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CONVERGENCE THEOREMS FOR THE BIRKHOFF INTEGRAL

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Abstract. We give sufficient conditions for the interchange of the operations of limit and the Birkhoff integral for a sequence (f_n) of functions from a measure space to a Banach space. In one result the equi-integrability of f_n's is involved and we assume $f_n \to f$ almost everywhere. The other result resembles the Lebesgue dominated convergence theorem where the almost uniform convergence of (f_n) to f is assumed.

Keywords: Birkhoff integral, convergence theorems, vector valued functions

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1. Introduction

Integration of vector valued functions is an important topic of mathematical analysis. A classical exposition of this theory can be found in [5] and [3]; see also the recent monograph [14] including the McShane and Kurzweil-Henstock integrals. The Birkhoff integral for Banach space valued functions, located strictly between the Bochner and Pettis integrals, was introduced in 1935 (see [1]). Lately, it has been investigated by several authors [2], [12], [9], [4], [10], [11]. A generalized version of the Birkhoff integral, invented by Dobrakov, has been studied in another recent article [13]. In our paper we will show some convergence theorems for the Birkhoff integral. One of them is due to Birkhoff and we recall it with the proof formulated in a new fashion. We give new sufficient conditions for the interchange of the operations of integral and limit. One theorem assumes equi-integrability of the functions of a sequence convergent almost everywhere. We also propose a version of the Lebesgue dominated convergence theorem for the absolute Birkhoff integral.

Let $\mathbb{N} = \{1, 2, \ldots\}$. Throughout the paper, $(\Omega, \mathfrak{S}, \mu)$ is a complete measure space with a σ-finite measure μ, and $(X, \|\cdot\|)$ is a Banach space over $\mathbb{R}$. Let us recall the original definition of the Birkhoff integral. By a *partition* of Ω we always mean a partition of Ω into (pairwise disjoint) countably many sets from $\mathfrak{S}$ of finite measure. For a given partition $\Gamma = (A_n)$ of Ω we say that a function $f\colon \Omega \to X$ is Γ-*summable* if the restrictions $f|A_n$ are bounded whenever $\mu(A_n) > 0$ and the set $J(f,\Gamma) = \Big\{\sum\limits_n f(t_n)\mu(A_n)\colon t_n \in A_n\Big\}$ consists of sums of unconditionally convergent series. The function f is called *Birkhoff integrable*, if for every $\varepsilon > 0$ there is a partition $\Gamma = (A_n)$ of Ω such that f is Γ-summable and $\operatorname{diam}(J(f,\Gamma)) < \varepsilon$. For an integrable function f, its *Birkhoff integral* is the unique element of the intersection

$$\bigcap \{\overline{\operatorname{Co}(J(f,\Gamma))}\colon\ f \text{ is } \Gamma\text{-summable}\}$$

where $\operatorname{Co}(A)$ stands for the convex hull of $A \subset X$. The integral will be denoted by $\int_\Omega f \,\mathrm{d}\mu$.

The above definition turns out to be equivalent with the version formulated by Fremlin [4] and with the notion introduced in [6], [7]. These equivalences were proved by B. Cascales and J. Rodríguez [2] (they assumed $\mu(\Omega) = 1$ but the theorem works for a σ-finite measure) and, independently, by the second author [10]. If Π and Γ are partitions of Ω, we say that Γ is finer than Π if each set from Γ is contained in some set from Π. Now, let us formulate the above-mentioned equivalences.

Proposition 1 ([2], [10]). *For a function $f\colon \Omega \to X$, the following conditions are equivalent:*

(i) *f is Birkhoff integrable;*

(ii) *there exists $x \in X$ such that for every $\varepsilon > 0$ there is a partition (A_i) of Ω such that for every choice $t_i \in A_i$ we have*

$$\Big\|\sum_i f(t_i)\mu(A_i) - x\Big\| < \varepsilon$$

and the series $\sum\limits_i f(t_i)\mu(A_i)$ is unconditionally covergent;

(iii) *there exists $y \in X$ such that for every $\varepsilon > 0$ there is a partition Π of Ω such that for any partition $\Gamma = (A_i)$ finer than Π and for every choice $t_i \in A_i$ we have*

$$\Big\|\sum_i f(t_i)\mu(A_i) - y\Big\| < \varepsilon$$

and the series $\sum\limits_i f(t_i)\mu(A_i)$ is unconditionally covergent.

Additionally, $x = y = \int_\Omega f \,\mathrm{d}\mu$.

Remark 2. Let us state another condition (ii′) equivalent to Birkhoff integrability. This is a Cauchy type condition associated with (ii) (compare also [10] and [2]). Namely, we have:

The function f is Birkhoff integrable if and only if for every $\varepsilon > 0$ there is a partition (A_i) of Ω such that

$$\Big\| \sum_n f(t_n)\mu(A_n) - \sum_n f(s_n)\mu(A_n) \Big\| < \varepsilon$$

for arbitrary choices $t_n, s_n \in A_i$, the series being unconditionally convergent.

We need the following useful characterization [8, Prop. 1.c.1]:

Fact 3. A series $\sum\limits_{i=1}^{\infty} x_i$ in X is unconditionally convergent if and only if, for every $\varepsilon > 0$ there is a positive integer k such that $\Big\| \sum\limits_{i\in S} x_i \Big\| < \varepsilon$ for every finite set $S \subset \mathbb{N} \setminus \{1, \ldots, k\}$.

Now, we give the convergence theorem due to Birkhoff [1] who only sketched the proof. We provide a new formal demonstration based on Proposition 1, Remark 2 and Fact 3.

Theorem 4. *Let $\mu(\Omega) < \infty$ and let $f_n\colon \Omega \to X$, $n \in \mathbb{N}$, be Birkhoff integrable. If (f_n) converges uniformly to f on Ω, then f is Birkhoff integrable and $\int_\Omega f \,\mathrm{d}\mu = \lim\limits_{n\to\infty} \int_\Omega f_n \,\mathrm{d}\mu$.*

Proof. We may assume that $\mu(\Omega) = 1$. To show the first assertion we use condition (ii′) from Remark 2. Let $\varepsilon > 0$. Since (f_n) converges to f uniformly, pick $N \in \mathbb{N}$ such that for all $n \geqslant N$ we have

$$\sup_{t\in\Omega} \|f_n(t) - f(t)\| \leqslant \frac{\varepsilon}{3}. \tag{1}$$

Since f_N is Birkhoff integrable, by (ii′) we can find a partition (E_i) of Ω such that

$$\Big\| \sum_i f_N(t_i)\mu(E_i) - \sum_i f_N(s_i)\mu(E_i) \Big\| \leqslant \frac{\varepsilon}{3} \tag{2}$$

for all $t_i, s_i \in E_i$ where the above series are unconditionally convergent.

First we will prove that for any $t_i \in E_i$ the series $\sum\limits_i f(t_i)\mu(E_i)$ is unconditionally convergent. To this aim we will use Fact 3. Fix any choice $t_i \in E_i$ and $\eta > 0$.

We will use (1) with $\varepsilon/3$ replaced by $\eta/2$, and N replaced by N_0. Since the series $\sum_i f_{N_0}(t_i)\mu(E_i)$ is unconditionally convergent, pick $k \in \mathbb{N}$ such that

$$\Big\| \sum_{i \in S} f_{N_0}(t_i)\mu(E_i) \Big\| < \frac{\eta}{2}$$

for every finite set $S \subset \mathbb{N} \setminus \{1, \dots, k\}$. Now, we have

$$\begin{aligned} \Big\| \sum_{i \in S} f(t_i)\mu(E_i) \Big\| &\leqslant \Big\| \sum_{i \in S} (f(t_i) - f_{N_0}(t_i))\mu(E_i) \Big\| + \Big\| \sum_{i \in S} f_{N_0}(t_i)\mu(E_i) \Big\| \\ &< \sum_{i \in S} \|f(t_i) - f_{N_0}(t_i)\|\mu(E_i) + \frac{\eta}{2} \leqslant \frac{\eta}{2} \sum_{i \in S} \mu(E_i) + \frac{\eta}{2} \leqslant \eta. \end{aligned}$$

Consequently, by Fact 3 the series $\sum_i f(t_i)\mu(E_i)$ is unconditionally convergent.

Observe that by (1) we get

$$\begin{aligned} \Big\| \sum_i f(t_i)\mu(E_i) - \sum_i f_N(t_i)\mu(E_i) \Big\| & \\ \leqslant \sum_i \|f(t_i) - f_N(t_i)\|\mu(E_i) \leqslant \frac{\varepsilon}{3} \sum_i \mu(E_i) = \frac{\varepsilon}{3}. \end{aligned} \tag{3}$$

Now, from (2) and (3) we derive a Cauchy type condition (ii′) (cf. Remark 2) for f. For any $t_i, s_i \in E_i$, $i \in \mathbb{N}$, we have

$$\begin{aligned} \Big\| \sum_i f(t_i)\mu(E_i) - \sum_i f(s_i)\mu(E_i) \Big\| &\leqslant \Big\| \sum_i f(t_i)\mu(E_i) - \sum_i f_N(t_i)\mu(E_i) \Big\| \\ + \Big\| \sum_i f_N(t_i)\mu(E_i) - \sum_i f_N(s_i)\mu(E_i) \Big\| &+ \Big\| \sum_i f_N(s_i)\mu(E_i) - \sum_i f(s_i)\mu(E_i) \Big\| \leqslant \varepsilon. \end{aligned}$$

Hence f is Birkhoff integrable. To show $\int_\Omega f_n \,\mathrm{d}\mu \to \int_\Omega f \,\mathrm{d}\mu$, let $\varepsilon > 0$ and choose $N \in \mathbb{N}$ as before. Fix $n \geqslant N$. Since f_n and f are Birkhoff integrable, by condition (iii) from Proposition 1 we can find a partition (F_i) such that for any $z_i \in F_i$ we have

$$\Big\| \sum_i f_n(z_i)\mu(F_i) - \int_\Omega f_n \,\mathrm{d}\mu \Big\| \leqslant \frac{\varepsilon}{3}, \tag{4}$$

$$\Big\| \sum_i f(z_i)\mu(F_i) - \int_\Omega f \,\mathrm{d}\mu \Big\| \leqslant \frac{\varepsilon}{3}, \tag{5}$$

where both the series are unconditionally convergent. As in the proof of (3) we get

$$\Big\|\sum_i f(z_i)\mu(F_i) - \sum_i f_n(z_i)\mu(F_i)\Big\| \leqslant \frac{\varepsilon}{3}. \tag{6}$$

Now from (4), (5), (6) it follows that

$$\begin{aligned}\Big\|\int_\Omega f_n\,\mathrm{d}\mu - \int_\Omega f\,\mathrm{d}\mu\Big\| &\leqslant \Big\|\int_\Omega f_n\,\mathrm{d}\mu - \sum_i f_n(z_i)\mu(F_i)\Big\| \\ &\quad + \Big\|\sum_i f_n(z_i)\mu(F_i) - \sum_i f(z_i)\mu(F_i)\Big\| \\ &\quad + \Big\|\sum_i f(z_i)\mu(F_i) - \int_\Omega f\,\mathrm{d}\mu\Big\| \leqslant \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.\end{aligned}$$

□

In the case when $X = \mathbb{R}$, the Birkhoff integral is reduced to the Lebesgue one, and Theorem 4 is well known. Note that the assumption $\mu(\Omega) < \infty$ cannot be omitted.

We say that Birkhoff integrable functions $f_n\colon \Omega \to X$, $n \in \mathbb{N}$, are *equi-Birkhoff integrable* if for every $\varepsilon > 0$ there is a partition (A_i) of Ω such that for every choice $t_i \in A_i$ the following conditions hold:

1° $\|\sum_i f_n(t_i)\mu(A_i) - \int_\Omega f_n\,\mathrm{d}\mu\| < \varepsilon$ for all $n \in \mathbb{N}$;

2° for every $\eta > 0$ there are $k \in \mathbb{N}$ and $n_0 \in \mathbb{N}$ such that $\|\sum_{i\in S} f_n(t_i)\mu(A_i)\| < \eta$ for every finite set $S \subset \mathbb{N} \setminus \{1,\dots,k\}$ and every $n \geqslant n_0$.

If a partition (A_i) and a choice $t_i \in A_i$ are fixed, and condition 2° is satisfied, we say that the series $\sum_i f_n(t_i)\mu(A_i)$, $n \in \mathbb{N}$, are *almost equi-unconditionally convergent* (in short, *AEU-convergent*).

Now, we will show that the equi-integrability of f_n's is more general than the uniform convergence of (f_n) if $\mu(\Omega) < \infty$ and f_n's are Birkhoff integrable.

Proposition 5. *Let $\mu(\Omega) < \infty$ and let $f_n\colon \Omega \to X$, $n \in \mathbb{N}$, be Birkhoff integrable. If (f_n) converges uniformly to $f\colon \Omega \to X$, then the functions f_n, $n \in \mathbb{N}$, are equi-Birkhoff integrable.*

Proof. Assume that $\mu(\Omega) = 1$. Let $\varepsilon > 0$. Pick $N_1 \in \mathbb{N}$ such that

$$\sup_{t\in\Omega} \|f_m(t) - f_n(t)\| < \frac{\varepsilon}{3} \tag{7}$$

for all $m, n \geqslant N_1$. By Theorem 4, f is Birkhoff integrable and $\int_\Omega f \,\mathrm{d}\mu = \lim\limits_{n\to\infty} \int_\Omega f_n \,\mathrm{d}\mu$. Pick $N_2 \in \mathbb{N}$ such that

$$\left\| \int_\Omega f_m \,\mathrm{d}\mu - \int_\Omega f_n \,\mathrm{d}\mu \right\| < \frac{\varepsilon}{3} \tag{8}$$

for all $m, n \geqslant N_2$. Put $N = \max\{N_1, N_2\}$ and $f_0 = f$. Since the functions $f_0, \ldots, f_N$ are Birkhoff integrable, using the equivalence (i) $\iff$ (iii) in Proposition 1 we find a partition (A_i) of Ω such that for every choice $t_i \in A_i$ and any $j \in \{0, \ldots, N\}$ we have

$$\left\| \sum_i f_j(t_i)\mu(A_i) - \int_\Omega f_j \,\mathrm{d}\mu \right\| < \frac{\varepsilon}{3}, \tag{9}$$

and the series $\sum\limits_i f_j(t_i)\mu(A_i)$, $j \in \{0, \ldots, N\}$, are unconditioanally convergent. Fix $t_i \in A_i$ and $n > N$. By (7) we have

$$\left\| \sum_i f_n(t_i)\mu(A_i) - \sum_i f_N(t_i)\mu(A_i) \right\| \leqslant \sum_i \|f_n(t_i) - f_N(t_i)\| \mu(A_i) < \frac{\varepsilon}{3}.$$

Hence by (9), (8) we obtain

$$\begin{aligned} \left\| \sum_i f_n(t_i)\mu(A_i) - \int_\Omega f_n \,\mathrm{d}\mu \right\| &\leqslant \left\| \sum_i f_n(t_i)\mu(A_i) - \sum_i f_N(t_i)\mu(A_i) \right\| \\ &+ \left\| \sum_i f_N(t_i)\mu(A_i) - \int_\Omega f_N \,\mathrm{d}\mu \right\| + \left\| \int_\Omega f_N \,\mathrm{d}\mu - \int_\Omega f_n \,\mathrm{d}\mu \right\| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

This together with (9) yields condition 1° of equi-integrability. It suffices to prove condition 2°. Thus let $\eta > 0$ and pick $n_0 \in \mathbb{N}$ such that

$$\sup_{t\in\Omega} \|f_n(t) - f(t)\| < \frac{\eta}{2}$$

for all $n \geqslant n_0$. Since $\sum\limits_i f(t_i)\mu(A_i)$ is unconditionally convergent, by Fact 3 pick $k \in \mathbb{N}$ such that $\|\sum\limits_{i\in S} f(t_i)\mu(A_i)\| < \eta/2$ for every finite set $S \subset \mathbb{N} \setminus \{1, \ldots, k\}$. Then for all $n \geqslant n_0$ and every S as above, we have

$$\left\| \sum_{i\in S} f_n(t_i)\mu(A_i) \right\| \leqslant \sum_{i\in S} \|f_n(t_i) - f(t_i)\| \mu(A_i) + \left\| \sum_{i\in S} f(t_i)\mu(A_i) \right\| < \frac{\eta}{2} + \frac{\eta}{2} = \eta.$$

□

In the next theorem we show that the equi-integrability of f_n's and the pointwise convergence of (f_n) guarantee the interchange of limit and integral. Results of that type are known for the vector-valued Kurzweil-Henstock and McShane integrals on $[a,b]$; see [14, Thm 3.5.2].

Theorem 6. *Assume that $(f_n)_{n\in\mathbb{N}}$ is a sequence of Birkhoff integrable functions from Ω to X, convergent almost everywhere to a function $f\colon \Omega\to X$. If the functions f_n, $n\in\mathbb{N}$, are equi-Birkhoff integrable then f is Birkhoff integrable and $\lim\limits_{n\to\infty}\int_\Omega f_n\,\mathrm{d}\mu=\int_\Omega f\,\mathrm{d}\mu$.*

Proof. Without loss of generality we may assume that $f_n\to f$ everywhere on Ω. Let $\varepsilon>0$. Since the functions f_n, $n\in\mathbb{N}$, are equi-Birkhoff integrable, pick a partition (A_i) of Ω such that for every choice $t_i\in A_i$ we have

$$(\forall n\in\mathbb{N})\ \Big\|\sum_i f_n(t_i)\mu(A_i)-\int_\Omega f_n\,\mathrm{d}\mu\Big\|<\frac{\varepsilon}{5},\tag{10}$$

$$\text{the series }\sum_i f_n(t_i)\mu(A_i),\ n\in\mathbb{N},\ \text{are AEU-convergent.}\tag{11}$$

First, observe that by Fact 3 it follows that, for a fixed choice $t_i\in A_i$, the series $\sum\limits_i f(t_i)\mu(A_i)$ is unconditionally convergent. Indeed, let $\eta>0$ and by (11) pick $k,n_0\in\mathbb{N}$ such that $\Big\|\sum\limits_{i\in S} f_n(t_i)\mu(A_i)\Big\|<\eta$ for every finite set $S\subset\mathbb{N}\setminus\{1,\dots,k\}$ and every $n\geqslant n_0$. Letting $n\to\infty$ we have $\Big\|\sum\limits_{i\in S} f(t_i)\mu(A_i)\Big\|\leqslant\eta$ for every finite set $S\subset\mathbb{N}\setminus\{1,\dots,k\}$.

Secondly, we will show that $\lim\limits_{n\to\infty}\int_\Omega f_n\,\mathrm{d}\mu$ exists. Let $\varepsilon>0$ and fix a choice $t_i\in A_i$. Arguing as before, we find $k,n_0\in\mathbb{N}$ such that $\Big\|\sum\limits_{i\in S} f(t_i)\mu(A_i)\Big\|\leqslant\varepsilon/5$ and $\Big\|\sum\limits_{i\in S} f_n(t_i)\mu(A_i)\Big\|\leqslant\varepsilon/5$ for every finite set $S\subset\mathbb{N}\setminus\{1,\dots,k\}$ and each $n\geqslant n_0$. It follows that

$$\Big\|\sum_{i>k} f(t_i)\mu(A_i)\Big\|\leqslant\frac{\varepsilon}{5}\ \text{and}\ \Big\|\sum_{i>k} f_n(t_i)\mu(A_i)\Big\|\leqslant\frac{\varepsilon}{5}\ \text{for all } n\geqslant n_0.\tag{12}$$

Since $f_n(t_i)\to f(t_i)$ for each $i\in\{1,\dots,k\}$, we can find $n_1\in\mathbb{N}$ such that

$$\|f_m(t_i)-f_n(t_i)\|\leqslant\frac{\varepsilon}{5k(\mu(A_i)+1)}\tag{13}$$

for all $m, n \geqslant n_1$ and $i \in \{1, \dots, k\}$. Put $N = \max\{n_0, n_1\}$. Using (10), (12), (13), for each $n \geqslant N$ we have

$$\Big\| \int_\Omega f_m \, d\mu - \int_\Omega f_n \, d\mu \Big\| \leqslant \Big\| \int_\Omega f_m \, d\mu - \sum_i f_m(t_i)\mu(A_i) \Big\| + \sum_{i \leqslant k} \|f_m(t_i) - f_n(t_i)\| \mu(A_i)$$
$$+ \Big\| \sum_{i>k} f_m(t_i)\mu(A_i) \Big\| + \Big\| \sum_{i>k} f_n(t_i)\mu(A_i) \Big\| + \Big\| \sum_i f_n(t_i)\mu(A_i) - \int_\Omega f_n \, d\mu \Big\| < 5 \cdot \frac{\varepsilon}{5} = \varepsilon.$$

This is a Cauchy condition, so $\lim\limits_{n\to\infty} \int_\Omega f_n \, d\mu = x$ exists.

Finally, we will show that f is Birkhoff integrable and $\int_\Omega f \, d\mu = x$. Let $\varepsilon > 0$. Consider the partition (A_i) and a choice $t_i \in A_i$ as before. Letting $n \to \infty$ in (13) we obtain

$$\|f(t_i) - f_n(t_i)\| \leqslant \frac{\varepsilon}{5k(\mu(A_i) + 1)} \tag{14}$$

for all $n \geqslant N$ and $i \in \{1, \dots, k\}$. Now, by (14), (12), (10), for every $n \geqslant N$ we have

$$\Big\| \sum_i f(t_i)\mu(A_i) - \int_\Omega f_n \, d\mu \Big\| \leqslant \sum_{i \leqslant k} \|f(t_i) - f_n(t_i)\| \mu(A_i) + \Big\| \sum_{i>k} f(t_i)\mu(A_i) \Big\|$$
$$+ \Big\| \sum_{i>k} f_n(t_i)\mu(A_i) \Big\| + \Big\| \sum_i f_n(t_i)\mu(A_i) - \int_\Omega f_n \, d\mu \Big\| \leqslant 4 \cdot \frac{\varepsilon}{5} < \varepsilon.$$

Letting $n \to \infty$ we get $\| \sum\limits_i f(t_i)\mu(A_i) - x\| \leqslant \varepsilon$. This together with the first part of the proof shows that $x = \int_\Omega f \, d\mu$. □

One can consider a notion analogous to the Birkhoff integral but, in the definition, the respective series $\sum\limits_n f(t_n)\mu(A_n)$ should be absolutely convergent. Then the corresponding versions of Proposition 1 and Remark 2 remain true. This notion will be called the *absolute Birkhoff integral*; it is still more general than the Bochner integral but essentially more restrictive than the Birkhoff integral (see [7] where this kind of integral was introduced for functions on $[0, 1]$, and called the Riemann-Lebesgue integral). Note that real-valued, absolutely Birkhoff integrable functions on Ω coincide with Lebesgue integrable ones [7, Thms 1.3 and 1.4].

A sequence (f_n) of functions $f_n \colon \Omega \to X$, $n \in \mathbb{N}$, is called convergent to $f \colon \Omega \to X$ *almost uniformly* if for every $\varepsilon > 0$ there exists an $E \in \mathfrak{S}$ such that $\mu(E) < \varepsilon$ and $(f_n|_{\Omega \setminus E})_{n \in \mathbb{N}}$ converges uniformly to $f|_{\Omega \setminus E}$; cf. [5, Def. 3.5.1].

Theorem 7. *Let $\mu(\Omega) < \infty$. Assume that functions $f_n\colon \Omega \to X$, $n \in \mathbb{N}$, are Birkhoff integrable and $\|f_n(t)\| \leqslant g(t)$ for all $n \in \mathbb{N}$ and almost all $t \in \Omega$ where $g\colon X \to \mathbb{R}$ is Lebesgue integrable. Then the functions f_n, $n \in \mathbb{N}$, are absolutely Birkhoff integrable. Moreover, if $f\colon \Omega \to X$ and (f_n) is convergent to f almost uniformly then f is absolutely Birkhoff integrable and $\lim\limits_{n\to\infty} \int_\Omega f_n \,\mathrm{d}\mu = \int_\Omega f \,\mathrm{d}\mu$.*

Proof. Let $E \in \mathfrak{S}$ be such that $\mu(E) = \mu(\Omega)$ and $\|f_n(t)\| \leqslant g(t)$ for all $n \in \mathbb{N}$ and $t \in E$. By assumption, g is absolutely Birkhoff integrable. So let $\varepsilon > 0$ and pick a partition Π_0 of Ω such that for any partition $\Gamma = (A_i)$ finer than Π_0 and for every choice $t_i \in A_i$ the series $\sum\limits_i g(t_i)\mu(A_i)$ is (absolutely) convergent. Fix $n \in \mathbb{N}$ and pick a partition Π_n of Ω finer than Π_0 such that for any partition $\Gamma = (A_i)$ finer than Π_n we have

$$\Bigl\|\sum_i f_n(t_i)\mu(A_i) - \sum_i f_n(s_i)\mu(A_i)\Bigr\| < \varepsilon$$

for arbitrary choices $t_i, s_i \in A_i$, the series being unconditionally convergent (cf. Remark 2). If $\Gamma = (A_i)$ is finer than Π_n, then the sets $A_i \cap E$ together with $\Omega \setminus E$ constitute a partition of Ω finer than Π_n. Hence without loss of generality we may assume that $E = \Omega$. Then

$$\sum_i \|f_n(t_i)\|\mu(A_i) \leqslant \sum_i g(t_i)\mu(A_i) < \infty$$

for any $\Gamma = (A_i)$ finer than Π_n and every choice $t_i \in A_i$. This implies that f_n is absolutely Birkhoff integrable. Since (f_n) is convergent to f almost uniformly, it also converges to f almost everywhere. Thus $\|f_n(t)\| \leqslant g(t)$ for almost all $t \in \Omega$. If we repeat the reasoning used above for f_n, we obtain

$$\sum_i \|f(t_i)\|\mu(A_i) < \infty$$

for any $\Gamma = (A_i)$ finer than Π_0 and every choice $t_i \in A_i$.

Now, we will show that f is absolutely Birkhoff integrable. Let $\varepsilon > 0$ and consider $\Pi_0 = (E_i)$ chosen as before. Since g is Π_0-summable, the restrictions $g|E_i$ are bounded whenever $\mu(E_i) > 0$. Let $J = \{i\colon \mu(E_i) > 0\}$. Since (f_n) is almost uniformly convergent to f, for every $i \in J$ pick a set $K_i \in \mathfrak{S}$ with $K_i \subset E_i$, $\mu(K_i) \leqslant \varepsilon/\Bigl(10\cdot 2^i \sup\limits_{t\in E_i}\|g(t)\| + 1\Bigr)$ and such that $f_n \to f$ uniformly on $E_i \setminus K_i$. Then for every choice $t_i \in K_i$ we have

$$\begin{aligned}(15)\qquad \sum_i \|f(t_i)\|\mu(K_i) &= \sum_{i\in J} \|f(t_i)\|\mu(K_i) \leqslant \sum_{i\in J} g(t_i)\mu(K_i)\\ &\leqslant \sum_{i\in J} g(t_i)\frac{\varepsilon}{10\cdot 2^i \sup\limits_{t\in E_i}\|g(t)\| + 1} \leqslant \sum_{i\in J}\frac{\varepsilon}{10\cdot 2^i} < \frac{\varepsilon}{10}.\end{aligned}$$

By Theorem 4, f is Birkhoff integrable on every set $E_i \setminus K_i$. Hence for every i pick a partition $(D_{ij})_j$ of $E_i \setminus K_i$ such that

$$\Big\| \sum_j f(t_{ij})\mu(D_{ij}) - \sum_j f(s_{ij})\mu(D_{ij}) \Big\| < \frac{\varepsilon}{5\cdot 2^i} \tag{16}$$

for any choices $t_{ij}, s_{ij} \in D_{ij}$. Consider a partition finer than Π_0 and $(K_i, D_{ij})_{ij}$ simultaneously. Then for any choices $t_i, s_i \in K_i$; $t_{ij}, s_{ij} \in D_{ij}$, by (15) and (16) we have

$$\begin{aligned}
&\Big\| \Big(\sum_i f(t_i)\mu(K_i) + \sum_{i,j} f(t_{ij})\mu(D_{ij}) \Big) - \Big(\sum_i f(s_i)\mu(K_i) + \sum_{i,j} f(s_{ij})\mu(D_{ij}) \Big) \Big\| \\
&\leqslant \sum_i \|f(t_i)\|\mu(K_i) + \sum_i \|f(s_i)\|\mu(K_i) + \Big\| \sum_{i,j} f(t_{ij})\mu(D_{ij}) - \sum_{i,j} f(s_{ij})\mu(D_{ij}) \Big\| \\
&\leqslant \frac{\varepsilon}{10} + \frac{\varepsilon}{10} + \sum_i \Big\| \sum_j f(t_{ij})\mu(D_{ij}) - \sum_j f(s_{ij})\mu(D_{ij}) \Big\| \leqslant \frac{\varepsilon}{5} + \sum_i \frac{\varepsilon}{5\cdot 2^i} \leqslant \frac{2}{5}\varepsilon.
\end{aligned}$$

This, by the corresponding version of Remark 2, implies that f is absolutely Birkhoff integrable.

Now, we shall prove that

$$\Big\| \int_F f_n \,\mathrm{d}\mu \Big\| \leqslant \int_F g \,\mathrm{d}\mu \quad \Big\| \int_F f \,\mathrm{d}\mu \Big\| \leqslant \int_F g \,\mathrm{d}\mu \tag{17}$$

for all $n \in \mathbb{N}$ and $F \in \mathfrak{S}$. Let $\varepsilon > 0$ and fix $n \in \mathbb{N}$, $F \in \mathfrak{S}$. Choose a partition (F_i) of F which guarantees that condition (ii) in the corresponding version of Proposition 1 holds true when one considers the absolute Birkhoff integrability of f_n and g. Then for every choice $z_i \in F_i$ we have

$$\begin{aligned}
\Big\| \int_F f_n \,\mathrm{d}\mu \Big\| &\leqslant \Big\| \sum_i f_n(z_i)\mu(F_i) \Big\| + \varepsilon \leqslant \sum_i \|f_n(z_i)\|\mu(F_i) + \varepsilon \\
&\leqslant \sum_i g(z_i)\mu(F_i) + \varepsilon \leqslant \int_E g \,\mathrm{d}\mu + 2\varepsilon.
\end{aligned}$$

Hence, by the arbitrariness of ε, we obtain the first inequality in (17). The proof of the second part of (17) is analogous.

To show that $\lim\limits_{n\to\infty} \int_\Omega f_n \,\mathrm{d}\mu = \int_\Omega f \,\mathrm{d}\mu$, consider $\varepsilon > 0$ and choose $\Pi_0 = (E_i)$ as in the proof of the absolute value Birkhoff integrability of f. Modifying that part of the proof, define the set J as before. Since g is absolutely continous, fix a function $\delta\colon (0,\varepsilon) \to (0,\infty)$ such that $\| \int_A g \,\mathrm{d}\mu \| < \eta$ whenever $A \in \mathfrak{S}$, $\mu(A) < \delta(\eta)$, $\eta \in (0,\varepsilon)$.

Then for every $i \in J$ pick a set $K_i \in \mathfrak{S}$ with $K_i \subset E_i$, $\mu(K_i) \leqslant \delta(\varepsilon/(5 \cdot 2^i))$ and such that $f_n \to f$ uniformly on $E_i \setminus K_i$. Put $K = \bigcup_i K_i$ and pick $N_0 \in \mathbb{N}$ such that if $K_0 = \bigcup_{i > N_0} (E_i \setminus K_i)$ then $\mu(K_0) < \delta(\varepsilon/5)$. Observe that $f_n \to f$ uniformly on $\bigcup_{i \leqslant N_0} (E_i \setminus K_i) = \Omega \setminus (K \cup K_0)$. By Theorem 6 pick $N \in \mathbb{N}$ such that for each $n > N$ we have

$$\left\| \int_{\Omega \setminus (K \cup K_0)} f_n \, \mathrm{d}\mu - \int_{\Omega \setminus (K \cup K_0)} f \, \mathrm{d}\mu \right\| < \frac{\varepsilon}{5}. \tag{18}$$

Hence, by (17) and (18), for each $n > N$ we obtain

$$\begin{aligned}
\left\| \int_\Omega f_n \, \mathrm{d}\mu - \int_\Omega f \, \mathrm{d}\mu \right\| &\leqslant \left\| \int_K f_n \, \mathrm{d}\mu - \int_K f \, \mathrm{d}\mu \right\| \\
&\quad + \left\| \int_{K_0} f_n \, \mathrm{d}\mu - \int_{K_0} f \, \mathrm{d}\mu \right\| + \left\| \int_{\Omega \setminus (K \cup K_0)} f_n \, \mathrm{d}\mu - \int_{\Omega \setminus (K \cup K_0)} f \, \mathrm{d}\mu \right\| \\
&\leqslant \sum_i \left\| \int_{K_i} f_n \, \mathrm{d}\mu \right\| + \sum_i \left\| \int_{K_i} f \, \mathrm{d}\mu \right\| + \left\| \int_{K_0} f_n \, \mathrm{d}\mu \right\| + \left\| \int_{K_0} f \, \mathrm{d}\mu \right\| + \frac{\varepsilon}{5} \\
&\leqslant 2 \sum_i \int_{K_i} g \, \mathrm{d}\mu + 2 \int_{K_0} g \, \mathrm{d}\mu + \frac{\varepsilon}{5} < 2 \sum_i \frac{\varepsilon}{5 \cdot 2^i} + \frac{2\varepsilon}{5} + \frac{\varepsilon}{5} \leqslant \varepsilon.
\end{aligned}$$

□

In a particular case we obtain the known Lebesgue type theorem for the Bochner integral (cf. [5, Thm 3.7.9]).

Corollary 8. *Let $\mu(\Omega) < \infty$. Assume that functions $f_n \colon \Omega \to X$, $n \in \mathbb{N}$, are strongly measurable, Birkhoff integrable, and $\|f_n(t)\| \leqslant g(t)$ for all $n \in \mathbb{N}$ and almost all $t \in \Omega$ where $g \colon \Omega \to \mathbb{R}$ is Lebesgue integrable. Then the functions f_n, $n \in \mathbb{N}$, are absolutely Birkhoff integrable, and if $f_n \to f$ almost everywhere, then f is absolutely Birkhoff integrable and $\lim\limits_{n \to \infty} \int_\Omega f_n \, \mathrm{d}\mu = \int_\Omega f \, \mathrm{d}\mu$.*

P r o o f. Note that the functions $t \mapsto \|f_n(t) - f(t)\|$, $n \in \mathbb{N}$, are measurable. By the Egorov theorem, (f_n) converges to f almost uniformly. So, Theorem 7 works. □

Now, we will give two examples which show that, in some cases, only one of the two results, Theorem 4 and Theorem 7, works.

Example 9. Let $\dim X = \infty$. By the Dvoretsky-Rogers theorem [8, Thm 1.c.2], pick an unconditionally convergent series $\sum\limits_{i=1}^{\infty} x_i$, with terms in X, such that $\sum\limits_{i=1}^{\infty} \|x_i\| = \infty$. Let $\Omega = \mathbb{N}$, $\mathfrak{S} = \mathscr{P}(\mathbb{N})$ (the power set of $\mathbb{N}$) and $\mu(\{i\}) = 2^{-i}$ for $i \in \mathbb{N}$. Define

$f\colon \mathbb{N} \to X$ by $f(i) = 2^i x_i$, $i \in \mathbb{N}$, and let $f_n = f$, $n \in \mathbb{N}$. Clearly $f_n \to f$ uniformly on $\mathbb{N}$, and f is Birkhoff integrable with $\int_{\mathbb{N}} f \,\mathrm{d}\mu = \sum_{i=1}^{\infty} x_i$. So, Theorem 4 works but Theorem 7 is not applicable since from $\sum_{i=1}^{\infty} \|x_i\| = \infty$ it follows that f is not absolutely Birkhoff integrable.

Example 10. Let $\Omega = (0,1]$, let $\mathfrak{S}$ denote the σ-algebra of Lebesgue measurable sets and let μ stand for the Lebesgue measure. Put $X = l_2(\Omega)$, the space of all functions φ from Ω to $\mathbb{R}$ that take non-zero values on countable subsets of Ω, with the norm $\|\varphi\| = (\sum_{x\in\Omega} \varphi^2(x))^{1/2}$. Define $e_t = \chi_{\{t\}}$, the characteristic function of $\{t\}$, $t \in \Omega$. For $n \in \mathbb{N}$ let $f_n\colon \Omega \to X$ be given by

$$f_n(t) = \sum_{i=1}^{n} e_t \cdot \chi_{(1/(i+1),1/i]}, \ t \in \Omega.$$

Then f_n converges almost uniformly to $f\colon \Omega \to X$ given by $f(t) = e_t$, $t \in \Omega$. Of course, $\|f_n(t)\| \leqslant 1$ for all $n \in \mathbb{N}$ and $t \in \Omega$. So, Theorem 7 works. We cannot use Theorem 4 because from $\sup_{t\in\Omega} \|f_n(t) - f(t)\| = 1$, $n \in \mathbb{N}$, it follows that (f_n) does not converge to f uniformly.

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