Guo-Fang Zhang

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SOME PROPERTIES OF RELATIVELY STRONG PSEUDOCOMPACTNESS

Guo-Fang Zhang, Siping

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Abstract. In this paper, we study some properties of relatively strong pseudocompactness and mainly show that if a Tychonoff space X and a subspace Y satisfy that $Y \subset \overline{\operatorname{Int} Y}$ and Y is strongly pseudocompact and metacompact in X, then Y is compact in X. We also give an example to demonstrate that the condition $Y \subset \overline{\operatorname{Int} Y}$ can not be omitted.

Keywords: relative topological properties, pseudocompact spaces, compact space

MSC 2010: 54D20, 54D30

1. Introduction

A. V. Arhangel'skii and H. M. M. Genedi [3] introduced the theory of relative topological properties in 1989. Many results on "absolute" topological properties can be interpreted as theorems on relative topological properties, which is a guideline of studying relative topology. In [2], it was shown that if Y is strongly pseudocompact and paracompact in X, then Y is compact in X. We know that pseudocompact metacompact spaces are compact [6], so it is natural to consider the following question:

Question. Let X be a Tychonoff space and Y a subspace of X such that Y is strongly pseudocompact and metacompact (strongly metacompact) in X. Is then Y compact in X?

In this paper, we provide an example to answer negatively the above question and also obtain the following theorem:

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Theorem. *Let Y be a subspace of a Tychonoff space X satisfying $Y \subset \overline{\operatorname{Int} Y}$, where $\overline{\operatorname{Int} Y}$ denotes the closure of $\operatorname{Int} Y$ in X. If Y is metacompact and strongly pseudocompact in X, then Y is compact in X.*

Throughout this paper, we assume that all spaces are T_1. Unless otherwise stated, when we say that a set U is open (closed), we mean it is open (closed) in X even if U is a subset of a subspace Y of X, $\overline{U}$ denotes the closure of U in X and $\operatorname{Int} U$ denotes the interior of U in X. For any set $A \subset X$ and collection $\mathscr{U}$ of subsets of X, $\operatorname{St}(A, \mathscr{U})$ denotes the set $\bigcup\{U \in \mathscr{U} \colon U \cap A \neq \emptyset\}$. A collection $\mathscr{U}$ of subsets of X is said to be *point finite* (*locally finite*) *on a subset* A [5] of X if for each $x \in A$ the collection $\{U \in \mathscr{U} \colon x \in U\}$ is finite (there is a neighborhood V of x in X such that $\{U \in \mathscr{U} \colon U \cap V \neq \emptyset\}$ is finite). Let $\mathscr{U}$ and $\mathscr{V}$ be be two collections of subsets of X. $\mathscr{V}$ is said to be a *partial refinement* of $\mathscr{U}$ if for each $V \in \mathscr{V}$ there is a $U \in \mathscr{U}$ such that $V \subset U$. If in the above definition $\mathscr{V}$ and $\mathscr{U}$ are two covers of X, we say that $\mathscr{V}$ is a *refinement* [4] of $\mathscr{U}$.

Other undefined notions and terminologies are as in [4].

2. Main results

Let Y be a subspace of a space X. Y is said to be *strongly pseudocompact in* X [2] (see also [1]) if every family $\mathscr{U} = \{U_\alpha \colon \alpha \in \Lambda\}$ of open subsets of X which is locally finite on Y and satisfies $U_\alpha \cap Y \neq \emptyset$ for every $\alpha \in \Lambda$, is finite. The following theorem characterizes strong pseudocompactness of Y in X in terms of collections similar to collections with the finite intersection property.

Theorem 2.1. *Let Y be a subspace of X. Then the following conditions are equivalent:*

(1) *Y is strongly pseudocompact in X.*

(2) *For every decreasing sequence $W_1 \supset W_2 \supset \ldots$ of open subsets of X which satisfies $W_i \cap Y \neq \emptyset$ for $i = 1, 2, \ldots$, $\Big(\bigcap\limits_{i=1}^{\infty} \overline{W_i}\Big) \cap Y \neq \emptyset$.*

(3) *If $\{V_i\}_{i=1}^{\infty}$ is a countable collection of open subsets of X such that $V_{i_1} \cap V_{i_2} \cap \ldots \cap V_{i_k} \cap Y \neq \emptyset$ for every finite set $\{i_1, i_2, \ldots, i_k\} \subset \{1, 2, \ldots\}$, then $\Big(\bigcap\limits_{i=1}^{\infty} \overline{V_i}\Big) \cap Y \neq \emptyset$.*

Proof. First we will show that (1) $\Rightarrow$ (2). Suppose that $\{W_i\}_{i=1}^{\infty}$ is a decreasing sequence of open subsets of X such that $W_i \cap Y \neq \emptyset$ for $i = 1, 2, \ldots$. By the definition of strong pseudocompactness of Y in X, $\{W_i\}_{i=1}^{\infty}$ is not locally finite at some point y_0 of Y. So every neighborhood U of y_0 in X meets infinitely many W_i's, hence $y_0 \in \bigcap\limits_{i=1}^{\infty} \overline{W_i}$.

To prove that (2) $\Rightarrow$ (3) it suffices to consider the decreasing sequence V_1, $V_1 \cap V_2, \ldots$. It is easy to see that the sequence satisfies condition (2), so $\left(\bigcap_{i=1}^{\infty} \overline{V_i}\right) \cap Y \neq \emptyset$.

Finally, we shall show that (3) $\Rightarrow$ (1). Suppose that $\{U_i\}_{i=1}^{\infty}$ is a collection of open subsets of X locally finite on Y and such that $U_i \cap Y \neq \emptyset$ for each i. Let $V_k = \bigcup_{i=k}^{\infty} U_i$. Then the collection $\{V_k\}_{k=1}^{\infty}$ satisfies condition (3), which implies that $\left(\bigcap_{k=1}^{\infty} \overline{V_k}\right) \cap Y \neq \emptyset$. So there is a $y_0 \in Y$ such that $y_0 \in \overline{V_k}$ for $k = 1, 2, \ldots$. Since $\{U_i\}_{i=1}^{\infty}$ is locally finite on Y, there exists a neighborhood U_{y_0} of y_0 in X and a finite subcollection $\{U_{i_1}, U_{i_2}, \ldots, U_{i_m}\}$ of $\{U_k\}_{k=1}^{\infty}$ such that U_{y_0} does not intersect any member of $\{U_k\}_{k=1}^{\infty}$ other than $U_{i_1}, U_{i_2}, \ldots, U_{i_m}$. Put $k_0 = \max\{i_1, i_2, \ldots, i_m\}$. Then $U_{y_0} \cap V_{k_0+1} = \emptyset$ and so $y_0 \notin \overline{V_{k_0+1}}$, a contradiction. Therefore, Y is strongly pseudocompact in X. □

Corollary 2.1. *Suppose that Y is a subspace of a Tychonoff space X such that Y is strongly pseudocompact in X. If $\{D_n\}_{n=1}^{\infty}$ is a sequence of open subsets of X such that $\overline{D_n \cap Y} \supset Y$ for $n = 1, 2, \ldots$, then $\overline{\bigcap_{n=1}^{\infty} (D_n \cap Y)} \supset Y$. In particular, if $\overline{D_n \cap Y} = Y$ for $n = 1, 2, \ldots$, then $\overline{\bigcap_{n=1}^{\infty} (D_n \cap Y)} = Y$.*

Proof. Let $\{D_n\}_{n=1}^{\infty}$ be a sequence of open subsets of X such that $\overline{D_n \cap Y} \supset Y$ for $n = 1, 2, \ldots$. Then for each $y \in Y$, if U is a neighborhood of y in X, we have $U \cap D_n \cap Y \neq \emptyset$ for $n = 1, 2, \ldots$. Let $V_1 = U$. So there exists a $y_1 \in Y$ such that $y_1 \in U \cap D_1 \cap V_1$. By the regularity of X, there is an open subset V_2 such that $y_1 \in V_2 \subset \overline{V_2} \subset U \cap D_1 \cap V_1$. Clearly $U \cap V_2$ is a neighborhood of y_1, so $U \cap V_2 \cap D_2 \cap Y \neq \emptyset$. Similarly there is a $y_2 \in Y$ and an open subset V_3 such that $y_2 \in V_3 \subset \overline{V_3} \subset U \cap D_2 \cap V_2$. Continuing the process, we can obtain a sequence $\{V_n\}_{n=1}^{\infty}$ of open subsets of X such that $V_1 \supset V_2 \supset \ldots$ and $V_n \cap Y \neq \emptyset$ for $n = 1, 2, \ldots$. By Theorem 2.1, $\left(\bigcap_{n=1}^{\infty} \overline{V_n}\right) \cap Y \neq \emptyset$ and so $U \cap \left(\bigcap_{n=1}^{\infty} D_n\right) \cap Y \neq \emptyset$. Since U is any neighborhood of y in X, $y \in \overline{\left(\bigcap_{n=1}^{\infty} D_n\right) \cap Y} = \overline{\bigcap_{n=1}^{\infty} (D_n \cap Y)}$. Hence $\overline{\bigcap_{n=1}^{\infty} (D_n \cap Y)} \supset Y$. If $\overline{D_n \cap Y} = Y$, it is easy to prove that $\overline{\bigcap_{n=1}^{\infty} (D_n \cap Y)} = Y$. □

Let Y be a subspace of a space X. Y is said to be *compact in* X [3] (see also [1]), if for every open cover of X there is a finite subfamily $\mathscr{H}$ such that $Y \subset \bigcup \mathscr{H}$. Y is said to be *metacompact in* X [5] provided every open cover $\mathscr{U}$ of X has an open partial refinement $\mathscr{V}$ point finite on Y. If in the above definition $\mathscr{V}$ covers X, then we say that Y *is strongly metacompact in* X [5]. The following theorem characterizes the relative version of the result that a pseudocompact metacompact space is compact.

Theorem 2.2. *Let Y be a subspace of a Tychonoff space X satisfying $Y \subset \overline{\operatorname{Int} Y}$. If Y is metacompact and strongly pseudocompact in X, then Y is compact in X.*

To prove Theorem 2.2, we need the following two lemmas.

Lemma 2.1. *Let Y be a subspace of a Tychonoff space X and let G be an open subset of X such that $G \cap Y \neq \emptyset$. If Y is strongly pseudocompact in X, then for every sequence $\{F_n\}_{n=1}^{\infty}$ of open subsets of X such that $\overline{F_n \cap G \cap Y} \supset G \cap Y$ for $n = 1, 2, \ldots$, we have $\overline{\bigcap\limits_{n=1}^{\infty} (F_n \cap G \cap Y)} \supset G \cap Y$. In particular, if $G \subset Y$ and $\overline{F_n \cap G} \supset G$ for $n = 1, 2, \ldots$, then $\overline{\bigcap\limits_{n=1}^{\infty} (F_n \cap G)} \supset G$.*

P r o o f. Let G be an open subset of X such that $G \cap Y \neq \emptyset$ and let $\{F_n\}_{n=1}^{\infty}$ be a sequence of open subsets of X such that $\overline{F_n \cap G \cap Y} \supset G \cap Y$ for $n = 1, 2, \ldots$. Then $\{(G \cap F_n) \cup (X - \overline{G \cap Y})\}_{n=1}^{\infty}$ is a collection of open subsets of X. It is easy to show that $\overline{[(G \cap F_n) \cup (X - \overline{G \cap Y})] \cap Y} \supset Y$ for $n = 1, 2, \ldots$. By Corollary 2.1, we have $\overline{\left[\bigcap\limits_{n=1}^{\infty} (G \cap F_n \cap Y)\right] \cup [(X - \overline{G \cap Y}) \cap Y]} \supset Y$.

Pick $y \in G \cap Y$, and let U be a neighborhood of y in X. Without loss of generality, we may assume that $U \subset G$, then $\left(U \cap \left[\bigcap\limits_{n=1}^{\infty} (G \cap F_n \cap Y)\right]\right) \cup [U \cap Y \cap (X - \overline{G \cap Y})] \neq \emptyset$, i.e., $U \cap \left[\bigcap\limits_{n=1}^{\infty} (G \cap F_n \cap Y)\right] \neq \emptyset$, so $y \in \overline{\bigcap\limits_{n=1}^{\infty} (G \cap F_n \cap Y)}$. It follows that $\overline{\bigcap\limits_{n=1}^{\infty} (F_n \cap G \cap Y)} \supset G \cap Y$. □

Lemma 2.2. *Let Y be a subspace of a Tychonoff space X satisfying $Y \subset \overline{\operatorname{Int} Y}$ and let $\mathscr{U}$ be a collection of open subsets of X covering Y and point finite on Y. If Y is strongly pseudocompact in X, then there exists a subset A of Y such that $\bar{A} \supset Y$ and $\mathscr{U}$ is locally finite on A.*

P r o o f. Let $\mathscr{U}$ be a collection of open subsets of X such that $\mathscr{U}$ is point finite on Y and $Y \subset \bigcup \mathscr{U}$. Put $A = \{x \in Y \colon \mathscr{U}$ is locally finite at $x\}$. We shall show that $\bar{A} \supset Y$. Pick $y \in Y$, and let V be an open neighborhood of y in X. It suffices to prove that $V \cap A \neq \emptyset$. Let $X_n = \{x \in Y \colon x$ is in at most n elements of $\mathscr{U}\}$. Then $\overline{X_n} \cap (Y - X_n) = \emptyset$ for each n. In fact, for each $z \in Y - X_n$, z is in at least $n+1$ elements of $\mathscr{U}$. Without loss of generality, we may assume that $U_1, U_2, \ldots, U_{n+1}$ are distinct elements of $\mathscr{U}$ such that $z \in \bigcap\limits_{i=1}^{n+1} U_i$. Then $\bigcap\limits_{i=1}^{n+1} U_i$ is a neighborhood of z which does not meet X_n, so $z \notin \overline{X_n}$. It follows that $\overline{X_n} \cap (Y - X_n) = \emptyset$. Hence $Y \cap \overline{X_n} = [(Y - X_n) \cup X_n] \cap \overline{X_n} = X_n$. For $n = 1, 2, \ldots$, let $E_n = V \cap X_n$.

Claim. $\mathrm{Int}(E_n) \neq \emptyset$ *for some* n.

Suppose that for each n, $\mathrm{Int}(E_n) = \emptyset$. Let $F_n = V - \overline{X_n}$. Then F_n is an open subset of X and $\overline{F_n \cap V \cap \mathrm{Int}\, Y} \supset V \cap \mathrm{Int}\, Y$ for $n = 1, 2, \ldots$. In fact, by $Y \cap \overline{X_n} = X_n$, $\overline{F_n \cap V \cap \mathrm{Int}\, Y} \supset \overline{(V \cap \mathrm{Int}\, Y) - E_n}$. Since $\mathrm{Int}(E_n) = \emptyset$ for each n, $\overline{(V \cap \mathrm{Int}\, Y) - E_n} \supset V \cap \mathrm{Int}\, Y$ for each n. Using the condition $Y \subset \overline{\mathrm{Int}\, Y}$ and Lemma 2.1, we have

$$\overline{\Big(\bigcap_{n=1}^{\infty} F_n\Big) \cap V \cap \mathrm{Int}\, Y} = \overline{\bigcap_{n=1}^{\infty} (F_n \cap V \cap \mathrm{Int}\, Y)} \supset V \cap \mathrm{Int}\, Y \neq \emptyset.$$

But $\bigcap_{n=1}^{\infty} F_n = V - \bigcup_{n=1}^{\infty} \overline{X_n} \subset V - Y$ and so $\overline{\Big(\bigcap_{n=1}^{\infty} F_n\Big) \cap V \cap \mathrm{Int}\, Y} = \emptyset$, a contradiction.

Let k be the least element such that $\mathrm{Int}(E_k) \neq \emptyset$. Then for each $x \in \mathrm{Int}(E_k) = V \cap \mathrm{Int}(X_k)$, $x \notin \mathrm{Int}(E_{k-1}) = V \cap \mathrm{Int}(X_{k-1})$ and so there exists a neighborhood U of x such that $U \subset E_k$ and $U \cap (X \setminus X_{k-1}) \neq \emptyset$. Pick $z \in U \cap (X \setminus X_{k-1})$, then $z \in \mathrm{Int}(E_k)$ and z is in k elements of $\mathscr{U}$. Assume that $U_1, U_2, \ldots, U_k$ are distinct elements of $\mathscr{U}$ such that $z \in \bigcap_{i=1}^{k} U_i$. Let $W = \mathrm{Int}(E_k) \cap \bigcap_{i=1}^{k} U_i$. By the definition of E_k, W can not intersect any element of $\mathscr{U}$ other than $U_1, U_2, \ldots, U_k$, hence $z \in A$. Since $E_n = V \cap X_n$ for each n, $z \in \mathrm{Int}(E_k) \subset V$. It follows that $z \in V \cap A$. This concludes the proof.

Proof of Theorem 2.2. Let $\mathscr{U}$ be an open cover of X. By the regularity of X and the metacompactness of Y in X, there is a collection $\mathscr{W}$ of open subsets of X covering Y such that $\mathscr{W}$ is point finite on Y and $\{\overline{W}\colon W \in \mathscr{W}\}$ refines $\mathscr{U}$. By Lemma 2.2, there is a subset A of Y such that $\mathscr{W}$ is locally finite on A and $\bar{A} \supset Y$. For each $y \in Y$, fix an open subset U_y of X such that U_y meets only finitely many members of $\mathscr{W}$. Let $U = \bigcup_{y \in Y} U_y$ and let V_1 be a nonempty open subset of U which meets only finitely many members of $\mathscr{W}$ and $V_1 \cap Y \neq \emptyset$. Inductively pick for each $i \in \mathbb{N}$, if possible, an open set $V_i \subset U - \bigcup_{k=1}^{i-1} \mathrm{St}(V_k, \mathscr{W})$ such that V_i meets only finitely many members of $\mathscr{W}$ and $V_i \cap Y \neq \emptyset$.

Claim. *The induction stops at some* i.

Suppose that the induction proceeds infinitely, we can obtain an infinite sequence $\{V_i\}_{i=1}^{\infty}$ of nonempty open subsets of U such that V_i meets only finitely many members of $\mathscr{W}$ and $V_i \cap Y \neq \emptyset$ for each i, also $V_{i+1} \subset U - \bigcup_{k=1}^{i} \mathrm{St}(V_k, \mathscr{W})$. Let $U_n = \bigcup_{k=n}^{\infty} V_k$. Then $\{U_n\}_{n=1}^{\infty}$ is a decreasing sequence of open subsets of X such that $U_n \cap Y \neq \emptyset$ for $n = 1, 2, \ldots$. By Theorem 2.1, $\Big(\bigcap_{n=1}^{\infty} \overline{U_n}\Big) \cap Y \neq \emptyset$ and so there exists a $y \in Y$ such that $y \in \bigcap_{n=1}^{\infty} \overline{U_n}$. Pick $W \in \mathscr{W}$ such that $y \in W$. Then for each natural number n, $W \cap U_n \neq \emptyset$, which implies there exist distinct natural numbers l, k such that $k < l$ and $W \cap V_k \neq \emptyset$, $W \cap V_l \neq \emptyset$. So $\mathrm{St}(V_k, \mathscr{W}) \cap V_l \supset W \cap V_l \neq \emptyset$, which contradicts $V_l \subset U - \bigcup_{i=1}^{l-1} \mathrm{St}(V_i, \mathscr{W})$. Hence, the sequence $\{V_i\}_{i=1}^{\infty}$ is finite.

Let the above sequence be $\{V_1, V_2, \ldots, V_m\}$. We claim that $A \subset \overline{\bigcup_{i=1}^{m} \mathrm{St}(V_i, \mathscr{W})}$. Otherwise, if there is a $y_0 \in A$ such that $y_0 \notin \overline{\bigcup_{i=1}^{m} \mathrm{St}(V_i, \mathscr{W})}$, then there exists an open neighborhood V of y_0 such that $V \cap \bigcup_{i=1}^{m} \mathrm{St}(V_i, \mathscr{W}) = \emptyset$. Put $V_{m+1} = V \cap U$. It is easy to see that $V_{m+1} \subset U - \bigcup_{i=1}^{m} \mathrm{St}(V_i, \mathscr{W})$ and $V_{m+1} \cap Y \neq \emptyset$, which is a contradiction.

If $\mathscr{H} = \{H \in \mathscr{W} \colon H \cap V_i \neq \emptyset$ for some $i \leqslant m\}$, then $\mathscr{H}$ is a finite subcollection of $\mathscr{W}$ such that $Y \subset \bar{A} \subset \overline{\bigcup \mathscr{H}} = \bigcup_{H \in \mathscr{H}} \overline{H}$. Since $\{\overline{W} \colon W \in \mathscr{W}\}$ refines $\mathscr{U}$, Y is compact in X.

The following example demonstrates that in Theorem 2.2 the condition $Y \subset \overline{\mathrm{Int}\, Y}$ is necessary.

Example 2.1. There exists a Tychonoff space X and its subspace Y such that Y is strongly pseudocompact and metacompact (strongly metacompact) in X, but Y is not compact in X.

Proof. Let $X = [0, \omega_1) \times [0, \omega_1) - \{(0,0)\}$. For each $\alpha \in (0, \omega_1)$, let $H_\alpha = [0, \omega_1) \times \{\alpha\}$ and $G_\alpha = \{\alpha\} \times [0, \omega_1)$. Define a topology on X as follows: for $\alpha \in (0, \omega_1)$, a neighborhood of $(0, \alpha)$ contains $(0, \alpha)$ and all but finitely many points of H_α. The neighborhood of $(\alpha, 0)$ contains $(\alpha, 0)$ and all but finitely many points of G_α. All other points of X are isolated. Let $Y = [(0, \omega_1) \times \{0\}] \cup [\{0\} \times (0, \omega_1)]$ with the subspace topology of X. □

Claim 1. *X is a Tychonoff space.*

Clearly X is T_1 and X has a base consisting of open-and-closed sets.

Claim 2. *Y is metacompact in X.*

In fact, X is a metacompact space because any open cover of X has a natural open refinement $\mathscr{V}$ such that each point x of X is in at most 2 members of $\mathscr{U}$. Thus Y is metacompact (strongly metacompact) in X.

Claim 3. *Y is not compact in X.*

Since the sets $A = \{(0,\alpha)\colon\ 0 < \alpha < \omega_1\}$ and $B = \{(\alpha,0)\colon\ 0 < \alpha < \omega\}$ are two disjoint closed sets of X which can not be separated in X, Y is not normal in X (Y is said to be *normal in* X [3] if for any two disjoint closed subsets A and B of X there exist two disjoint open subsets U and V in X such that $A\cap Y \subset U$ and $B\cap Y\subset V$). By Theorem 5.1 of [1] (see also [3]), Y is not compact in X.

Claim 4. *Y is strongly pseudocompact in X.*

Let $\{V_i\}_{i=1}^{\infty}$ be a collection of nonempty open subsets of X such that $V_i\cap Y\neq\emptyset$ for each i. It suffices to prove that $\{V_i\}_{i=1}^{\infty}$ is not locally finite at some point of Y. Assume that $\{V_i\}_{i=1}^{\infty}$ is locally finite at each point of Y. Let $V_1^{(1)} = V_1$. Pick $y_1 \in V_1^{(1)}\cap Y$, then there are infinitely many members of $\{V_i\}_{i=2}^{\infty}$ which can not contain y_1. Otherwise, $\{V_i\}_{i=1}^{\infty}$ is not locally finite at y_1. Without loss generality, pick $V_1^{(2)}, V_2^{(2)},\ldots$ such that $y_1\notin V_k^{(2)}$ for $k=1,2,\ldots$, where $V_k^{(2)}\in\{V_i\}_{i=2}^{\infty}$ for each k, and let B_{y_1} be a basic neighborhood of y_1 in X such that $B_{y_1}\subset V_1^{(1)}$. Pick $y_2\in V_1^{(2)}\cap Y$, similarly there are infinitely many members of $\{V_i^{(2)}\}_{i=2}^{\infty}$ which can not contain y_2. Otherwise, $\{V_i\}_{i=1}^{\infty}$ is not locally finite at y_2. We pick $V_1^{(3)}, V_2^{(3)},\ldots$ such that $y_2\notin V_k^{(3)}$ for $k=1,2,\ldots$, where $V_k^{(3)}\in\{V_i^{(2)}\}_{i=2}^{\infty}$ for each k, and let B_{y_2} be a basic neighborhood of y_2 in X such that $B_{y_2}\subset V_1^{(2)}$. Continuing the process, so we can obtain an infinite sequence of distinct points of Y: $y_1, y_2,\ldots$ such that $B_{y_n}\subset V_1^{(n)}$, where B_{y_n} is a basic neighborhood of y_n in X and $V_1^{(n)}\in\{V_i\}_{i=1}^{\infty}$ for $n=1,2,\ldots$.

We shall show that there is a $y_0\in Y$ such that $\{B_{y_n}\}_{n=1}^{\infty}$ is not locally finite at y_0, and so $\{V_i\}_{i=1}^{\infty}$ is not locally finite at y_0 which contradicts our assumption. Clearly, there are infinitely many elements of $\{y_n\}_{n=1}^{\infty}$ which are contained in $(0,\omega_1)\times\{0\}$ or $\{0\}\times(0,\omega_1)$. Without loss of generality, we suppose that there are infinitely many elements of $\{y_n\}_{n=1}^{\infty}$ which are contained in $(0,\omega_1)\times\{0\}$ and denote these elements by $y_{n_1} = (\alpha_{n_1},0)$, $y_{n_2} = (\alpha_{n_2},0),\ldots$. Let

$$B_{y_{n_j}} = \{\alpha_{n_j}\}\times[0,\omega_1)\setminus\{(\alpha_{n_j},\beta_1^{\alpha_{n_j}}),(\alpha_{n_j},\beta_2^{\alpha_{n_j}}),\ldots,(\alpha_{n_j},\beta_{i_j}^{\alpha_{n_j}})\},$$

where $\beta_k^{\alpha_{n_j}}\in(0,\omega_1)$ for $1\leqslant k\leqslant i_j$ and $i_j\in\mathbb{N}$ for $j=1,2,\ldots$. Put

$$\beta = \sup\{\beta_1^{\alpha_{n_1}},\ldots,\beta_{i_1}^{\alpha_{n_1}},\beta_1^{\alpha_{n_2}},\ldots,\beta_{i_2}^{\alpha_{n_2}},\ldots\}.$$

Pick $y_0 = (0,\beta+1)$, then any neighborhood of y_0 meets infinitely many members of $\{B_{y_{n_j}}\}_{j=1}^{\infty}$. It follows that $\{V_i\}_{i=1}^{\infty}$ is not locally finite at y_0. □

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References

[1] *A. V. Arhangel'skii*: From classic topological invariant to relative topological properties. Sci. Math. Japon. *55* (2001), 153–201.
[2] *A. V. Arhangel'skii*: Location type properties: relative strong pseudocompactness. Trudy Matem. Inst. RAN *193* (1992), 28–30.
[3] *A. V. Arhangel'skii, H. M. M. Genedi*: Beginning of the theory of relative topological properties. General Topology: Space and Mapping. MGU, Moscow, 1989, pp. 3–48. (In Russian.)
[4] *R. Engelking*: General Topology. Sigma Series in Pure Mathematics. Heldermann, Berlin, 1989.
[5] *E. M. Grabner, G. C. Grabnor, K. Miyazaki*: On properties of relative metacompactness and paracompactness type. Topol. Proc. *25* (2000), 145–177.
[6] *B. M. Scott*: Pseudocompact metacompact spaces are compact. Topology, Proc. Conf. *4* (1979), 577–587.

Author's address: G.-F. Zhang, College of Mathematics, Jilin Normal University, Siping, Jilin Province 136 000, P. R. China, e-mail: `guofangzhang@126.com`.