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CRITERIA FOR TESTING WALL'S QUESTION

Jiří Klaška, Brno

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Abstract. In this paper we find certain equivalent formulations of Wall's question and derive two interesting criteria that can be used to resolve this question for particular primes.

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1. Introduction

In 1960, D. D. Wall published a well-known paper [6] concerning the modular periodicity of a Fibonacci sequence. In this paper an interesting problem was formulated, often referred to as Wall's question (see [6, p. 528]), which has remained unsolved up to the present. Let us outline this problem.

Let $(F_n)_{n=0}^{\infty}$ denote the Fibonacci sequence defined by $F_{n+2} = F_{n+1} + F_n$ with $F_0 = 0$, $F_1 = 1$. Let $m > 0$ be an arbitrary integer. Reducing F_n modulo m and taking the least nonnegative residues, we obtain the sequence $(F_n \bmod m)_{n=0}^{\infty}$, which is periodic. A positive integer $k(m)$ is called the period of the Fibonacci sequence modulo m if it is the smallest positive integer for which $F_{k(m)} \equiv 0 \pmod{m}$ and $F_{k(m)+1} \equiv 1 \pmod{m}$. For a fixed prime p, Wall proved that, if $k(p) = k(p^s) \neq k(p^{s+1})$, then $k(p^t) = p^{t-s}k(p)$ for $t \geqslant s > 0$. Wall asked whether $k(p) = k(p^2)$ is possible. This is still an open question.

In [6] Wall noted that for $p < 10^4$, a counterexample of $k(p) \neq k(p^2)$ does not exist. According to [7], $k(p) \neq k(p^2)$ for $p < 10^9$. Using extensive search by computer, in [2] this result was extended to $p < 10^{14}$. Finally, according to the last report from 2007 (see [4]) there exists no such prime $p < 2 \times 10^{14}$. Finding the answer to Wall's question can be extremely difficult. In 1992, Zhi-Hong Sun and Zhi-Wei Sun [5] showed that, if $p \nmid xyz$ and $x^p + y^p = z^p$, then $k(p) = k(p^2)$. Consequently, an affirmative answer to Wall's question implies the first case of Fermat's last theorem.

It is well known that $k(p) = k(p^2)$ if and only if $F_{p-(5|p)} \equiv 0 \pmod{p^2}$ where $(a|b)$ denotes the Legendre symbol of a and b. Crandall, Dilcher, and Pomerance [1] called primes $p > 5$ satisfying $F_{p-(5|p)} \equiv 0 \pmod{p^2}$ the Wall-Sun-Sun primes. These are sometimes also called Fibonacci-Wieferich primes. See [4] for example. It has been conjectured that there are infinitely many Wall-Sun-Sun primes, but the conjecture remains unproven.

2. Wall's question and its equivalent formulations

It is well known that F_n can be computed by taking the powers of a matrix. Namely, if

$$F = \begin{bmatrix} F_0 & F_1 \\ F_1 & F_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \quad \text{then } F^n = \begin{bmatrix} F_{n-1} & F_n \\ F_n & F_{n+1} \end{bmatrix}. \tag{2.1}$$

Consequently, $k(p)$ is the period of $(F_n \bmod p)_{n=0}^{\infty}$ if and only if $k(p)$ is the smallest positive integer k for which $F^k \equiv E \pmod{p}$ and $k(p^2)$ is the period of $(F_n \bmod p^2)_{n=0}^{\infty}$ if and only if $k(p^2)$ is the smallest positive integer l satisfying $F^l \equiv E \pmod{p^2}$, where E is the 2×2 identity matrix. For any prime p, let us now define the integer matrix $A_p = [a_{ij}]$ such that

$$A_p = \frac{1}{p}(F^{k(p)} - E). \tag{2.2}$$

From (2.1) it follows that

$$A_p = \begin{bmatrix} a_{11} & a_{21} \\ a_{21} & a_{11} + a_{21} \end{bmatrix}. \tag{2.3}$$

Lemma 2.1. *For any prime p we have $k(p) \neq k(p^2)$ if and only if $A_p \not\equiv 0 \pmod{p}$.*

Proof. This follows from (2.2). □

Lemma 2.2. *Let $p \neq 5$. Then $A_p \equiv 0 \pmod{p}$ if and only if $\det A_p \equiv 0 \pmod{p}$.*

Proof. Let $p \neq 2$. Put $k = k(p)$. From (2.2) and (2.3) it follows that

$$\det F^k = 1 + p(2a_{11} + a_{21}) + p^2 \det A_p \quad \text{where } \det A_p = a_{11}^2 + a_{11}a_{21} - a_{21}^2. \tag{2.4}$$

Since $\det F = -1$, (2.4) implies $2a_{11} + a_{21} \equiv 0 \pmod{p}$ and $\det A_p \equiv -5a_{11}^2 \pmod{p}$. Consequently, we have $a_{11} \equiv 0 \pmod{p}$ if and only if $a_{21} \equiv 0 \pmod{p}$, and thus, $\det A_p \equiv 0 \pmod{p}$ implies $A_p \equiv 0 \pmod{p}$. The validity of the converse implication is evident. On the other hand, for $p = 2$ we can easily verify that $A_2 \not\equiv 0 \pmod{2}$ and $\det A_2 \not\equiv 0 \pmod{2}$. □

Remark 2.3. For $p = 5$ we have $A_5 \not\equiv 0 \pmod 5$ and $\det A_5 \equiv 0 \pmod 5$.

Our next considerations will take place in the following framework. Let L_p be the splitting field of the Fibonacci characteristic polynomial $f(x) = x^2 - x - 1$ over the field of p-adic numbers $\mathbb{Q}_p$ and let α, β be the roots of $f(x)$ in L_p. Denote by O_p the ring of integers of L_p. Clearly $\alpha, \beta \in O_p$. Since the discriminant of $f(x)$ is equal to 5, it follows that, for $p \neq 5$, $L_p/\mathbb{Q}_p$ does not ramify and so the maximal ideal of O_p is generated by p. Moreover, if $L_p = \mathbb{Q}_p$, then $\alpha, \beta \in \mathbb{Z}_p$, where $\mathbb{Z}_p$ is the ring of p-adic integers.

For a unit $\varepsilon \in O_p$ we denote by $\operatorname{ord}_{p^t}(\varepsilon)$ the least positive rational integer h such that $\varepsilon^h \equiv 1 \pmod{p^t}$. Since $\varepsilon^h \equiv 1 \pmod p$ implies $\varepsilon^{ph} \equiv 1 \pmod{p^2}$, we have

$$\text{either} \quad \operatorname{ord}_{p^2}(\varepsilon) = \operatorname{ord}_p(\varepsilon) \quad \text{or} \quad \operatorname{ord}_{p^2}(\varepsilon) = p \cdot \operatorname{ord}_p(\varepsilon). \tag{2.5}$$

Furthermore, it is not difficult to prove that if $p > 2$ and $\operatorname{ord}_p(\varepsilon) \neq \operatorname{ord}_{p^2}(\varepsilon)$, then for any $t \in \mathbb{N}$ we have $\operatorname{ord}_{p^t}(\varepsilon) = p^{t-1} \operatorname{ord}_p(\varepsilon)$. More generally, if $\varepsilon \neq \pm 1$ and $s \in \mathbb{N}$ is the largest integer such that $\operatorname{ord}_{p^s}(\varepsilon) = \operatorname{ord}_p(\varepsilon)$, then for any $t \geqslant s$, we have $\operatorname{ord}_{p^t}(\varepsilon) = p^{t-s} \operatorname{ord}_p(\varepsilon)$.

Lemma 2.4. *Let $p \neq 5$. We have either $\operatorname{ord}_{p^t}(\alpha) = \operatorname{ord}_{p^t}(\beta)$ or $\operatorname{ord}_{p^t}(\alpha) = 2\operatorname{ord}_{p^t}(\beta)$ or $2\operatorname{ord}_{p^t}(\alpha) = \operatorname{ord}_{p^t}(\beta)$.*

Proof. From Viète's equation $\alpha\beta = -1$ in L_p it follows that $\alpha = \pm 1$ if and only if $\beta = \pm 1$. Hence, if $\alpha^r = 1$, then $\beta^r = \pm 1$, and consequently, $\beta^{2r} = 1$. This implies $\operatorname{ord}_{p^t}(\beta) \mid 2\operatorname{ord}_{p^t}(\alpha)$. By analogy, we can obtain $\operatorname{ord}_{p^t}(\alpha) \mid 2\operatorname{ord}_{p^t}(\beta)$. □

Corollary 2.5. *For any prime $p \neq 5$ we have*

$$\operatorname{ord}_{p^2}(\alpha) \equiv 0 \pmod p \quad \textit{if and only if} \quad \operatorname{ord}_{p^2}(\beta) \equiv 0 \pmod p. \tag{2.6}$$

Proof. This is a consequence of Lemma 2.4 if $p \neq 2$. For $p = 2$, the polynomial $f(x)$ is irreducible over $\mathbb{Q}_2$ and so $\operatorname{ord}_{2^t}(\alpha) = \operatorname{ord}_{2^t}(\beta)$. □

In Theorem 2.6 we generalize [3, Lemma 2.4] also to the case of $f(x)$ being irreducible over $\mathbb{Q}_p$.

Theorem 2.6. *Let $p \neq 5$. Then $k(p^t) = \operatorname{lcm}(\operatorname{ord}_{p^t}(\alpha),\ \operatorname{ord}_{p^t}(\beta))$ for any $t \in \mathbb{N}$.*

Proof. Over L_p we can write $F_n = A\alpha^n + B\beta^n$ for suitable $A, B \in L_p$. The coefficients A, B are uniquely determined by the equations $A + B = 0$ and $A\alpha + B\beta = 1$ over L_p. The determinant of the matrix of this system is equal to $\beta - \alpha$. As $\alpha \not\equiv \beta \pmod{p}$, the Cramer rule gives $A = -(\beta-\alpha)^{-1}$, $B = (\beta-\alpha)^{-1}$. Moreover, A, B are units in O_p. Let $k = k(p^t)$. Then $[A\alpha^k + B\beta^k, A\alpha^{k+1} + B\beta^{k+1}] \equiv [A + B, A\alpha + B\beta] \pmod{p^t}$. This system can be reduced to an equivalent form

$$\begin{bmatrix} 1 & 1 \\ \alpha & \beta \end{bmatrix} \begin{bmatrix} A(\alpha^k - 1) \\ B(\beta^k - 1) \end{bmatrix} \equiv \begin{bmatrix} 0 \\ 0 \end{bmatrix} \pmod{p^t}. \tag{2.7}$$

As the determinant of the matrix in (2.7) is not divisible by p, (2.7) has only one solution

$$A(\alpha^k - 1) \equiv 0 \pmod{p^t}, \quad B(\beta^k - 1) \equiv 0 \pmod{p^t}.$$

This implies $\alpha^k \equiv 1 \pmod{p^t}$ and $\beta^k \equiv 1 \pmod{p^t}$. Thus, we have $\operatorname{ord}_{p^t}(\alpha) \mid k$ and $\operatorname{ord}_{p^t}(\beta) \mid k$, which implies $\operatorname{lcm}(\operatorname{ord}_{p^t}(\alpha),\ \operatorname{ord}_{p^t}(\beta)) \mid k$. As A, B are not divisible by p, the periods of the sequences $(A\alpha^n \bmod p^t)_{n=0}^{\infty}$ and $(B\beta^n \bmod p^t)_{n=0}^{\infty}$ are $\operatorname{ord}_{p^t}(\alpha)$ and $\operatorname{ord}_{p^t}(\beta)$. Consequently, the period k of $(A\alpha^n + B\beta^n \bmod p^t)_{n=0}^{\infty}$ divides $\operatorname{lcm}(\operatorname{ord}_{p^t}(\alpha), \operatorname{ord}_{p^t}(\beta))$ and the theorem follows. □

Theorem 2.7. *Let $p \neq 5$. Then $k(p) \neq k(p^2)$ if and only if*

$$\operatorname{ord}_{p^2}(\alpha) \equiv 0 \pmod{p} \ \textit{ and } \ \operatorname{ord}_{p^2}(\beta) \equiv 0 \pmod{p}. \tag{2.8}$$

Proof. It follows from (2.8) that $\operatorname{lcm}(\operatorname{ord}_{p^2}(\alpha), \operatorname{ord}_{p^2}(\beta)) \equiv 0 \pmod{p}$ and, by Theorem 2.6, we have $k(p^2) \equiv 0 \pmod{p}$. Using Theorem 2.6 for $t = 1$ and recalling that (p) is the maximal ideal of O_p, we have $k(p) \not\equiv 0 \pmod{p}$, which together with $k(p^2) \equiv 0 \pmod{p}$, gives $k(p) \neq k(p^2)$.

Conversely, if $k(p) \neq k(p^2)$, then $k(p^2) = p \cdot k(p)$. From Theorem 2.6 it now follows that $\operatorname{lcm}(\operatorname{ord}_{p^2}(\alpha), \operatorname{ord}_{p^2}(\beta)) \equiv 0 \pmod{p}$. This implies that $\operatorname{ord}_{p^2}(\alpha) \equiv 0 \pmod{p}$ or $\operatorname{ord}_{p^2}(\beta) \equiv 0 \pmod{p}$, which together with (2.6) proves (2.8). □

Remark 2.8. If $p = 5$, then $k(p) \neq k(p^2)$ and $k(5^t) = 4 \cdot 5^t$ for any $t \in \mathbb{N}$. See [6].

Our results can be summarized in the following theorem.

Theorem 2.9. *Let $p \neq 5$ and let s be the number of roots α, β of $f(x)$ in O_p whose order modulo p^2 is divisible by p. Then there are the following possibilities:*

Case $s = 0$: $k(p) = k(p^2)$, or equivalently $A_p \equiv 0 \pmod p$.

Case $s = 1$: This case is impossible.

Case $s = 2$: $k(p) \neq k(p^2)$, or equivalently $\det A_p \not\equiv 0 \pmod p$.

Proof. By Theorem 2.6 we have that $s = 0$ if and only if $k(p) = k(p^2)$. Lemma 2.1 states that $k(p) = k(p^2)$ if and only if $A_p \equiv 0 \pmod p$, which is equivalent to $\det A_p \equiv 0 \pmod p$ by Lemma 2.2. By Corollary 2.5 we see that the case of $k = 1$ is impossible. The proof is complete. □

Our results reduce Wall's question to solving the following equivalent problem. Is there at least one root $\alpha \in O_p$ of $f(x)$ for which $\mathrm{ord}_{p^2}(\alpha) \not\equiv 0 \pmod p$ or is this never possible?

Now we derive two interesting criteria that can be used, without computing the roots of $f(x)$ in O_p, to decide whether $k(p) = k(p^2)$ or not. Let $p \neq 5$. Put $q = |O_p/(p)|$. Then $q = p^t$ where $t = [L_p : \mathbb{Q}_p] \in \{1, 2\}$. If $f(x)$ is irreducible over $\mathbb{Q}_p$, then $O_p/(p)$ is a field with p^2 elements. If $f(x)$ is not irreducible over $\mathbb{Q}_p$, then $f(x)$ has both roots in the ring $\mathbb{Z}_p$ and $O_p/(p)$ is a field with p elements. For the proof of our criteria, we shall need the following lemma.

Lemma 2.10. *We have $\mathrm{ord}_{p^2}(\alpha) \not\equiv 0 \pmod p$ if and only if $\alpha^{q-1} \equiv 1 \pmod{p^2}$.*

Proof. Put $s = \mathrm{ord}_{p^2}(\alpha)$. Clearly, $[O_p/(p^2)]^\times$ has $q(q-1)$ elements and so $s \mid q(q-1)$. Let $p \nmid s$. As $q = p^t$, we have $s \mid q - 1$, and $\alpha^{q-1} \equiv 1 \pmod{p^2}$ follows. On the other hand, let $\alpha^{q-1} \equiv 1 \pmod{p^2}$. Then $s \mid q - 1$. As $p \nmid q - 1$, we have $\mathrm{ord}_{p^2}(\alpha) \not\equiv 0 \pmod p$. □

Theorem 2.11. *Let $p \neq 5$, $u \in O_p$ be such that $f(u) \equiv 0 \pmod p$. Then $k(p) = k(p^2)$ if and only if*

$$u^{2q} - u^q - 1 \equiv 0 \pmod{p^2}, \tag{2.9}$$

or equivalently

$$f(u) + (u^q - u)f'(u) \equiv 0 \pmod{p^2}, \tag{2.10}$$

where f' is the derivative of the Fibonacci characteristic polynomial f.

Proof. Let $u \in O_p$, $u^2 - u - 1 \equiv 0 \pmod p$. Then we have $u \equiv \alpha \pmod p$ or $u \equiv \beta \pmod p$. We can assume $u \equiv \alpha \pmod p$. Then $u^q \equiv \alpha^q \pmod{p^2}$. If $k(p) = k(p^2)$, then $u^q \equiv \alpha^q \equiv \alpha \pmod{p^2}$ and $u^{2q} - u^q - 1 \equiv \alpha^2 - \alpha - 1 = 0 \pmod{p^2}$.

On the other hand, assume $u^{2q} - u^q - 1 \equiv 0 \pmod{p^2}$. Let $u^q = \alpha + pv$. Then $(\alpha + pv)^2 - (\alpha + pv) - 1 \equiv pv(2\alpha - 1) \equiv 0 \pmod{p^2}$. Now $p \neq 5$ implies $2\alpha - 1 \not\equiv 0 \pmod{p}$ and so $v \equiv 0 \pmod{p}$. Consequently, $u^q \equiv \alpha \pmod{p^2}$ and $\alpha^{q-1} \equiv u^{q(q-1)} \equiv 1 \pmod{p^2}$. This, together with Lemma 2.10, yields $\operatorname{ord}_{p^2}(\alpha) \not\equiv 0 \pmod{p}$ and $k(p) = k(p^2)$ follows by Theorem 2.7 and Corollary 2.5.

Furthermore, let $u = \alpha + pw$. Then (2.10) is equivalent to

$$(\alpha^q - \alpha)(2\alpha + 2pw - 1) \equiv 0 \pmod{p^2}. \tag{2.11}$$

If $k(p) = k(p^2)$, then $\alpha^q \equiv \alpha \pmod{p^2}$ and (2.11) follows.

Conversely, assume (2.11). As $p \neq 5$, we have $2\alpha + 2pw - 1 \equiv 2u - 1 \equiv f'(\alpha) \not\equiv 0 \pmod{p}$. Consequently, (2.11) gives $\alpha^q - \alpha \equiv 0 \pmod{p^2}$. This, together with Lemma 2.10, implies $k(p) = k(p^2)$ as required. □

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Author's address: Jiří Klaška, Department of Mathematics, Brno University of Technology, Technická 2, 616 69 Brno, Czech Republic, e-mail: klaska@fme.vutbr.cz.