we have $\chi(\pi)\subset\chi_P$, whence $\chi(\pi)\cap(\chi(\pi))^{-1}\subset\chi_P\cap\chi_P^{-1}=\pi_P=\pi$.

Assume now that the premise of (33) is satisfied. Then $(h_1,h_2)\in\gamma$, $(h_1,g_1)\in\chi(\pi)$ and $(h_2,g_2)\in\chi(\pi)$. Consequently, $(h_1,h_2)\in\gamma_P$, $(h_1,g_1)\in\chi_P$ and $(h_2,g_2)\in\chi_P$,

[4] Recall that $\sigma\circ\varrho=\{(a,c)\mid(\exists b)(a,b)\in\varrho\wedge(b,c)\in\sigma\}$, $f_R(\varrho)=\varrho\cup\triangle_A$, $f_t(\varrho)=\bigcup\limits_{n=1}^{\infty}\varrho^n$, where $\varrho^n=\underbrace{\varrho\circ\varrho\circ\ldots\circ\varrho}_{n}$, ϱ,σ are binary relations on A, and $\triangle_A=\{(a,a)\mid a\in A\}$.

i.e., $\mathrm{pr}_1 P(h_1) \cap \mathrm{pr}_1 P(h_2) \neq \emptyset$, $\mathrm{pr}_1 P(h_1) \subset \mathrm{pr}_1 P(g_1)$ and $\mathrm{pr}_1 P(h_2) \subset \mathrm{pr}_1 P(g_2)$, whence $\mathrm{pr}_1 P(g_1) \cap \mathrm{pr}_1 P(g_2) \neq \emptyset$. So, $(g_1, g_2) \in \gamma_P = \gamma$, which means that the condition (33) is satisfied. The necessity is proved.

To prove the sufficiency, assume that the pair (γ, π) of binary relations satisfies all the conditions of the theorem and consider the triplet $(\chi(\pi), \gamma, \pi)$. Then $\pi = \pi^{-1} \subset (\chi(\pi))^{-1}$, because $\pi \subset \chi(\pi)$. Therefore $\pi \subset \chi(\pi) \cap (\chi(\pi))^{-1}$, which, together with the condition (b), gives $\pi = \chi(\pi) \cap (\chi(\pi))^{-1}$. This means that the triplet $(\chi(\pi), \gamma, \pi)$ satisfies all the conditions of Theorem 1. So, $(\chi(\pi), \gamma, \pi)$, and consequently, (γ, π) is projection representable. The sufficiency is proved. $\square$

Problem 3. *Find the necessary and sufficient conditions under which the pair (γ, π) of binary relations will be faithful projection representable.*

Applying the method of mathematical induction to (32) we can prove the following proposition.

Proposition 5. *The condition $(g_1, g_2) \in \chi(\pi)$, where $g_1, g_2 \in G$, means that the system of conditions*

$$\left.\begin{array}{l} g_1 = x_0 \,\wedge\, g_2 = x_n, \\ \bigwedge\limits_{i=0}^{n-1} \left(\begin{pmatrix} x_i \equiv t_i((y_i \overset{k_{s_i}}{\underset{k_{1_i}}{\oplus}} z_{1_i}^{s_i})[\overline{w}_i]), \\ \\ x_{i+1} = \mu_{k_i}(\overset{k_{s_i}}{\underset{k_{1_i}}{\oplus}} z_{1_i}^{s_i})[\overline{w}_i] \end{pmatrix} \vee x_i \equiv t_i(x_{i+1}) \right) \end{array}\right\} \tag{34}$$

is valid for some $n \in \mathbb{N}$, $x_i, y_i, z_i \in G$, $\overline{w}_i \in G^n$, $t_i \in T_n(G)$, $k_i \in \{1, \ldots, n\}$.

In the sequel the formula

$$\bigwedge_{i=m}^{n} \left(\begin{pmatrix} x_i \equiv t_i((y_i \overset{k_{s_i}}{\underset{k_{1_i}}{\oplus}} z_{1_i}^{s_i})[\overline{w}_i]), \\ \\ x_{i+1} = \mu_{k_i}(\overset{k_{s_i}}{\underset{k_{1_i}}{\oplus}} z_{1_i}^{s_i})[\overline{w}_i] \end{pmatrix} \vee x_i \equiv t_i(x_{i+1}) \right)$$

will be denoted by $\mathfrak{M}(m, n)$.

The inclusion $\chi(\pi) \cap (\chi(\pi))^{-1} \subset \pi$ means that for all $g_1, g_2 \in G$ we have

$$(g_1, g_2) \in \chi(\pi) \,\wedge\, (g_2, g_1) \in \chi(\pi) \longrightarrow g_1 \equiv g_2,$$

which, according to Proposition 5, can be written as the system of conditions $(A_{n,m})_{n,m \in \mathbb{N}}$, where

$$A_{n,m}\colon\ \mathfrak{M}(0, n-1) \,\wedge\, \mathfrak{M}(n+1, n+m) \,\wedge\, x_0 = x_{n+m} \longrightarrow x_0 \equiv x_n.$$

The system $(A_{n,m})_{n,m\in\mathbb{N}}$ is equivalent to the system $(A_n)_{n\in\mathbb{N}}$, where

$$A_n\colon\ \mathfrak{M}(0,n-1)\ \wedge\ x_0=x_n\ \longrightarrow\ x_0\equiv x_1.$$

Consider now the implication (33). According to (34) the condition $(h_1,g_1)\in\chi(\pi)$ means that

$$h_1=x_0\ \wedge\ \mathfrak{M}(0,n-1)\ \wedge\ x_n=g_1 \tag{35}$$

for some $x_i,y_i,z_{k_i},t_i,k_i,\overline{w}_i$. Similarly, the condition $(h_2,g_2)\in\chi(\pi)$ means that

$$h_2=x_{n+1}\ \wedge\ \mathfrak{M}(n+1,n+m)\ \wedge\ x_{n+m+1}=g_2 \tag{36}$$

for some $x_i,y_i,z_{k_i},t_i,k_i,\overline{w}_i$. So, (33) can be written as the system $(B_{n,m})_{n,m\in\mathbb{N}}$ of conditions

$$B_{n,m}\colon\ x_0\top x_{n+1}\ \wedge\ \mathfrak{M}(0,n-1)\ \wedge\ \mathfrak{M}(n+1,n+m)\ \longrightarrow\ x_n\top x_{n+m+1}.$$

In this way we have proved

Theorem 4. *A pair (γ,π) of binary relations on a representable Menger $(2,n)$-semigroup $\mathcal{G}$ is projection representable if and only if*
(a) *γ is an l-cancellative 0-quasi-equivalence,*
(b) *π is an l-regular equivalence,*
(c) *the conditions $(A_n)_{n\in\mathbb{N}}$ and $(B_{n,m})_{n,m\in\mathbb{N}}$ are satisfied.*

Theorem 5. *A pair (χ,π) of binary relations is (faithful) projection representable for a representable Menger $(2,n)$-semigroup $\mathcal{G}$ if and only if χ is an l-regular and v-negative quasi-order such that $\pi=\chi\cap\chi^{-1}$.*

P r o o f. The necessity of these conditions follows from the proof of Theorem 1. To prove their sufficiency, for every element $g\in G$ we define an n-place function $P_a(g)\colon\ \mathfrak{A}^*\to G$, where $a\in G$, putting

$$P_a(g)(x_1^n)=\begin{cases} g[x_1^n] & \text{if } a\sqsubset g[x_1^n]\ \text{ and }\ x_1^n\in G^n,\\ g & \text{if } a\sqsubset g\ \text{ and }\ x_1^n=e_1^n,\\ g\overset{i_s}{\underset{i_1}{\oplus}}y_1^s & \text{if } a\sqsubset g\overset{i_s}{\underset{i_1}{\oplus}}y_1^s\ \text{ and }\ x_i=\mu_i^*(\overset{i_s}{\underset{i_1}{\oplus}}y_1^s),\\ & i=1,\dots,n,\ \text{for some } y_1^s\in G^s,\\ & i_1,\dots,i_s\in\{1,\dots,n\}.\end{cases} \tag{37}$$

Since, for $h_1 = h_2 = a \in G$, the function $P_{(h_1,h_2)}(g)$ defined by (16) coincides with the function $P_a(g)$, from Propositions 1 – 3 it follows that the mapping $P_a\colon\ g \mapsto P_a(g)$ is a representation of $\mathscr{G}$ by n-place functions. Further, analogously as in the proof of Theorem 1, we can prove that $P_0 = \sum\limits_{a\in G} P_a$ is a representation of $\mathscr{G}$ for which $\chi = \chi_{P_0}$ and $\pi = \pi_{P_0}$. So, the pair (χ, π) is projection representable for $\mathscr{G}$.

Let us show that (χ, π) is faithful projection representable. In [11] it is proved that each representable Menger $(2,n)$-semigroup has a faithful representation by n-place functions. Let Λ be such representation. Then obviously $\chi_\Lambda = G \times G$ and $\pi_\Lambda = G \times G$.

Consider the representation $P = \Lambda + P_0$. Since Λ is a faithful representation, P is also faithful. Moreover $\chi_P = \chi_\Lambda \cap \chi_{P_0} = G \times G \cap \chi = \chi$ and $\pi_P = \pi_\Lambda \cap \pi_{P_0} = G \times G \cap \pi = \pi$. So, (χ, π) is faithful projection representable for $\mathscr{G}$. $\square$

In the same manner, using the construction (37), we can prove the following theorem.

Theorem 6. *A binary relation χ is (faithful) projection representable for a representable Menger $(2,n)$-semigroup if and only if it is an l-regular, v-negative quasi-order.*

Theorem 7. *A binary relation π is (faithful) projection representable for a representable Menger $(2,n)$-semigroup if and only if it is an l-regular equivalence such that $\chi(\pi) \cap (\chi(\pi))^{-1} \subset \pi$.*

Proof. Consider the pair $(\chi(\pi), \pi)$ of binary relations, where $\chi(\pi)$ is defined by (32). In a similar way as in the proof of Theorem 3, we can prove that this pair satisfies all the conditions of Theorem 5, whence we conclude the validity of Theorem 7. $\square$

Since, as it was showed above, the inclusion $\chi(\pi) \cap (\chi(\pi))^{-1} \subset \pi$ is equivalent to the system of conditions $(A_n)_{n\in\mathbb{N}}$, the last theorem can be rewritten in the form:

Theorem 8. *A binary relation π is (faithful) projection representable for a representable Menger $(2,n)$-semigroup if and only if it is an l-regular equivalence and the system of conditions $(A_n)_{n\in\mathbb{N}}$ is satisfied.*

Consider on a Menger $(2,n)$-semigroup $\mathscr{G}$ the binary relation χ_0 defined in the following way:

$$\chi_0 = f_t(f_R(\delta_2) \circ \delta_1) = \bigcup_{n=1}^{\infty} \left((\delta_2 \cup \triangle_G) \circ \delta_1\right)^n, \tag{38}$$

where f_t and f_R are reflexive and transitive closure operations.

Proposition 6. *χ_0 is the least l-regular and v-negative quasi-order on $\mathscr{G}$.*

The proof of this proposition is analogous to the proof of Proposition 3.

Theorem 9. *A binary relation γ is projection representable for a representable Menger $(2,n)$-semigroup if and only if it is an l-cancellative 0-quasi-equivalence and the following implication*

$$h_1 \top h_2 \wedge h_1 \sqsubset_0 g_1 \wedge h_2 \sqsubset_0 g_2 \longrightarrow g_1 \top g_2 \tag{39}$$

is satisfied for all $h_1, h_2, g_1, g_2 \in G$, where $h \sqsubset_0 g$ means $(h,g) \in \chi_0$.

P r o o f. The necessity of (39) can be proved analogous as the necessity of (33) in the proof of Theorem 3. To prove the sufficiency we consider the pair (χ_0, γ). By Proposition 6, this pair satisfies all demands of Theorem 2, whence we conclude the validity of Theorem 9. □

Problem 4. *Find the necessary and sufficient conditions under which γ will be faithful projection representable.*

Basing on the formula (38) we can prove the following proposition:

Proposition 7. *¿From $(g_1, g_2) \in \chi_0$, where $g_1, g_2 \in G$, it follows that the system of conditions*

$$g_1 = x_0 \wedge g_2 = x_n \wedge \bigwedge_{i=0}^{n-1} \left(\begin{pmatrix} x_i = t_i((y_i \overset{k_{s_i}}{\underset{k_{1_i}}{\oplus}} z_{1_i}^{s_i})[\overline{w}_i]), \\ \\ x_{i+1} = \mu_{k_i}(\overset{k_{s_i}}{\underset{k_{1_i}}{\oplus}} z_{1_i}^{s_i})[\overline{w}_i] \end{pmatrix} \vee x_i = t_i(x_{i+1}) \right)$$

is valid for $n \in \mathbb{N}$, $x_i, y_i, z_i \in G$, $\overline{w}_i \in G^n$, $t_i \in T_n(G)$, $k_i \in \{1, \ldots, n\}$.

Denoting by $\mathfrak{N}(m,n)$ the formula

$$\bigwedge_{i=m}^{n} \left(\begin{pmatrix} x_i = t_i((y_i \overset{k_{s_i}}{\underset{k_{1_i}}{\oplus}} z_{1_i}^{s_i})[\overline{w}_i]), \\ \\ x_{i+1} = \mu_{k_i}(\overset{k_{s_i}}{\underset{k_{1_i}}{\oplus}} z_{1_i}^{s_i})[\overline{w}_i] \end{pmatrix} \vee x_i = t_i(x_{i+1}) \right),$$

and using the same argumentation as in the proof of Theorem 4, we can prove that the implication (39) is equivalent to the system of conditions $(C_{n,m})_{n,m \in \mathbb{N}}$, where

$$C_{n,m}\colon\ x_0 \top x_{n+1} \wedge \mathfrak{N}(0, n-1) \wedge \mathfrak{N}(n+1, n+m) \longrightarrow x_0 \top x_{n+m+1}.$$

So, the following theorem is true:

Theorem 10. *A binary relation γ is projection representable for a representable Menger $(2,n)$-semigroup if and only if it is an l-cancellative 0-quasi-equivalence and the system of conditions $(C_{n,m})_{n,m\in\mathbb{N}}$ is satisfied.*

4. Projection representable relations on $(2,n)$-semigroups

Let χ, γ and π be three binary relations on a $(2,n)$-semigroup $(G;\underset{1}{\oplus},\ldots,\underset{n}{\oplus})$. Similarly as in the case of Menger $(2,n)$-semigroups we say that the triplet (χ,γ,π) is *(faithful) projection representable for a $(2,n)$-semigroup* $(G;\underset{1}{\oplus},\ldots,\underset{n}{\oplus})$ if there exists such (faithful) representation P of $(G;\underset{1}{\oplus},\ldots,\underset{n}{\oplus})$ by n-place functions for which $\chi=\chi_P$, $\gamma=\gamma_P$ and $\pi=\pi_P$. Analogously we define the projection representable pairs and separate relations.

It is not difficult to verify that our Theorem 1 formulated for representable Menger $(2,n)$-semigroups is also valid for representable $(2,n)$-semigroups. The proof of this version of Theorem 1 is analogous to the proof of the previous version, but in the proof of the sufficiency instead of the representation P we must consider the representation $P^\bullet$, which is the sum of the family of representations $(P^\bullet_{(h_1,h_2)})_{(h_1,h_2)\in\gamma}$, where for every $g\in G$ $P^\bullet_{(h_1,h_2)}(g)\colon\ \mathfrak{A}_0^*\to G$, ($\mathfrak{A}_0^*=\mathfrak{A}_0\cup\{(e_1,\ldots,e_n)\}$, see page 6) is a partial n-place function such that

$$x_1^n\in \mathrm{pr}_1\, P^\bullet_{(h_1,h_2)}(g)\longleftrightarrow\begin{cases} h_1\sqsubset g\ \vee\ h_2\sqsubset g & \text{if } x_1^n=e_1^n,\\ h_1\sqsubset g\overset{i_s}{\underset{i_1}{\oplus}}y_1^s\ \vee\ h_2\sqsubset g\overset{i_s}{\underset{i_1}{\oplus}}y_1^s & \text{if }\ x_i=\mu_i^*(\overset{i_s}{\underset{i_1}{\oplus}}y_1^s),\\ & i=1,\ldots,n,\ \text{for}\\ & \text{some } y_1^s\in G^s \text{ and}\\ & i_1\ldots,i_s\in\{1,\ldots,n\}\end{cases}$$

and

$$P^\bullet_{(h_1,h_2)}(g)(x_1^n)=\begin{cases} g & \text{if } x_1^n=e_1^n,\\ g\overset{i_s}{\underset{i_1}{\oplus}}y_1^s & \text{if }\ x_i=\mu_i^*(\overset{i_s}{\underset{i_1}{\oplus}}y_1^s),\\ & i=1,\ldots,n,\ \text{for some } y_1^s\in G^s\\ & \text{and }\ i_1\ldots,i_s\in\{1,\ldots,n\}.\end{cases}$$

Also Theorem 2 is valid for $(2,n)$-semigroups. Moreover, problems analogous to Problem 1 and Problem 2 can be posed for $(2,n)$-semigroups, too.

Theorem 3 will be valid for $(2,n)$-semigroups if we replace the relation $\chi(\pi)$ by the relation

$$\chi^\bullet(\pi) = f_t(f_R(\delta_2)\circ\pi) = \bigcup_{n=1}^{\infty}\left((\delta_2\cup\triangle_G)\circ\pi\right)^n, \tag{40}$$

i.e., if we delete δ_1 from the formula (32).

Proposition 5 for $(2,n)$-semigroups has the following form:

Proposition 8. *The condition $(g_1,g_2)\in\chi^\bullet(\pi)$, where $g_1,g_2\in G$, means that the system of conditions*

$$g_1 = x_0 \,\wedge\, g_2 = x_n \,\wedge\, \bigwedge_{i=0}^{n-1}\left(\left(\begin{array}{l} x_i \equiv y_i \overset{k_{s_i}}{\underset{k_{1_i}}{\oplus}} z_{1_i}^{s_i}, \\ \\ x_{i+1} = \mu_{k_i}(\overset{k_{s_i}}{\underset{k_{1_i}}{\oplus}} z_{1_i}^{s_i}) \end{array}\right) \vee\, x_i \equiv x_{i+1}\right)$$

is valid for some $n\in\mathbb{N}$, $x_i,y_i,z_i\in G$, $k_i\in\{1,\dots,n\}$.

Denoting by $\mathfrak{X}(m,n)$ the formula

$$\bigwedge_{i=m}^{n}\left(\left(\begin{array}{l} x_i \equiv y_i \overset{k_{s_i}}{\underset{k_{1_i}}{\oplus}} z_{1_i}^{s_i}, \\ \\ x_{i+1} = \mu_{k_i}(\overset{k_{s_i}}{\underset{k_{1_i}}{\oplus}} z_{1_i}^{s_i}) \end{array}\right) \vee\, x_i \equiv x_{i+1}\right)$$

and using the same argumentation as in the proof of Theorem 4, we can prove

Theorem 11. *A pair (γ,π) of binary relations on a representable $(2,n)$-semigroup is projection representable if and only if γ is an l-cancellative 0-quasi-equivalence, π is an l-regular equivalence, and the systems of conditions $A_n^\bullet$ and $B_{n,m}^\bullet$, where*

$$\begin{array}{l} A_n^\bullet\colon\ \mathfrak{X}(0,n-1)\,\wedge\, x_0 = x_n \longrightarrow x_0 \equiv x_1, \\ B_{n,m}^\bullet\colon\ x_0 \top x_{n+1}\,\wedge\,\mathfrak{X}(0,n-1)\,\wedge\,\mathfrak{X}(n+1,n+m) \longrightarrow x_n \top x_{n+m+1} \end{array}$$

are satisfied.

Theorem 5 is valid for $(2,n)$-semigroups too, but in the proof, the representation P_a defined by (37) must be replaced by the representation $P_a^\bullet$, where

$$P_a^\bullet(g)(x_1^n) = \begin{cases} g & \text{if } a \sqsubset g \text{ and } x_1 = e_1^n, \\ g \overset{i_s}{\underset{i_1}{\oplus}} y_1^s & \text{if } a \sqsubset g \overset{i_s}{\underset{i_1}{\oplus}} y_1^s \text{ and } x_i = \mu_i^*(\overset{i_s}{\underset{i_1}{\oplus}} y_1^s), \\ & i = 1,\dots,n, \text{ for some } y_1^s \in G^s, \\ & \text{and } i_1,\dots,i_s\in\{1,\dots,n\}. \end{cases}$$

For $(2,n)$-semigroups Theorem 6 has the same form as for Menger $(2,n)$-semigroup, in Theorem 7 the relation $\chi(\pi)$ must be replaced by $\chi^{\bullet}(\pi)$, and in Theorem 8 instead of A_n we must use $A_n^{\bullet}$.

Further, using the same argumentation as in the proof of Proposition 4 we can prove that the relation

$$\chi_0^{\bullet} = f_t(f_R(\delta_2)) = \bigcup_{n=1}^{\infty} (\delta_2 \cup \triangle_G)^n ,$$

where f_t and f_R are the reflexive and the transitive closure operations, is the least l-regular and v-negative quasi-order on a given $(2,n)$-semigroup. Using this relation, we can prove the analog of Theorem 10 for $(2,n)$-semigroups. The analog of Problem 4 can be posed too.

Proposition 7 for $(2,n)$-semigroups has the following form:

Proposition 9. *The condition $(g_1,g_2) \in \chi_0^{\bullet}$, where $g_1,g_2 \in G$, means that the system of conditions*

$$g_1 = x_0 \wedge g_2 = x_n \wedge \bigwedge_{i=0}^{n-1} \left(\begin{pmatrix} x_i = y_i \overset{k_{s_i}}{\underset{k_{1_i}}{\oplus}} z_{1_i}^{s_i}, \\ \\ x_{i+1} = \mu_{k_i}(\overset{k_{s_i}}{\underset{k_{1_i}}{\oplus}} z_{1_i}^{s_i}) \end{pmatrix} \vee x_i = x_{i+1} \right)$$

is valid for $n \in \mathbb{N}$, $x_i, y_i, z_i \in G$.

Further, denoting by $\mathfrak{B}(m,n)$ the formula

$$\bigwedge_{i=m}^{n} \left(\begin{pmatrix} x_i = y_i \overset{k_{s_i}}{\underset{k_{1_i}}{\oplus}} z_{1_i}^{s_i}, \\ \\ x_{i+1} = \mu_{k_i}(\overset{k_{s_i}}{\underset{k_{1_i}}{\oplus}} z_{1_i}^{s_i}) \end{pmatrix} \vee x_i = x_{i+1} \right)$$

and using the same argumentation as in the proof of Theorem 10, we can prove

Theorem 12. *A binary relation γ is projection representable for a representable $(2,n)$-semigroup if and only if it is an l-cancellative 0-quasi-equivalence and the system of conditions $(C_{n,m}^{\bullet})_{n,m\in\mathbb{N}}$, where*

$$C_{n,m}^{\bullet}\colon\ x_0 \top x_{n+1} \wedge \mathfrak{B}(0,n-1) \wedge \mathfrak{B}(n+1,n+m) \longrightarrow x_0 \top x_{n+m+1}$$

is satisfied.

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