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DIRECT PRODUCT DECOMPOSITIONS OF BOUNDED COMMUTATIVE RESIDUATED ℓ-MONOIDS

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Abstract. The notion of bounded commutative residuated ℓ-monoid (BCR ℓ-monoid, in short) generalizes both the notions of MV-algebra and of BL-algebra. Let $\mathscr{A}$ be a BCR ℓ-monoid; we denote by $\ell(\mathscr{A})$ the underlying lattice of $\mathscr{A}$. In the present paper we show that each direct product decomposition of $\ell(\mathscr{A})$ determines a direct product decomposition of $\mathscr{A}$. This yields that any two direct product decompositions of $\mathscr{A}$ have isomorphic refinements. We consider also the relations between direct product decompositions of $\mathscr{A}$ and states on $\mathscr{A}$.

Keywords: bounded commutative residuated ℓ-monoid, lattice, direct product decomposition, internal direct factor

MSC 2010: 06D35, 06F05, 03G25

1. Introduction

A bounded commutative residuated ℓ-monoid (BCR ℓ-monoid, in short) is an algebra $\mathscr{A} = (A; \odot, \rightarrow, \vee, \wedge, 1, 0)$ of type $(2,2,2,2,0,0)$ satisfying certain axioms (cf. Dvurečenskij and Rachůnek [3], [4]; cf. also Section 2 for a detailed definition). The algebra $\ell(\mathscr{A}) = (A; \vee, \wedge, 1, 0)$ is a lattice with the greatest element 1 and the least element 0; we say that $\ell(\mathscr{A})$ is the underlying lattice of $\mathscr{A}$.

Particular cases of BCR ℓ-monoids are MV-algebras (cf. Cignoli, D'Ottaviano and Mundici [2]) and BL-algebras (cf. Hájek [5]). On the other hand, the notion of BCR ℓ-monoid is a particular case of the notion of the commutative residuated ℓ-monoid. This is a dual of the notion of the DRL-monoid which was introduced and studied by Swamy [13].

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Direct product decompositions of MV-algebra were dealt with by the author [7]; for the case of pseudo MV-algebras and pseudo effect algebras cf. [8] or [9], respectively.

Two-factor direct product decompositions of dually residuated lattice ordered monoids were investigated by Rachůnek and Šalounová [12].

Let $\mathscr{A}$ be a BCR ℓ-monoid. In the present paper we prove that each direct product decomposition of the lattice $\ell(\mathscr{A})$ determines a direct product decomposition of $\mathscr{A}$. Any two internal direct product decompositions of $\mathscr{A}$ have a common refinement. Hence any two direct product decompositions of $\mathscr{A}$ have isomorphic refinements. We consider also the relations between direct product decompositions of $\mathscr{A}$ and states on $\mathscr{A}$.

2. Preliminaries

We recall the definition of a BCR ℓ-monoid (cf. [3]).

A *BCR ℓ-monoid* is an algebra $\mathscr{A} = (A; \odot, \rightarrow, \vee, \wedge, 1, 0)$ of type $(2,2,2,2,0,0)$ which satisfies the following conditions:

(i) $(A; \odot, 1)$ is a commutative monoid.
(ii) $(A; \vee, \wedge.0, 1)$ is a lattice with the least element 0 and the greatest element 1.
(iii) The operation $\odot$ distributes over the operations $\vee$ and $\wedge$.
(iv) $x \odot y \leqslant z$ if and only if $x \leqslant y \rightarrow z$ for any $x, y, z \in A$.
(v) The identity $(x \rightarrow y) \odot x = x \wedge y$ is valid in A.

For each $x, y \in A$ we put

$$x^- = x \rightarrow 0,$$
$$d(x,y) = (x \rightarrow y) \wedge (y \rightarrow x).$$

The following basic rules are consequences of the axioms (i)–(v) (cf. e.g. [3]):

(b1) $x \leqslant y \Leftrightarrow x \rightarrow y = 1$.
(b2) $x \rightarrow (y \wedge z) = (x \rightarrow y) \wedge (x \rightarrow z)$.
(b3) $d(x,y) = (x \vee y) \rightarrow (x \wedge y)$.
(b4) $x \odot y = 0 \Leftrightarrow y \leqslant x^-$.

Since 0 is the least element of $\ell(\mathscr{A})$, from (b4) we obtain

($*_1$) $x \odot 0 = 0$ for $x \in A$.

Further, (v) implies $(1 \rightarrow x) \odot 1 = 1 \wedge x$, hence

($*_2$) $1 \rightarrow x = x$ for $x \in A$.

Since $x \vee 1 = 1$ for each $x \in A$, in view of (iii) we get, for each $x, y \in A$,

$$(x \odot y) \vee (1 \odot y) = 1 \odot y,$$
$$(x \odot y) \vee y = y.$$

Therefore

($*_3$) $x \odot y \leqslant y$ and $x \odot y \leqslant x$ for each $x, y \in A$.

In view of [3], Section 3 we have

($*_4$) $x_1 \leqslant x_2$ and $y_1 \leqslant y_2$ imply $x_1 \odot y_1 \leqslant x_2 \odot y_2$ for each $x_1, x_2, y_1, y_2 \in A$.

Also, according to [3],

($*_5$) the lattice $\ell(\mathscr{A})$ is distributive.

Let I be a nonempty set and for each $i \in I$ let $\mathscr{A}_i$ be a BCR ℓ-monoid. The direct product $\prod\limits_{i\in I} \mathscr{A}_i$ is defined in the usual way. If $I = \{1, 2, \ldots, n\}$, then we apply also the notation $\mathscr{A}_1 \times \ldots \times \mathscr{A}_n$. The elements of $\prod\limits_{i\in I} \mathscr{A}_i$ are written in the form $x = (x_i)_{i\in I}$; x_i is the *component* of x in $\mathscr{A}_i$. We write also $x_i = x(\mathscr{A}_i)$.

Let $\mathscr{A}$ be a BCR ℓ-monoid. An isomorphism of the form

$$\varphi\colon\ \mathscr{A} \to \prod_{i\in I} \mathscr{A}_i \tag{1}$$

is a *direct product decomposition* of $\mathscr{A}$. If $a \in A$ and $\varphi(a) = (a_i)_{i\in I}$ then instead of $\varphi(a)(\mathscr{A}_i) = a_i$ we write shortly $a(\mathscr{A}_i) = a_i$.

For each $i \in I$ we put

$$A_{i0} = \{a \in A\colon\ a(\mathscr{A}_j) = 1(\mathscr{A}_j) \quad \text{for each } j \in I \setminus \{i\}\}.$$

Let $x^i \in A_i$, where A_i is the underlying set of $\mathscr{A}_i$. We denote by $\varphi_i(x^i)$ the element of A_{i0} whose i-th component is x^i; i.e., we have

$$\varphi_i(x^i)(\mathscr{A}_i) = x^i.$$

Let 0^i be the least element of $\ell(\mathscr{A}_i)$; we set $\varphi_i(0^i) = c_i$. Then A_{i0} is the interval $[c_i, 1]$ of $\ell(\mathscr{A})$. The set A_{i0} is closed with respect to the operations $\odot, \rightarrow, \vee$ and $\wedge$. It is easy to verify that the algebra

$$\mathscr{A}_{i0} = (A_{i0}; \odot, \rightarrow, \vee, \wedge, 1, c_i)$$

is a *BCR* ℓ-monoid and that the mapping

$$\varphi_i\colon\ \mathscr{A}_i \to \mathscr{A}_{i0} \tag{2}$$

is an isomorphism.

For each $a \in A$ we set

$$\varphi_0(a) = (\varphi_i(a_i))_{i \in I}.$$

Then in view of (1) and (2) we conclude that the mapping

$$\varphi_0\colon \mathscr{A} \to \prod_{i \in I} \mathscr{A}_{i0} \tag{3}$$

is a direct product decomposition of $\mathscr{A}$.

We say that $\mathscr{A}_{i0}$ $(i \in I)$ are *internal direct factors* of $\mathscr{A}$ and that (3) is an *internal direct product decomposition* of $\mathscr{A}$.

For a similar terminology concerning groups cf., e.g., Kurosh [11].

Further, we apply the analogous terminology and notation in the case when instead of $\mathscr{A}$ and $(\mathscr{A}_{i0})_{i \in I}$ we deal with a bounded lattice L and an indexed system $(L_i)_{i \in I}$ of bounded lattices. The greatest element and the least element of L are denoted by 1 and by 0, respectively; the symbols 1^i and 0^i have analogous meanings with respect to the lattice L_i for $i \in I$.

We recall that in the terminology of [10] concerning internal direct product decompositions of partially ordered sets, we now deal with the case when the element 1 of the lattice $L = \ell(\mathscr{A})$ is taken as the central element in the direct product decomposition under consideration (according to [10], any element of L could be taken as central for such decompositions of the lattice L).

3. Two-factor direct product decompositions

Again, let $\mathscr{A}$ be a BCR ℓ-monoid and $L = \ell(\mathscr{A})$. In this section we assume that L has a two-factor direct product decomposition

$$\varphi\colon\ L \to L_1 \times L_2. \tag{1}$$

Since the lattice L is bounded, in view of (1) we obtain that the lattice L_i is bounded as well, where $i \in \{1, 2, \}$; let 1^i and 0^i be the greatest and the least element of L_i, respectively. We put

$$\varphi^{-1}((1^1, 0^2)) = p, \quad \varphi^{-1}((0^1, 1^2)) = q.$$

Then we have

$$p \vee q = 1, \quad p \wedge q = 0. \tag{2}$$

Let $t \in A$, $\varphi(t) = (t_1, t_2)$. Further, let φ_0 be as in Section 2. Then $\varphi_0(t) = (\overline{t}_1, \overline{t}_2)$, where

$$\varphi(\overline{t}_1) = (t_1, 1), \quad \varphi(\overline{t}_2) = (1, t_2).$$

Therefore

$$\overline{t}_1 = p \vee t, \quad \overline{t}_2 = q \vee t, \quad \overline{t}_1 \wedge \overline{t}_2 = t.$$

Applying the notation from Section 2, we have an internal direct product decomposition

$$(1') \qquad \varphi_0\colon\ L_{10} \times L_{20}.$$

Clearly, L_{10} is the interval $[p, 1]$ of L; similarly, L_{20} is the interval $[q, 1]$ of L.

Lemma 3.1. *$p \odot q = 0$, $p \odot p = p$ and $q \odot q = q$.*

Proof. From the relation $p \wedge q = 0$ and from $(*_3)$ we obtain $p \odot q = 0$.
Further, from $p \vee q = 1$ we get

$$(p \odot p) \vee (p \odot q) = p \odot 1,$$

thus $p \odot p = p$. Similarly, $q \odot q = q$. □

Lemma 3.2. *The interval $[p, 1]$ of $\ell(\mathscr{A})$ is closed with respect to the operation $\odot$.*

Proof. This is a consequence of the relation $p \odot p = p$ and of $(*_4)$. □

Lemma 3.3. *The interval $[p, 1]$ of $\ell(\mathscr{A})$ is closed with respect to the operation $\to$.*

Proof. Let $y, z \in [p, 1]$. We have to verify that the relation $p \leqslant y \to z$ is valid. In view of (iv) it suffices to show that $p \odot y \leqslant z$.
According to 3.1, $(*_4)$ and $(*_3)$ we get

$$p = p \odot p \leqslant p \odot y \leqslant p,$$

whence $p \odot y = p$. Therefore $p \odot y \leqslant z$. □

Lemma 3.4. *The algebra $\mathscr{A}_1 = ([p, 1]; \odot, \to, \vee, \wedge, 1, p)$ is a BCR ℓ-monoid.*

Proof. This is a consequence of 3.2 and 3.3. □

An analogous result holds for the algebra $\mathscr{A}_2 = ([q, 1], \odot, \to, \vee, \wedge, 1, q)$.

Lemma 3.5. *For each $x \in A$ let us put $\varphi_1(x) = x \vee p$. Then for each $x, y \in A$ we have*

a) $\varphi_1(x \vee y) = \varphi_1(x) \vee \varphi_1(y)$;
b) $\varphi_1(x \wedge y) = \varphi_1(x) \wedge \varphi_1(y)$;
c) $\varphi_1(x \odot y) = \varphi_1(x) \odot \varphi_1(y)$.

Proof. The relation a) is obvious. In view of the distributivity of $\ell(\mathscr{A})$, b) is valid. The condition (iii) implies that c) holds. □

We clearly have $\varphi_1(x) = x$ for each $x \in [p, 1]$, hence φ_1 is a surjective mapping of A onto $[p, 1]$.

For the mapping $\varphi_2(x) = x \vee q$ we have an analogous result.

Consider the algebra $\mathscr{A}^* = (A; \odot, \vee, \wedge, 1, 0)$. Let φ_0 be as in $(1')$. Then in view of 3.5 we obtain

Lemma 3.6. *The mapping*

$$\varphi_0\colon \mathscr{A}^* \to \mathscr{A}_1^* \times \mathscr{A}_2^* \tag{$1''$}$$

is an internal direct product decomposition of $\mathscr{A}^*$ (where $\mathscr{A}_1^*$ and $\mathscr{A}_2^*$ are defined analogously to $\mathscr{A}^*$).

Now let us deal with the operation $\to$.

Let $y, z \in A$. We put $X = \{x \in A\colon x \odot y \leqslant z\}$. Then according to (iv) we get

$$y \to z = \max X. \tag{3}$$

Consider the set

$$X_1 = \{t \in [p, 1]\colon t \odot \varphi_1(y) \leqslant \varphi_1(z)\}.$$

Analogously to (3),

$$\varphi_1(y) \to \varphi_1(z) = \max X_1. \tag{$3'$}$$

In view of 3.6, we have

Lemma 3.7. *Let $x \in A$. Then $x \odot y \leqslant z$ if and only if $\varphi_1(x) \odot \varphi_1(y) \leqslant \varphi_1(z)$ and $\varphi_2(x) \odot \varphi_2(y) \leqslant \varphi_2(z)$.*

Put $X_0 = \{\varphi_1(x)\colon\ x \in X\}$. Applying 3.6 again, we get

$$\varphi_1(y \to z) = \max X_0. \tag{3''}$$

Also, $\varphi_1(x) = x \vee p \in X_1$ for each $x \in X$, hence

$$X_0 \subseteq X_1. \tag{4}$$

Let $v \in X_1$. Hence $v \odot \varphi_1(y) \leqslant \varphi_1(z)$. Since $v \in [p, 1]$, we obtain $v = \varphi_1(v)$, thus

$$\varphi_1(v) \odot \varphi_1(y) \leqslant \varphi_1(z). \tag{5}$$

We take any fixed $t \in X$. In view of 3.7,

$$\varphi_2(t) \odot \varphi_2(y) \leqslant \varphi_2(z). \tag{6}$$

According to Lemma 3.6 there exists $u \in A$ such that

$$\varphi_1(u) = \varphi_1(v), \quad \varphi_2(u) = \varphi_2(t).$$

Then in view of (5), (6) and 3.7 we conclude that u is an element of X. Therefore $\varphi_1(u) \in X_0$. Since $\varphi_1(u) = v$, we get $v \in X_0$. Hence $X_1 \subseteq X_0$. Summarizing, we have $X_1 = X_0$. Thus from (3′) and (3″) we obtain

Lemma 3.8. $\varphi_1(y \to z) = \varphi_1(y) \to \varphi_1(z)$.

Similarly, the relation

$$\varphi_2(y \to z) = \varphi_2(y) \to \varphi_2(z) \tag{7}$$

is valid.

Now from Lemma 3.6, Lemma 3.8 and (7) we conclude

Lemma 3.9. *The mapping*

$$\varphi_0\colon \mathscr{A} \to \mathscr{A}_1 \times \mathscr{A}_2$$

is an internal direct product decomposition of $\mathscr{A}$.

We have verified that each two-factor direct product decomposition of the lattice $\ell(\mathscr{A})$ determines a two-factor internal direct product decomposition of the BCR ℓ-monoid $\mathscr{A}$.

In the next section we will extend this result to the case when the direct product decomposition of $\ell(\mathscr{A})$ can have more than two factors.

We remark that Lemma 3.9 is related to Proposition 2.1 in Dvurečenskij and Rachůnek [4]. Applying the terminology used at the end of Section 2 above, the differences between the two results are as follows:

1) In 3.9 we deal with internal direct product decompositions having the central element 1 (i.e., we have direct factors whose underlying sets are of the form $[p, 1]$ while in 2.1 of [4], the central element is 0 (i.e., the factors are defined on intervals of type $[0, e]$).

2) On the direct factor, we work with the original binary operation $\rightarrow$ (as defined in $\mathscr{A}$), while in 2.1 of [4], new operations $\rightarrow_e$ are introduced.

In connection with the above situation let us also mention the well-known fact that if L is a distributive lattice with $a, b, u, v \in L$ such that

$$[u, v] = L, \quad a \wedge b = u, \quad a \vee b = v,$$

then the mapping $\psi\colon L \rightarrow [a, v] \times [b, v]$ defined by

$$\psi(x) = (x \vee a, x \vee b) \quad \text{for each } x \in L$$

yields a direct product decomposition of L. The corresponding dual result also holds.

4. The general case

Assume that $\mathscr{A}$ is a BCR ℓ-monoid and that for the corresponding lattice $\ell(\mathscr{A})$ we have a direct product decomposition

$$\varphi\colon \ell(\mathscr{A}) \rightarrow \prod_{i \in I} L_i. \tag{1}$$

We suppose that I has at least two elements.

Let i be a fixed element of I. Put $I^i = \{j \in I\colon j \neq i\}$ and

$$L_i' = \prod_{j \in I^i} L_j.$$

For $a \in A$ we put

$$a(L_i') = (a(L_j))_{j \in I^i},$$
$$\varphi^i(a) = (a(L_i), a(L_j))_{j \in I^i}).$$

Then we have a two factor direct product decomposition

$$(1') \qquad \varphi^i\colon \ell(\mathscr{A}) \to L_i \times L_i'.$$

We construct L_{i0}, L_{i0}' and φ_0^i as in Section 2. In this way we obtain a two-factor internal direct product decomposition

$$(1'') \qquad \varphi_0^i\colon \ell(\mathscr{A}) \to L_{i0} \times L_{i0}'.$$

In view of Lemma 3.9 we conclude that

1) the algebra $(L_{i0}; \odot, \to, \vee, \wedge, 1, v^i)$ is a BCR ℓ-monoid; it will be denoted by $\mathscr{A}_{i0}$,

2) the algebra $(L_{i0}'; \odot, \to, \vee, \wedge, 1, v^{i1})$ is a BCR ℓ-monoid which will be denoted by $\mathscr{A}_{i0}'$;

3) the mapping

$$(1''') \qquad \varphi_0^i\colon \mathscr{A} \to \mathscr{A}_{i0} \times \mathscr{A}_{i0}'$$

is an internal direct product decomposition of $\mathscr{A}$.

Let $a \in A$ and $i \in I$. By virtue of $(1''')$ we can consider the component $a(\mathscr{A}_{i0})$ of a in $\mathscr{A}_{i0}$.

Now we put $\varphi_0(a) = (a(\mathscr{A}_{i0}))_{i\in I}$.

Theorem 4.1. *The mapping*

$$\varphi_0\colon \mathscr{A} \to \prod_{i\in I} \mathscr{A}_{i0}$$

is an internal direct product decomposition of $\mathscr{A}$.

Proof. Let $i \in I$. In view of $(1''')$, the mapping

$$a \to a(\mathscr{A}_{i0})$$

is a homomorphism of $\mathscr{A}$ onto $\mathscr{A}_{i0}$. This implies that φ_0 is a homomorphism of $\mathscr{A}$ into $\prod_{i\in I} \mathscr{A}_{i0}$.

According to (1) and the definitions from Section 2, φ_0 yields an internal direct product decomposition of $\ell(\mathscr{A})$. Hence the mapping φ_0 is a bijection. Thus φ_0 is an isomorphism of $\mathscr{A}$ onto $\prod_{i\in I} \mathscr{A}_{i0}$. Moreover, in view of the above mentioned fact concerning $\ell(\mathscr{A})$, φ_0 is also an internal direct product decompostion of $\mathscr{A}$. □

Let φ_0 be as in 4.1. Further, let

$$\psi_0\colon \mathscr{A} \to \prod_{j\in J} \mathscr{B}_{j0}$$

be another internal direct prodct decomposition of $\mathscr{A}$. We say that ψ_0 is a refinement of φ_0 if for each $i \in I$ there exists a subset $J(i)$ of J such that we have an internal direct product decomposition

$$\mathscr{A}_{i0} \to \prod_{j\in J(i)} \mathscr{B}_{j0}.$$

An analogous terminology will be applied for internal direct product decompositions of bounded lattices.

Now let φ_0 and ψ_0 be any internal direct product decompositions of $\mathscr{A}$. Then

$$\varphi_0\colon \ell(\mathscr{A}) \to \prod_{i\in I} \ell(\mathscr{A}_{i0}),$$
$$\psi_0\colon \ell(\mathscr{A}) \to \prod_{j\in J} \ell(\mathscr{A}_{j0})$$

are internal direct product decompositions of the lattice $\ell(\mathscr{A})$. According to the well-known result of Hashimoto [6], any two internal direct product decompositions of a bounded lattice L have a common refinement. From this it also follows that the system of all internal direct factors of L is a Boolean algebra. Therefore in view of Theorem 4.1 we obtain

Theorem 4.2. *Any two internal direct product decompositions of a BCR ℓ-monoid $\mathscr{A}$ have a common refinement. The system of all internal direct factors of $\mathscr{A}$ is a Boolean algebra.*

Let $\mathscr{A}$ be a BCR ℓ-monoid. Consider direct product decompositions

$$\alpha\colon \mathscr{A} \to \prod_{i\in I} \mathscr{A}_i,$$
$$\beta\colon \mathscr{A} \to \prod_{j\in J} \mathscr{B}_j$$

of $\mathscr{A}$. We say that α and β are isomorphic if there exists a bijection $\chi\colon I \to J$ such that $\mathscr{A}_i \simeq B_{\chi(i)}$ for each $i \in I$.

The following assertion is obvious.

Lemma 4.3. *Let α, β and γ be direct product decompositions of a BCR ℓ-monoid $\mathscr{A}$. Assume that α is isomorphic to β and γ is a refinement of α. Then there exists a direct product decomposition δ of $\mathscr{A}$ such that δ is a refinement of β and γ is isomorphic to δ.*

If α is a direct product decomposition of a BCR ℓ-monoid $\mathscr{A}$, then we denote by α_0 the corresponding internal direct product decomposition of $\mathscr{A}$ (cf. the notation φ and φ_0 in Section 2). It is obvious that α is isomorphic to α_0.

From Theorem 4.1 and Lemma 4.3 we obtain (cf. Fig. 1, where γ_0 denotes the common refinement of α_0 and β_0)

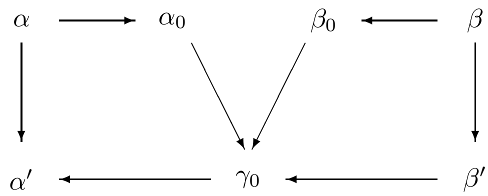

Fig. 1

Proposition 4.4. *Any two direct product decompositions of a BCR ℓ-monoid have isomorphic refinements.*

5. States on direct products

As above, let $\mathscr{A} = (A; \odot, \rightarrow, \vee, \wedge, 1, 0)$ be a BCR ℓ-monoid.

Definition 5.1 (Cf. [3]). A mapping s of the set A into the interval $[0, 1]$ of reals is called a state on $\mathscr{A}$ if the following conditions are satisfied:

(S1) $s(x) + s(x \rightarrow y) = s(y) + s(y \rightarrow x)$ for each $x, y, z \in A$;
(S2) $s(0) = 0$ and $s(1) = 1$.

Assume that s is a state on $\mathscr{A}$. Then in view of Proposition 4.2 in [3], for each $x, y \in A$ we have

(S6) $x \leqslant y \Rightarrow s(x) \leqslant s(y)$;
(S13) $s(x) + s(y) = s(x \vee y) + s(x \wedge y)$.

Applying the standard terminology of lattice theroy (cf. Birkhoff [1]), from (S13) we conclude that s is a *valuation* on the lattice $\ell(\mathscr{A})$.

We will use the notation from Section 2 and Section 3.

Proposition 5.2. *Assume that*

$$\varphi_0\colon \mathscr{A} \to \mathscr{A}_{10} \times \mathscr{A}_{20}$$

is an internal direct product decomposition of $\mathscr{A}$*. Let* s *be a state on* $\mathscr{A}$*. Then the mapping* s *is uniquely determined by the values* $s(t)$*, where* t *runs over the set* $A_{10} \cup A_{20}$.

Proof. The mapping φ_0 yields also a direct product decomposition of the lattice $\ell(\mathscr{A})$; we have

$$\varphi_0\colon \ell(\mathscr{A}) \to \ell(\mathscr{A}_{10}) \times \ell(\mathscr{A}_{20}).$$

Let p and q be as in Section 3; hence $\ell(\mathscr{A}_{10})$ is an interval $[p,1]$ of $\ell(\mathscr{A})$; similarly $\ell(\mathscr{A}_{20})$ is an interval $[q,1]$ of $\ell(\mathscr{A})$.

For $x \in A$ we put $p_1 = p \vee x$ and $q_1 = q \vee x$. Then $p_1, q_1 \in A_{10} \cup A_{20}$ and

$$p_1 \vee q_1 = 1, \quad p_1 \wedge q_1 = x.$$

Thus in view of (S13) we obtain

$$s(p_1) + s(q_1) = 1 + s(x),$$
$$s(x) = s(p_1) + s(q_1) - 1.$$

□

By the obvious induction, from Proposition 5.2 we get

Proposition 5.3. *Assume that*

$$\varphi_0\colon \mathscr{A} \to \mathscr{A}_{10} \times \ldots \times \mathscr{A}_{1n}$$

is an internal direct product decomposition of $\mathscr{A}$*. Let* s *be a state on* $\mathscr{A}$*. Then the mapping* s *is uniquely determined by the values* $s(t)$*, where* t *runs over the set* $A_{10} \cup \ldots \cup A_{n0}$.

Let the assumptions of Proposition 5.2 be fulfilled and let p, q be as in the proof of 5.2. Then $p \vee q = 1$ and $p \wedge q = 0$, whence in view of (S13) we get

(1) $$s(p) + s(q) = 1.$$

Further, according to (S6), for each $p_1 \in [p,1]$ and each $q_1 \in [q,1]$ we have

(2) $$s(p_1) \in [s(p), 1], \quad s(q_1) \in [s(q), 1].$$

Having in mind the relations (1) and (2) we consider the following construction. Assume that r_1, r_2 are non-negative integers with $r_1 + r_2 = 1$.

Suppose that s_1 is a mapping of the interval $[p, 1]$ of $\ell(\mathscr{A})$ into the interval $[r_1, 1]$ of reals such that for any $p_1, p_2 \in [p, 1]$ we have

$$s_1(p_1) + s_1(p_1 \to p_2) = s_1(p_2) + s_1(p_2 \to p_1),$$
$$s_1(p) = r_1, \quad s_1(1) = 1.$$

Further, suppose that $s_2 \colon [q, 1] \to [r_2, 1]$ has analogous properties.

Recall (cf. Section 3) that for $x \in A$ we have $\varphi_0(x) = (x \vee p, x \vee q)$. For each $x \in A$ we put

$$s(x) = s_1(x \vee p) + s_2(x \vee q) - 1. \tag{3}$$

Proposition 5.4. *Let s be as in* (3)*. Then s is a state on $\mathscr{A}$.*

Proof. By easy calculation we verify that $s(0) = 0$ and $s(1) = 1$.

Let $x, y \in A$. Put $x \vee p = p_1, x \vee q = q_1, y \vee p = p_2, y \vee q = q_2$. In view of 3.9,

$$(x \to y) \vee p = (x \vee p) \to (y \vee p) = p_1 \to p_2.$$

Analogously we have

$$(x \to y) \vee q = q_1 \to q_2, \quad (y \to x) \vee p = p_2 \to p_1, \quad (y \to x) \vee q = q_2 \to q_1.$$

Therefore

$$s(x) = s_1(p_1) + s_2(q_1) - 1,$$
$$s(y) = s_1(p_2) + s_2(q_2) - 1,$$
$$s(x \to y) = s_1(p_1 \to p_2) + s_2(q_1 \to q_2) - 1,$$
$$s(y \to x) = s_1(p_2 \to p_1) + s_2(q_2 \to q_1) - 1.$$

Using these relations and the above mentioned assumptions concerning s_1 and s_2 we obtain that (S1) holds. □

Similarly to Propositions 5.2 and 5.3, Proposition 5.4 can be generalized for n-factor direct product decompositions.

Now let us suppose that s is a state on a BCR ℓ-monoid and that

$$\varphi_0 \colon \mathscr{A} \to \prod_{i \in I} \mathscr{A}_{i0}$$

is an internal direct product decomposition of $\mathscr{A}$ such that the set I is infinite.

We apply the notation as in the previous section. The case $\operatorname{card} A = 1$ being trivial we suppose that $\operatorname{card} A > 1$; then without loss of generality we can assume that $\operatorname{card} A_{i0} > 1$ for each $i \in I$.

For $i \in I$, v^i is the least element of A_{i0} and 1 is the greatest element of A_{i0}. Hence $v^i < 1$.

We prove the following result:

Proposition 5.5. *Let φ_0 and s be as above. Put*

$$I_0 = \{i \in I\colon\ s(v^i) = 1\}.$$

Then $\operatorname{card}(I \setminus I_0) \leqslant \aleph_o$.

Before proving Proposition 5.5 we need some auxiliary considerations.

Let $i \in I$. There exists $q^i \in A$ such that

$$q^i(\mathscr{A}_{i0}) = 1, \quad q^i(\mathscr{A}_{j0}) = v^i \quad \text{for each } j \in I \setminus \{i\}.$$

Hence $q^i \neq 0$. If $i(1)$ and $i(2)$ are distinct elements of I, then

$$q^{i(1)} \wedge q^{i(2)} = 0, \quad q^{i(1)} \vee q^{i(2)} = 1.$$

Let I_0 be as in 5.5. Further, for each $n \in \mathbb{N}$ we set

$$I_n = \Big\{i \in I\colon\ \frac{1}{n+1} < s(q^i) \leqslant \frac{1}{n}\Big\}.$$

Thus the sets $I_0, I_1, I_2, \ldots$ are mutually disjoint.

Lemma 5.6. *Let k be a positive integer. Then the set I_k is finite.*

P r o o f. By way of contradiction, assume that the set I_k is infinite. Then there exists a system of distinct elements $\{i(k,n)\}_{n\in\mathbb{N}}$ belonging to I_k. Let $m \in \mathbb{N}$. We denote

$$t_m = q^{i(k,1)} \vee \ldots \vee q^{i(k,m)}.$$

Since the elements $q^{i(k,1)}, \ldots, q^{i(k,m)}$ are mutually orthogonal, from (S13) and by induction we obtain

$$s(t_m) = s(^{i(k,1)}) + \ldots + s(q^{i(k,m)}).$$

In view of the definition of I_k,

$$\frac{1}{k+1} < s(q^{i(k,1)}), \ \ldots, \ \frac{1}{k+1} < s(q^{i(k,m)}),$$

whence $s(t_m) > m/(k+1)$. For $m > k+1$ we get $s(t_m) > 1$, which is a contradiction. $\square$

P r o o f of Proposition 5.5. Put $I^* = \bigcup_{n\in\mathbb{N}} I_n$. According to Lemma 5.6 we obtain $\operatorname{card} I^* \leqslant \aleph_0$. For each $i \in I$ we have

$$v^i \wedge q^i = 0, \quad v^i \vee q^i = 1.$$

Then in view of (S13) we get $S(v^i) + S(q^i) = 1$, whence

$$s(v^i) = 1 \Leftrightarrow s(q^i) = 0.$$

This yields $I \setminus I_0 = I^*$. Therefore $\operatorname{card}(I \setminus I_0) \leqslant \aleph_0$. □

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