Miroslav Pavlović
A formula for the Bloch norm of a C^1-function on the unit ball of $\mathbb{C}^n$

Czechoslovak Mathematical Journal, Vol. 58 (2008), No. 4, 1039–1043

Persistent URL: `http://dml.cz/dmlcz/140437`

A FORMULA FOR THE BLOCH NORM OF A C^1-FUNCTION ON THE UNIT BALL OF $\mathbb{C}^n$

Miroslav Pavlović, Beograd

(Received October 20, 2006)

Abstract. For a C^1-function f on the unit ball $\mathbb{B} \subset \mathbb{C}^n$ we define the Bloch norm by $\|f\|_{\mathfrak{B}} = \sup \|\tilde{d}f\|$, where $\tilde{d}f$ is the invariant derivative of f, and then show that

$$\|f\|_{\mathfrak{B}} = \sup_{\substack{z,w\in\mathbb{B}\\ z\neq w}} (1-|z|^2)^{1/2}(1-|w|^2)^{1/2}\frac{|f(z)-f(w)|}{|w-P_w z - s_w Q_w z|}.$$

Keywords: Bloch norm, Möbius transformation

MSC 2010: 32A18, 32A37

Let $\mathbb{B} = \mathbb{B}_n$ denote the unit ball of $\mathbb{C}^n$. For a complex-valued function $f \in C^1(\mathbb{B})$, let $\tilde{d}f$ denote the "invariant" derivative of f,

$$\tilde{d}f(a) = d(f \circ \varphi_a)(0), \qquad a \in \mathbb{B},$$

where φ_a denotes the biholomorphic automorphism of $\mathbb{B}$ such that $\varphi_a(0) = a$ and $\varphi_a(\varphi_a(z)) \equiv z$, and $dg(0)$ denotes the euclidean derivative of g at 0 treated as an $\mathbb{R}$-linear operator from $\mathbb{C}^n$ into $\mathbb{C}^n$. "Invariant" means that

$$\|\tilde{d}(f\circ\psi)(z)\| \equiv \|(\tilde{d}f)(\psi(z))\|,$$

for every $\psi \in \mathscr{M}(\mathbb{B})$, where $\mathscr{M}(\mathbb{B})$ denotes the group of all biholomorphic automorphisms of $\mathbb{B}$, and $\|\tilde{d}f(b)\|$ denotes the norm of the linear operator $\tilde{d}f(b)$. This relation is proved in the same way as Theorem 4.1.2 in [4]. The Bloch norm of f is given by

$$\|f\|_{\mathfrak{B}} = \sup_{\mathbb{B}} \|\tilde{d}f\|.$$

Supported by MNZŽS Serbia, Project No. 144010.

If f is real-valued, then

$$\|\tilde{d}f(a)\| = |\widetilde{\nabla} f(a)|,$$

where $\widetilde{\nabla} f(a)$ is the invariant gradient of f,

$$\widetilde{\nabla} f(a) = \nabla(f \circ \varphi_a)(0), \qquad a \in \mathbb{B}.$$

Here ∇f denotes the euclidean gradient of f, the modulus of which can be given by

$$|\nabla f(z)|^2 = 2 \sum_{j=1}^{n} \left|\frac{\partial f}{\partial z_j}\right|^2 + \left|\frac{\partial f}{\partial \bar{z}_j}\right|^2, \qquad z = (z_1, \dots, z_n) \in \mathbb{C}^n.$$

If f is holomorphic, then

$$\|\tilde{d}f(a)\| = |\widetilde{D} f(a)|,$$

where

$$\widetilde{D} f(a) = D(f \circ \varphi_a)(0) \quad \text{and } Df = \Big(\frac{\partial f}{\partial z_1}, \dots, \frac{\partial f}{\partial z_n}\Big).$$

Note the formula

$$|\widetilde{D} f(a)|^2 = (1 - |a|^2) \left(|Df(a)|^2 - |\langle \bar{a}, Df(a)\rangle|^2\right),$$

where

$$\langle w, z\rangle = \sum_{j=1}^{n} w_j z_j.$$

In [2], extending a result of Holland and Walsh [1], Nowak proved that if f is holomorphic in $\mathbb{B}$, then $\|f\|_{\mathfrak{B}} < \infty$ if and only if

$$\sup_{\substack{z, w \in \mathbb{B} \\ z \neq w}} (1 - |z|^2)^{1/2} (1 - |w|^2)^{1/2} \frac{|f(z) - f(w)|}{|w - P_w z - s_w Q_w z|} < \infty,$$

where P_w is the orthogonal projection of $\mathbb{C}^n$ onto the subspace spanned by w, $Q_w z = z - P_w z$ and $s_w = (1 - |w|^2)^{1/2}$. Here we prove

Theorem 1. *If $f \in C^1(\mathbb{B})$, then*

$$\|f\|_{\mathfrak{B}} = \sup_{\substack{z,w\in\mathbb{B}\\ z\neq w}} (1-|z|^2)^{1/2}(1-|w|^2)^{1/2}\frac{|f(z)-f(w)|}{|w-P_w z - s_w Q_w z|}. \tag{1}$$

In the case $n = 1$, formula (2) reduces to

$$\|f\|_{\mathfrak{B}} = \sup_{\substack{z,w\in\mathbb{B}\\ z\neq w}} (1-|z|^2)^{1/2}(1-|w|^2)^{1/2}\frac{|f(z)-f(w)|}{|w-z|}, \tag{2}$$

which is proved in [3] in a different context. For the proof of Theorem 1 we need some formulas. We have

$$\varphi_a(z) = \frac{a - P_a z - s_a Q_a z}{1-\langle z, \bar{a}\rangle},$$
$$1 - |\varphi_a(z)|^2 = \frac{(1-|a|^2)(1-|z|^2)}{|1-\langle z,\bar{a}\rangle|^2},$$

and

$$d\varphi_a(0)h = -s_a^2 P_a h - s_a Q_a h = -s_a h - (s_a - s_a^2)P_a h, \qquad h \in \mathbb{C}^n \tag{3}$$

(see [4, Theorem 2.2.2]). From (3) we can obtain the inequality

$$(1-|a|^2)\,\|df(a)\| \leqslant \|\tilde{d}f(a)\|, \qquad a \in \mathbb{B}. \tag{4}$$

Indeed, by (3) and the chain rule, we have

$$\tilde{d}f(a)h = d(f\circ\varphi_a)(0)h = -df(a)(s_a h + (s_a - s_a^2)P_a h).$$

Hence

$$|\tilde{d}f(a)h| \geqslant s_a|df(a)h| - (s_a - s_a^2)|df(a)P_a h| \geqslant s_a|df(a)h| - (s_a - s_a^2)\|df(a)\|\,|h|.$$

Now choose h with $|h| = 1$ so that $|df(a)h| = \|df(a)\|$; it follows that

$$\|\tilde{d}f(a)\| \geqslant |\tilde{d}f(a)h| \geqslant s_a^2\|df(a)\|,$$

which gives (4). The pseudo-hyperbolic metric on $\mathbb{B}$ is defined by

$$\varrho(z,a) = |\varphi_a(z)| = |\varphi_z(a)|.$$

This metric is $\mathscr{M}(\mathbb{B})$-invariant in the sense that $\varrho(\psi(a), \psi(b)) = \varrho(a, b)$ for all $\psi \in \mathscr{M}(\mathbb{B})$. Hence (2) can be written as

$$\|f\|_{\mathfrak{B}} = \sup_{\substack{z,w\in\mathbb{B}\\ z\neq w}} \frac{(1-|z|^2)^{1/2}(1-|w|^2)^{1/2}}{|1-\langle z,\overline{w}\rangle|} \frac{|f(z)-f(w)|}{\varrho(z,w)}. \tag{5}$$

To prove (5) consider the operator

$$Lf(a) = \limsup_{z\to a} \frac{|f(z)-f(a)|}{\varrho(z,a)}.$$

As a consequence of the $\mathscr{M}(\mathbb{B})$-invariance of ϱ, we have that L is $\mathscr{M}(\mathbb{B})$-invariant, i.e., that $L(f\circ\psi) = (Lf)\circ\psi$ for $\psi \in \mathscr{M}(\mathbb{B})$. Since the same holds for $\|\tilde{d}f\|$ and since

$$\|\tilde{d}f(0)\| = \|df(0)\| = \limsup_{z\to 0} \frac{|f(z)-f(0)|}{|z|} = \limsup_{z\to 0} \frac{|f(z)-f(0)|}{\varrho(z,0)},$$

we see that

$$\|\tilde{d}f(a)\| = Lf(a).$$

Now assuming that

$$\frac{(1-|z|^2)^{1/2}(1-|w|^2)^{1/2}}{|1-\langle z,\overline{w}\rangle|} \frac{|f(z)-f(w)|}{\varrho(z,w)} \leqslant 1,$$

we let w tend to z and get $\|f\|_{\mathfrak{B}} = \sup_{\mathbb{B}} Lf \leqslant 1$, which proves part "$\leqslant$" of (5).

In order to prove the reverse inequality assume that $\|f\|_{\mathfrak{B}} \leqslant 1$, i.e., that $\|\tilde{d}f(z)\| \leqslant 1$ for all $z \in \mathbb{B}$. Then, by (4),

$$\|df(z)\| \leqslant (1-|z|^2)^{-1},$$

and hence, by integration,

$$|f(z)-f(0)| \leqslant |z|\frac{1}{2}\log\frac{1+|z|}{1-|z|}, \qquad z\in\mathbb{B}.$$

From this and the elementary inequality

$$\frac{1}{2}\log\frac{1+t}{1-t} \leqslant (1-t^2)^{-1/2}, \qquad 0\leqslant t<1$$

(see, e.g., [3]) we find that

$$|f(z)-f(0)| \leqslant |z|(1-|z|^2)^{-1/2}, \qquad z\in\mathbb{B}. \tag{6}$$

Since $\|f \circ \varphi_a\|_{\mathfrak{B}} = \|f\|_{\mathfrak{B}} \leqslant 1$, we can apply (6) to $f \circ \varphi_a$ to get

$$|f(\varphi_a(z)) - f(a)| \leqslant |z|(1 - |z|^2)^{-1/2}, \qquad z \in \mathbb{B}.$$

Hence, by the substitution $w = \varphi_a(z)$,

$$|f(w) - f(a)| \leqslant |\varphi_a(w)|(1 - |\varphi_a(w)|^2)^{-1/2} = \varrho(a, w)\frac{(1 - |a|^2)^{1/2}(1 - |w|^2)^{1/2}}{|1 - \langle w, \bar{a}\rangle|},$$

which implies the desired inequality.

References

[1] *F. Holland and D. Walsh*: Criteria for membership of Bloch space and its subspace, BMOA. Math. Ann. *273* (1986), 317–335.

[2] *M. Nowak*: Bloch space and Möbius invariant Besov spaces on the unit ball of $\mathbb{C}^n$. Complex Variables Theory Appl. *44* (2001), 1–12.

[3] *M. Pavlović*: On the Holland-Walsh characterization of Bloch functions. Proc. Edinb. Math. Soc. *51* (2008), 439–441.

[4] *W. Rudin*: Function Theory in the Unit Ball of $\mathbf{C}^n$. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Science], vol. 241, Springer-Verlag, New York, 1980.

Author's address: Miroslav Pavlović, Matematički fakultet, Studentski trg 16, 110 01 Beograd, p.p. 550, Serbia,e-mail: pavlovic@matf.bg.ac.yu.