Czechoslovak Mathematical Journal, 59 (134) (2009), 1–22

ON SUPER VERTEX-GRACEFUL UNICYCLIC GRAPHS

Sin-Min Lee, Elo Leung, and Ho Kuen Ng, California

(Received June 16, 2005)

Abstract. A graph G with p vertices and q edges, vertex set $V(G)$ and edge set $E(G)$, is said to be super vertex-graceful (in short SVG), if there exists a function pair (f, f^{+}) where f is a bijection from $V(G)$ onto P, f^{+} is a bijection from $E(G)$ onto Q, $f^{+}((u,v)) = f(u) + f(v)$ for any $(u,v) \in E(G)$,

$$Q = \begin{cases} \{\pm 1, \ldots, \pm \frac{1}{2}q\}, & \text{if } q \text{ is even,} \\ \{0, \pm 1, \ldots, \pm \frac{1}{2}(q-1)\}, & \text{if } q \text{ is odd,} \end{cases}$$

and

$$P = \begin{cases} \{\pm 1, \ldots, \pm \frac{1}{2}p\}, & \text{if } p \text{ is even,} \\ \{0, \pm 1, \ldots, \pm \frac{1}{2}(p-1)\}, & \text{if } p \text{ is odd.} \end{cases}$$

We determine here families of unicyclic graphs that are super vertex-graceful.

Keywords: graceful, edge-graceful, super edge-graceful, super vertex-graceful, amalgamation, trees, unicyclic graphs

MSC 2010: 05C78

1. Introduction

All graphs in this paper are finite simple graphs with no loops or multiple edges.

A graph G with p vertices and q edges is graceful if there is an injective mapping $f\colon V(G) \to \{0, 1, \ldots, q\}$ such that $f^{*}\colon E(G) \to \{1, 2, \ldots, q\}$ defined by $f^{*}((u,v)) = |f(u) - f(v)|$ is surjective. Graceful graph labelings were first introduced by Alex Rosa around 1967 as a means of attacking the problem of cyclically decomposing a complete graph into other graphs. A well-known conjecture of Ringel and Kotzig is that all trees are graceful. Since Rosa's original article, more than six hundred papers have been written on graph labelings (see [2]).

A dual concept of graceful labeling on graphs, known as edge-graceful labeling, was introduced by S. P. Lo [17] in 1985. G is said to be edge-graceful if the edges are labeled by $1, 2, \ldots, q$ so that the vertex sums are distinct $\bmod p$.

A necessary condition for edge-gracefulness is (Lo [17])

$$q(q+1) \equiv \frac{p(p-1)}{2} \pmod{p}.$$

Finding edge-graceful labelings of graphs is related to solving a system of linear Diophantine equations. In general it is difficult to find an edge-graceful labeling of a graph. Several classes of graphs have been shown to be edge-graceful [10], [11], [12], [13], [14], [15], [16], [17], [18], [19], [20], [21], [22], [23]. For a survey of result on these labelings, see Gallian [2].

Lee [7] conjectured that all odd-order trees are edge-graceful. In [19] the concept of super edge-graceful graph was introduced to work on this conjecture.

The first author introduced super vertex-gracefulness, a dual concept of super edge-gracefulness in [8]. Consider a graph G with vertex set $V(G)$ and edge set $E(G)$, $p = |V(G)|$ and $q = |E(G)|$. G is said to be super vertex-graceful (in short SVG), if there exists a function pair (f, f^+) which assigns integer labels to the vertices and edges such that both $f\colon V(G) \to P$ and $f^+\colon E(G) \to Q$ are onto, $f^+((u,v)) = f(u) + f(v)$ where $(u,v) \in E(G)$,

$$Q = \begin{cases} \{\pm 1, \ldots, \pm \frac{1}{2}q\}, & \text{if } q \text{ is even,} \\ \{0, \pm 1, \ldots, \pm \frac{1}{2}(q-1)\}, & \text{if } q \text{ is odd,} \end{cases}$$

and

$$P = \begin{cases} \left\{\pm 1, \ldots, \pm \frac{1}{2}p\right\}, & \text{if } p \text{ is even,} \\ \{0, \pm 1, \ldots, \pm \frac{1}{2}(p-1)\}, & \text{if } p \text{ is odd.} \end{cases}$$

Figure 1 shows that the star St(4) is SVG. From [8], St(5) is not.

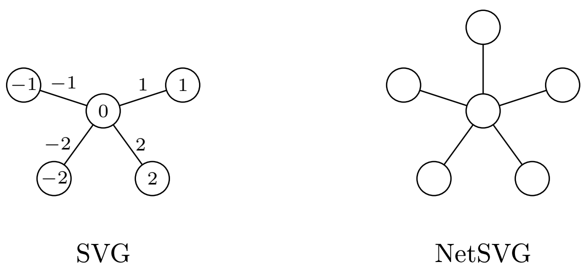


Figure 1

In [8], Lee showed that every graph is an induced subgraph of a super vertex-graceful graph. From this, one could conclude that there does not exist a Kuratowski type characterization for super vertex-graceful graphs. In [9], the first and second authors considered trees that are SVG. In this paper we will show that any ring-worm is an induced subgraph of an SVG ring-worm.

The present paper was motivated by the desire to determine the structure of super vertex-graceful unicyclic graphs. At present, no characterization of SVG unicyclic graphs is known. An open problem on super vertex-graceful unicyclic graphs is proposed at the end of the last section.

2. Super vertex-graceful unicyclic graphs of order at most 6

In [8], Lee showed that

Theorem 2.0. *If G is an SVG (p,q)-graph with vertex labeling f and degree sequence $\{d(v_i)\colon\ i = 1, 2, \ldots, p\}$, then $\sum(d(v_i) - 1)f(v_i) = 0$.*

We can apply this result and exhaustion to show

Theorem 2.1. *The unicyclic graph of order 3 is SVG. Both unicyclic graphs of order 4 are non-SVG. Among the 5 unicyclic graphs of order 5, only three are SVG.*

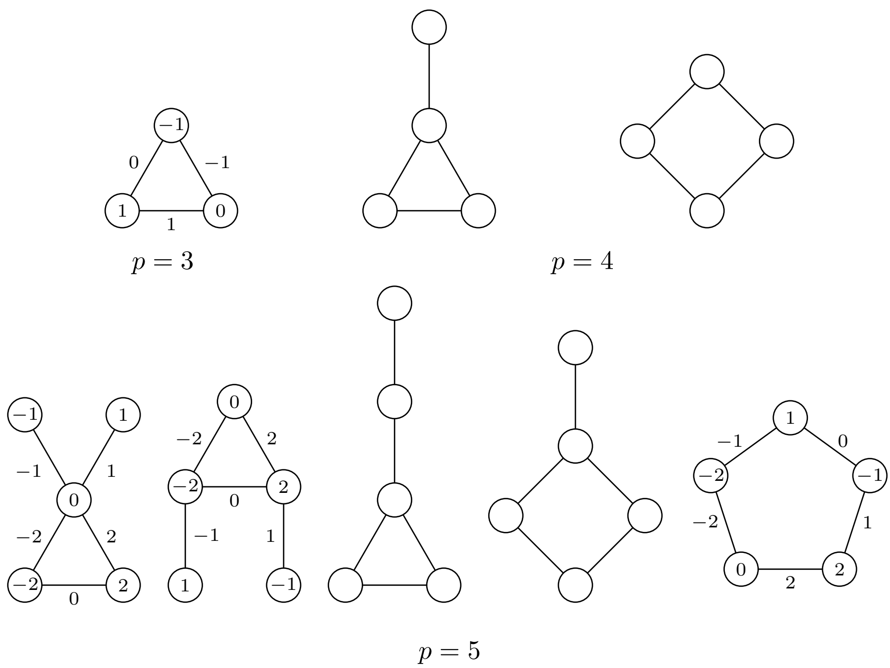


Figure 2

Theorem 2.2. *Among the* 13 *unicyclic graphs of order* 6*, only two are SVG.*

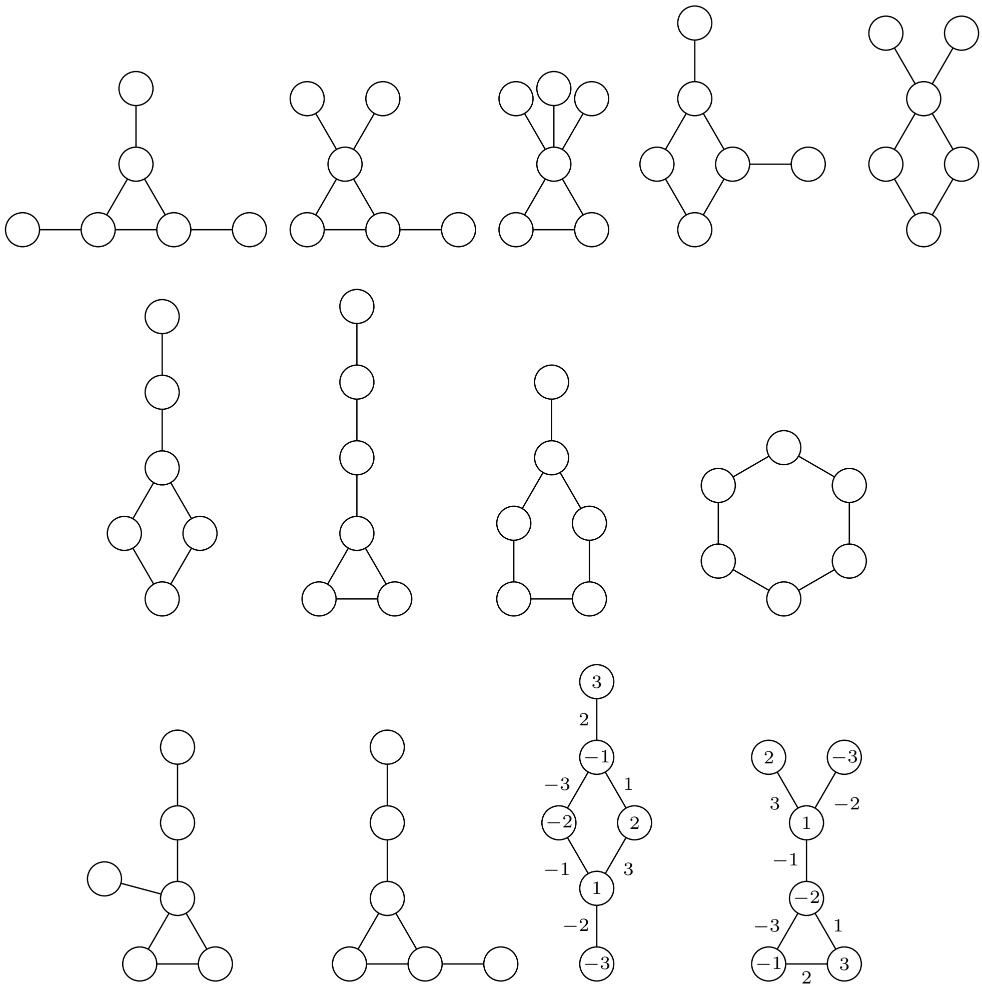


Figure 3

3. Construction theorems of SVG unicyclic graphs

We can construct certain SVG unicyclic graphs from SVG trees.

Theorem 3.1. *Let G be a super vertex-graceful tree of odd order. If f is an SVG vertex labeling of G, and two vertices u and v have labels $f(u) = -f(v)$, then the new uncyclic graph obtained by connecting u and v in G is SVG.*

Example 1. The following tree is SVG with 7 vertices. Applying the above result, we can obtain three SVG unicyclic graphs (Figure 4).

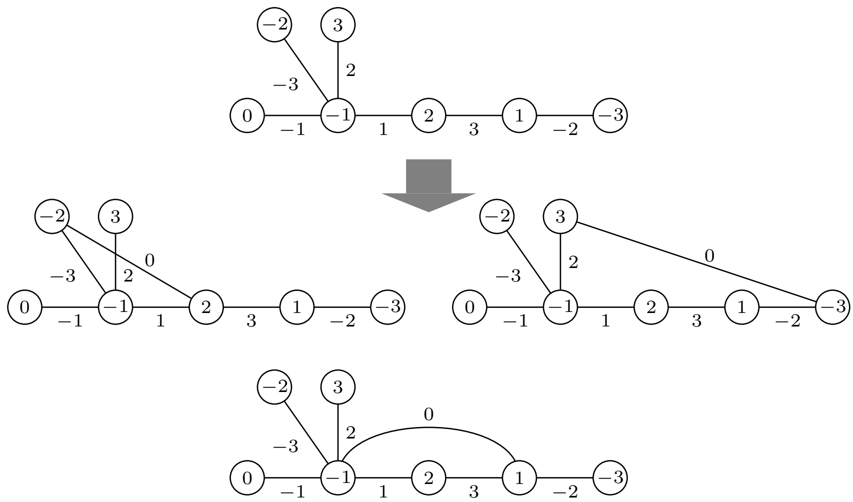


Figure 4

For a graph G, we denote by $D(G)$ the resulting graph upon deleting all its vertices of degree one. A caterpillar is a tree T such that $D(T)$ is a path. Similar to caterpillars in trees, we introduce the concept of ring-worms among unicyclic graphs.

A ring-worm is a unicyclic graph $U_n(a_1, a_2, \dots, a_n)$ such that $D(U_n(a_1, a_2, \dots, a_n)) = C_n$ and $a_i \geqslant 0$ for $i = 1, \dots, n$. A ring-worm is shown in Figure 5. We can view $U_n(a_1, a_2, \dots, a_n)$ as $S_1 \cup S_2 \cup \dots \cup S_n$, where S_i is the star with $V(Si) = \{c_i, x_{i,1}, x_{i,2}, \dots, x_{i,a_i}\}$, center c_i, and $a_i + 2$ edges, and in each case S_i shares an edge with S_{i+1}. The cycle with vertices $\{c_1, c_2, \dots, c_n\}$ will be called the spine of $U_n(a_1, a_2, \dots, a_n)$. This ring-worm has $n + a_1 + a_2 + \dots + a_n$ vertices and $n + a_1 + a_2 + \dots + a_n$ edges. Notice that c_1 and c_n are connected.

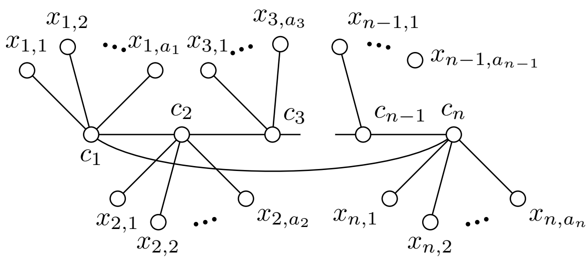
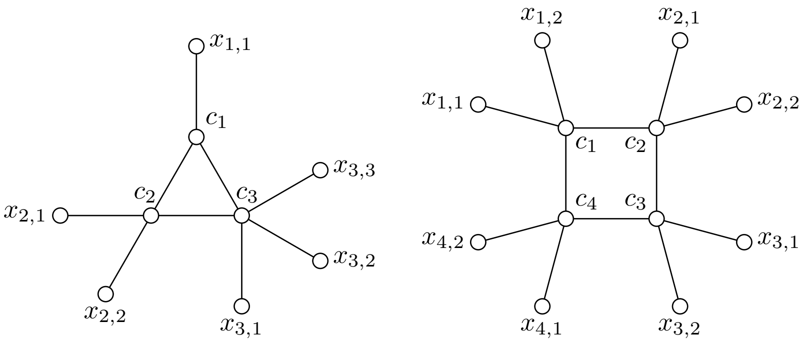


Figure 5

Figure 6 shows the two ring-worms $U_3(1,2,3)$ and $U_4(2,2,2,2)$.



$U_3(1,2,3)$ $U_4(2,2,2)$

Figure 6

Extending the definition, we will use $[d]^b$ as the notation that b paths of length d are appended to a vertex of the spine.

Corollary 3.2. *The ring-worm $U_{2k+1}(1,1,0^{2k-1})$ is SVG for all $k \geqslant 1$.*

Proof. Let $V(P_{2k+3}) = \{c_1, c_2, \ldots, c_{2k+3}\}$. It is shown in [8] that the path P_n is super vertex-graceful for all odd $n \geqslant 3$. (See Figure 7 for some examples.) By Theorem 3.1, we obtain the result by joining the vertices c_2 and c_{2k+2}. $\square$

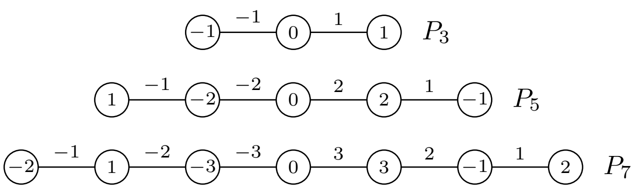


Figure 7

Corollary 3.3. *The unicyclic graph $U_{2k+1}([d-1],[d-1],0^{2k-1})$ formed by appending a path of length $d-1$ to each of two adjacent vertices of the cycle C_{2k+1} is SVG for all $d \geqslant 2$ and $k \geqslant 1$.*

Proof. Let $V(P_{2k+2d-1}) = \{c_1, c_2, \ldots, c_{2k+2d-1}\}$ and let the path $P_{2k+2d-1}$ be labeled as in Figure 7. By Theorem 3.1, we see that $U_{2k+1}([d],[d],0^{2k-1})$ obtained by joining the vertex c_d with c_{2k+d} is SVG. $\square$

However, a unicyclic graph obtained by appending an edge to a cycle is not SVG.

Theorem 3.4. *The ring-worm $U_n(1, 0^{n-1})$ is not SVG for all $n \geqslant 3$.*

Proof. Let $V(U_n(1, 0^{n-1})) = \{c_1, x_{1,1}, c_2, \ldots, c_n\}$. If $U_n(1, 0^{n-1})$ is SVG with a SVG labeling f, then by Theorem 2.0, we have $2f(c_1)+f(c_2)+f(c_3)+\ldots+f(c_n) = 0$. However, we have $f(V(U_n(1, 0^{n-1}))) = P$ and so $f(c_1)+f(x_{1,1})+f(c_2)+f(c_3)+\ldots+f(c_n) = 0$. Subtracting the equations, we obtain $f(c_1) - f(x_{1,1}) = 0$, *i.e.*, $f(c_1) = f(x_{1,1})$, which contradicts that f is a bijection. Hence the ring-worm $U_n(1, 0^{n-1})$ is not SVG. □

In [8], it was shown that one could obtain many super vertex-graceful graphs of odd order by the following construction.

Theorem 3.5. *Let G be a super vertex-graceful graph of odd order. If two edges are appended to the vertex of G with label 0, then the new graph is super vertex-graceful.*

Let (G_1, u) and (G_2, v) be two graphs with fixed vertices u, v respectively. The amalgamation of (G_1, u) and (G_2, v) is the graph which is the disjoint union of G_1 and G_2 with u and v identified. We will denote the resulting graph by $\mathrm{Amal}((G_1, u), (G_2, v))$. It is obvious that u is a cut-vertex of the amalgamation. We will leave out u or v if the context is clear.

We can extend the above result as follows:

Corollary 3.6. *Let G be a super vertex-graceful graph of odd order and with vertex u labeled 0. The amalgamation $\mathrm{Amal}((G, u), \mathrm{St}(2k, c))$ of (G, u) and $(\mathrm{St}(2k), c)$, where c is a center of the star, is super vertex-graceful for all $k \geqslant 1$.*

Theorem 3.7. *The unicyclic graph $\mathrm{Amal}(C_{2n+1}, \mathrm{St}(2k, c))$ is SVG for all $n \geqslant 1$ and $k \geqslant 1$.*

4. Super vertex-graceful unicyclic graphs of diameter 2 and 3

Recall that a tree is called a spider if it has a center vertex c of degree $k > 1$ and each other vertex either is a leaf or has degree 2. Thus a spider is an amalgamation of k paths with various lengths. If it has x_1 paths of length a_1, x_2 paths of length $a_2, \ldots$, we denote the spider by $\mathrm{SP}(a_1^{x_1}, a_2^{x_2}, \ldots, a_m^{x_m})$ where $a_1 < a_2 < \ldots < a_m$ and $x_1 + x_2 + \ldots + x_m = k$. (See Figure 8.)

In [8], it was shown that

Example 2. A star $\mathrm{St}(n)$ is super vertex-graceful if and only if n is even.

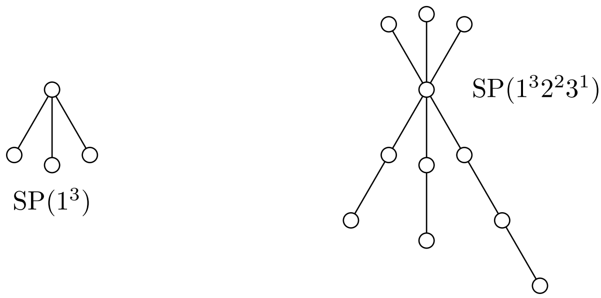


Figure 8

We observe that all SVG trees of diameter 2 can be constructed through amalgamation of $\mathrm{St}(2k)$ and $\mathrm{St}(2)$. (See Figure 9 for examples.)

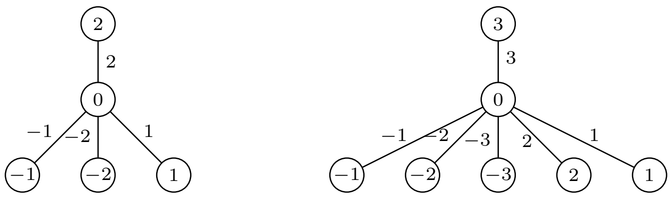


Figure 9

Theorem 4.1. *The SVG uncyclic graphs of diameter 2 have the form* $U_3(1^{2k}, 0^2)$ *where* $k \geqslant 1$.

Proof. We show that the unicyclic graph $\mathrm{Amal}(C_3, \mathrm{St}(m, c))$ is SVG if and only if m is even.

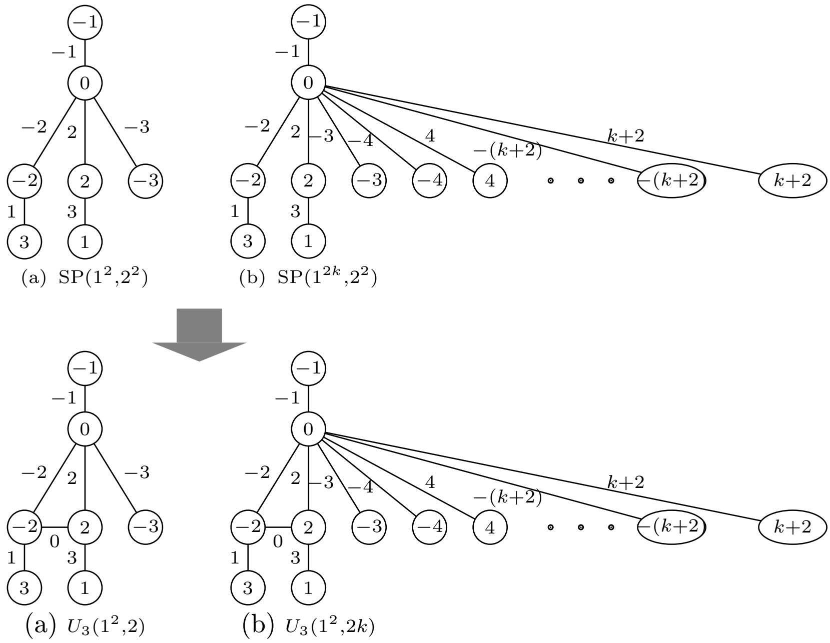
If m is even, the result follows from Corollary 3.6. If m is odd, then the graph has an even number of vertices and edges, and thus 0 can be neither a vertex label nor an edge label. The center of $\mathrm{St}(m, c)$ is adjacent to every other vertex. No matter what its label is, it must be adjacent to the vertex with its negative label, giving an edge label of 0. □

Theorem 4.2. *The unicyclic graph* $U_3(1^2, 2k)$ *of diameter* 3 *is SVG for all* $k \geqslant 1$.

Proof. In [22], we showed that the spider $\mathrm{SP}(1^{2k}, 2^2)$ is SVG. Applying Theorem 3.1, we conclude that $U_3(1^2, 2k)$ is SVG for all $k \geqslant 1$. (See Figure 10.) □

Theorem 4.3. *The unicyclic graph* $U_3(1^2, (2k)[2]^2)$ *of diameter* 3 *is SVG for all* $k \geqslant 1$.

Proof. (See Figure 11.)



(a) SP(1^2,2^2) (b) SP(1^{2k},2^2)

(a) $U_3(1^2,2)$ (b) $U_3(1^2,2k)$

Figure 10

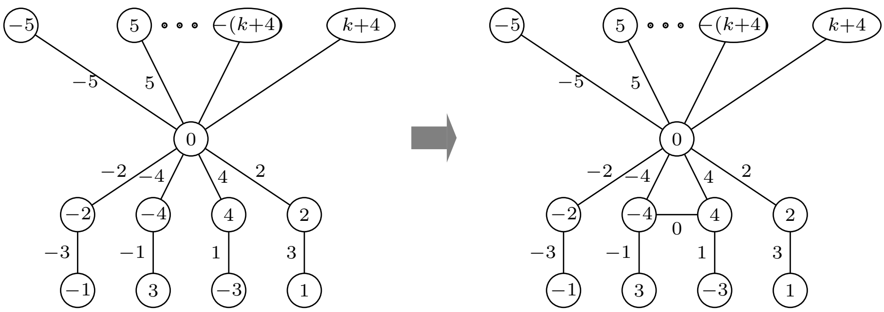


Figure 11

Theorem 4.4. *The unicyclic graph $U_4(0^3, m)$ of diameter 3 is not SVG for any even $m \geqslant 2$.*

P r o o f. Let $V(U_4(0^3, m)) = \{c_1, x_{1,1}, \ldots, x_{1,m}, c_2, c_3, c_4\}$. If $U_4(0^3, m)$ is SVG with a SVG labeling f, then by Theorem 2.0, we have $(m+1)f(c_1) + f(c_2) + f(c_3) + f(c_4) = 0$. However, we have $f(V(U_4(0^3, m))) = P$ and so $f(c_1) + f(x_{1,1}) + \ldots + f(x_{1,m}) + f(c_2) + f(c_3) + f(c_4) = 0$. Subtracting the two equations, we obtain $mf(c_1) - f(x_{1,1}) - \ldots - f(x_{1,m}) = 0$, *i.e.*, $mf(c_1) = f(x_{1,1}) + \ldots + f(x_{1,m})$. If m is even, then 0 is not in P. The center c_1 of $\mathrm{St}(m, c_1)$ is adjacent to every other vertex except c_3. Thus $f(c_3) = -f(c_1)$. No matter what its label is, it must be adjacent to a vertex with a label making the induced edge label having magnitude exceeding $\frac{1}{2}m + 2$. Hence the ring worm $U_4(0^3, m)$ is not SVG. $\square$

Thus, there are infinitely many trees of diameter 3 that are not SVG.

5. Super vertex-graceful unicyclic graphs of diameter 4

There exist infinitely many unicyclic graphs of diameter 4 that are SVG.

Theorem 5.1. *The unicyclic graph $U_3(0^2, (2n)[2]^2)$ is SVG for all $n \geqslant 0$.*

Figure 12 shows an example of the construction.

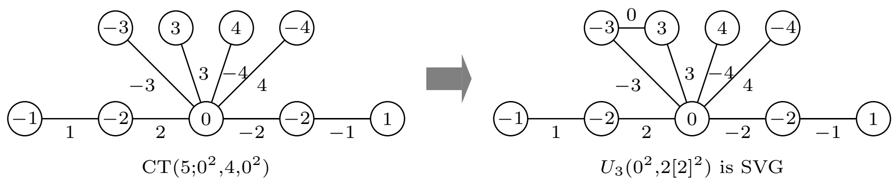


Figure 12

Theorem 5.2. *The unicyclic graph $U_4(0^2, 1, (2n)[2])$ is SVG for all $n \geqslant 1$.*

P r o o f. (See Figure 13.) $\square$

Theorem 5.3. *The unicyclic graph $U_4(0, 1^2, 2n+1)$ is SVG for all $n \geqslant 0$.*

P r o o f. (See Figure 14.) $\square$

Theorem 5.4. *The unicyclic graph $U_4(0^2, 1, (2n)[2])$ is SVG for all $n \geqslant 1$.*

P r o o f. (See Figure 15.) $\square$

Theorem 5.5. *The unicyclic graph $U_4(0^3, (2n+1)[2]^2)$ is SVG for all $n \geqslant 0$.*

P r o o f. (See Figure 16.) $\square$

Theorem 5.6. *The unicyclic graph $U_5(0^4, (2n)[2]^2)$ is SVG for all $n \geqslant 1$.*

P r o o f. (See Figure 17.) $\square$

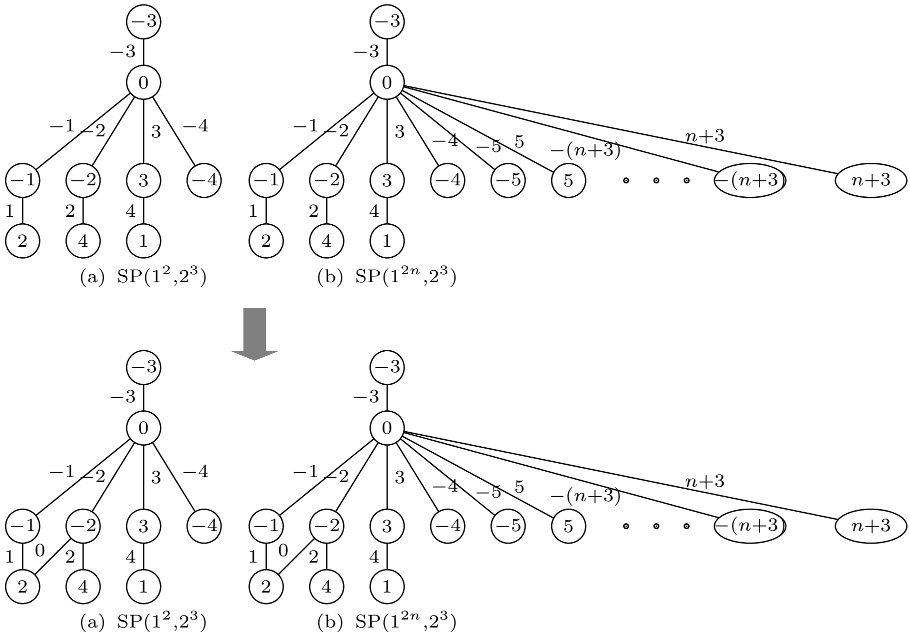


Figure 13

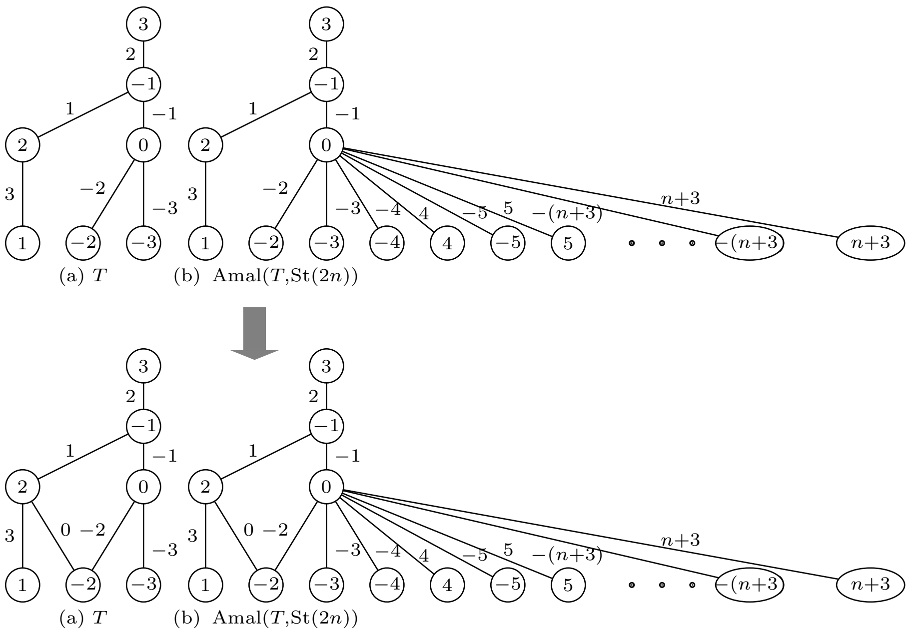


Figure 14

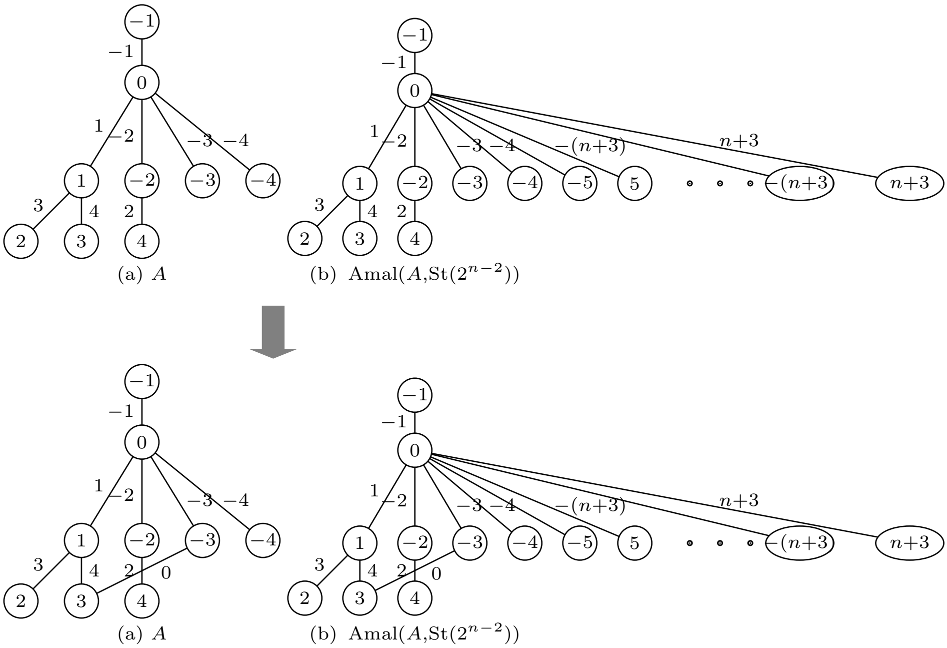


Figure 15

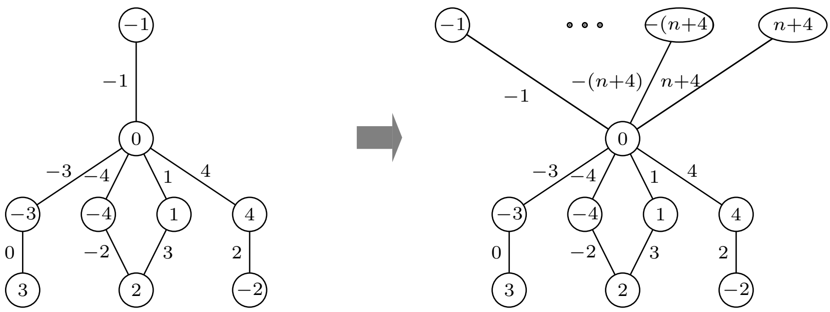


Figure 16

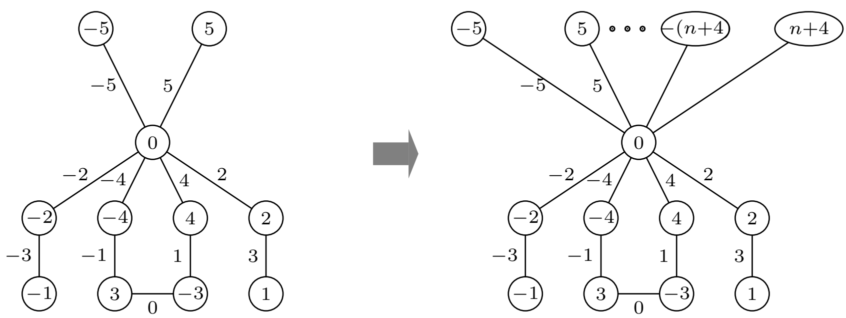


Figure 17

6. SVG unicyclic graphs with large diameters

In this section we show that we can have SVG unicyclic ring-worms with arbitrarily large diameters.

Theorem 6.1. *$U_{2m+1}((2n)^{2m+1})$ is SVG for any $m \geqslant 1$ and $n \geqslant 0$.*

Proof. Consider $n = 0$. $U_{2m+1}((2n)^{2m+1})$ is simply the cycle C_{2m+1}. We use the SVG labeling in [19]. Let the vertices of the cycle be $v_0, v_1, v_2, \ldots, v_{2m}$, in this order. Define vertex labels by $f(v_{2i}) = i$, for $i = 0, \ldots, m$, and $f(v_{2i+1}) = -m + i$, for $i = 0, \ldots, m-1$.

Consider $n = 1$. We add a pair of edges to each of the vertices of C_{2m+1}. To be SVG, we need additional vertex labels and additional edge labels $\pm(m+1)$, $\pm(m+2), \ldots, \pm(3m+1)$.

We pair the numbers $m+1, m+2, \ldots, 3m+1$ as follows. Pair $m+1$ with $2m+1$, $2m+2$ with $3m+1$, $m+2$ with $2m, 2m+3$ with $3m$, $m+3$ with $2m-1$, $2m+4$ with $3m-1, \ldots$ Note that the differences between the pairs are $m, m-1, m-2$, $m-3, m-4, m-5, \ldots$ respectively. Since we have $(2m+1)$ numbers in this sequence, we will eventually have m pairs of numbers, with differences from m to 1 inclusive, and one number will be left by itself.

Now we begin labeling the new vertices. The two new vertices adjacent to the one labeled-m on the cycle are labeled $(m+1)$ and $-(2m+1)$, giving edge labels $2m+1$ and $-(m+1)$. The two new vertices adjacent to the one labeled $-m$ on the cycle are labeled $-(m+1)$ and $(2m+1)$, giving edge labels $-(2m+1)$ and $m+1$. The two new vertices adjacent to the one labeled $m-1$ on the cycle are labeled $(2m+2)$ and $-(3m+1)$, giving edge labels $3m+1$ and $-(2m+2)$. The two new vertices adjacent to the one labeled $-(m-1)$ on the cycle are labeled $-(2m+2)$ and $(3m+1)$, giving edge labels $-(3m+1)$ and $2m+2$. In general, for the two new vertices adjacent to the one labeled k on the cycle, and for the two new vertices adjacent to the one labeled-k on the cycle, we use $\pm$ (the pair of numbers found in the previous paragraph that have difference k), with the signs arranged so that the four new vertex labels are the same as the four new edge labels. Finally the two new vertices adjacent to the one labeled 0 on the cycle are labeled $\pm$ (the remaining number in the previous paragraph), with the same numbers as the new edge labels.

In general, by continuing this process another $(n-1)$ times, we obtain an SVG labeling for $U_{2m+1}((2n)^{2m+1})$. $\square$

Corollary 6.2. *$U_{2m+1}((2n)^{2m}, 2n+2k)$ is SVG for any $m \geqslant 1, n \geqslant 0$, and $k \geqslant 0$.*

Proof. Use the above Theorem to construct an SVG labeling for $U_{2m+1} \times ((2n)^{2m+1})$. Use Theorem 3.5 to add an even number of edges to the vertex on the spine that is labeled 0. $\square$

Theorem 6.3. *$U_6(1,0,1,1,0,1)$ is SVG.*

Proof. The following diagram gives an SVG labeling. $\square$

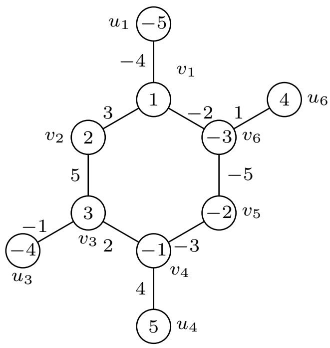


Figure 18

Theorem 6.4. *$U_{4k+2}(1^k, 0^2, 1^{k-2}, 2, 1^k, 0^2, 1^{k-2}, 2)$ is SVG for any $k \geqslant 2$.*

Proof. Let the vertices on the spine be $v_1, v_2, \ldots, v_{4k+2}$, in this order. There is one pendant edge at each of $v_1, v_2, \ldots, v_k, v_{k+3}, v_{k+4}, \ldots, v_{2k}, v_{2k+2}, v_{2k+3}, \ldots, v_{3k+1}, v_{3k+4}, v_{3k+5}, \ldots, v_{4k+1}$. There are two pendant edges at each of v_{2k+1} and v_{4k+2}. Let the pendant edges be (u_1, v_1), (u_2, v_2), …, (u_k, v_k), (u_{k+3}, v_{k+3}), (u_{k+4}, v_{k+4}), …, (u_{2k}, v_{2k}), (u_{2k+2}, v_{2k+2}), (u_{2k+3}, v_{2k+3}), …, (u_{3k+1}, v_{3k+1}), (u_{3k+4}, v_{3k+4}), (u_{3k+5}, v_{3k+5}), …, (u_{4k+1}, v_{4k+1}), (a_{2k+1}, v_{2k+1}), (b_{2k+1}, v_{2k+1}), (a_{4k+2}, v_{4k+2}), and (b_{4k+2}, v_{4k+2}).

Label the vertices $v_1, v_2, \ldots, v_{2k+1}$ by $1, 2, \ldots 2k+1$. Label the vertices $v_{2k+2}, v_{2k+3}, \ldots, v_{4k+2}$ by $-1, -2, \ldots, -(2k+1)$. Label the vertices $u_1, u_2, \ldots, u_k$ by $-(4k+1), -4k, \ldots, -(3k+2)$. Label the vertices $u_{k+3}, u_{k+4}, \ldots, u_{2k}$ by $-(3k+1), -(3k+2), \ldots, -(2k+4)$. Label the vertices $u_{2k+2}, u_{2k+3}, \ldots, u_{3k+1}$ by $(4k+1), 4k, \ldots, (3k+2)$. Label the vertices $u_{3k+4}, u_{3k+5}, \ldots, u_{4k+1}$ by $(3k+1), (3k+4), \ldots, (2k+4)$. Label a_{2k+1} and b_{2k+1} by $-(2k+2)$ and $-(2k+3)$. Label a_{4k+2} and b_{4k+2} by $2k+2$ and $2k+3$. Thus the vertex labels are $\pm 1, \pm 2, \ldots, \pm(4k+1)$.

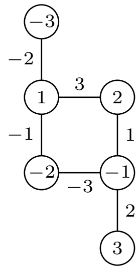
A direct calculation shows that the edges on the spine, $(v_1, v_2), (v_2, v_3), \ldots, (v_{4k+2}, v_1)$ have labels $3, 5, 7, \ldots, 4k+1, 2k, -3, -5, -7, \ldots, -(4k+1), -2k$. The following table gives the labels of the pendant edges.

Pendant edges	Labels
(u_1, v_1)	$-4k$
(u_2, v_2)	$-(4k-2)$
$\ldots$	
(u_k, v_k)	$-(2k+2)$
(u_{k+3}, v_{k+3})	$-(2k-2)$
(u_{k+4}, v_{k+4})	$-(2k-4)$
$\ldots$	
(u_{2k}, v_{2k})	-4
(u_{2k+2}, v_{2k+2})	$4k$
(u_{2k+3}, v_{2k+3})	$4k-2$
$\ldots$	
(u_{3k+1}, v_{3k+1})	$2k+2$
(u_{3k+4}, v_{3k+4})	$2k-2$
(u_{3k+5}, v_{3k+5})	$2k-4$
$\ldots$	
(u_{4k+1}, v_{4k+1})	4
(a_{2k+1}, v_{2k+1})	-1
(b_{2k+1}, v_{2k+1})	-2
(a_{4k+2}, v_{4k+2})	1
(b_{4k+2}, v_{4k+2})	2

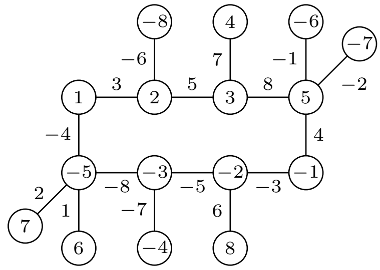
Thus the edge labels are $\pm 1, \pm 2, \ldots, \pm(4k+1)$. □

Theorem 6.5. *$U_4(1,0,1,0)$, $U_8(0,1,1,2,0,1,1,2)$, $U_{12}(0,2,0,0,1,3,0,2,0,0,1,3)$, and $U_{16}(0,1,1,1,0,1,1,3,0,1,1,1,0,1,1,3)$ are SVG.*

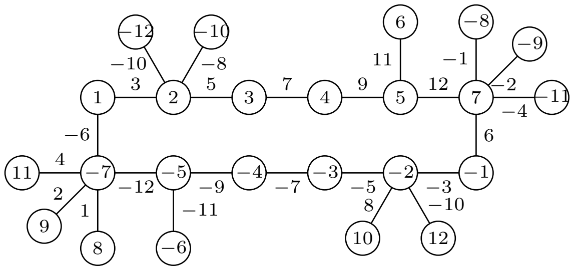
P r o o f. The following diagrams give SVG labelings for the above graphs. □



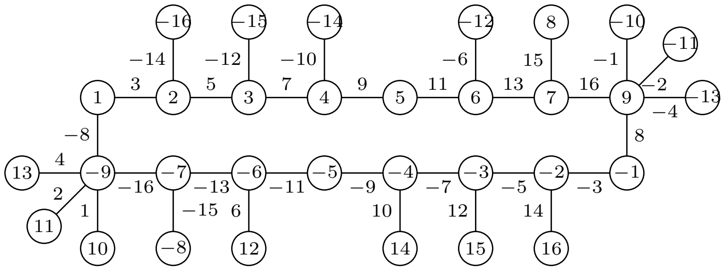
$U_4(1,0,1,0)$



$U_8(0,1,1,2,0,1,1,2)$



U_{12}(0,2,0,0,1,3,2,0,0,1,3)



U_{16}(0,1,1,1,0,1,1,3,0,1,1,1,0,1,1,3)

Theorem 6.6. *$U_{4k}(0, 1^{k-1}, 0^2, 1^{k-5}, 2, 1, 3, 0, 1^{k-1}, 0^2, 1^{k-5}, 2, 1, 3)$ is SVG for any $k \geqslant 5$.*

P r o o f. Let the vertices on the spine be $v_1, v_2, \ldots, v_{4k}$, in this order. There is one pendant edge at each of $v_2, v_3, \ldots, v_k, v_{k+3}, v_{k+4}, \ldots, v_{2k-3}, v_{2k-1}, v_{2k+2}, v_{2k+3}, \ldots, v_{3k}, v_{3k+3}, v_{3k+4}, \ldots, v_{4k-3}, v_{4k-1}$. Let the other ends of these pendant edges be u_2, u_3, …, u_k, u_{k+3}, u_{k+4}, …, u_{2k-3}, u_{2k-1}, u_{2k+2}, u_{2k+3}, …, u_{3k}, u_{3k+3}, u_{3k+4}, …, u_{4k-3}, u_{4k-1} respectively. There are two pendant edges at each of v_{2k-2} and v_{4k-2}. Let the other ends of these pendant edges be a_{2k-2}, b_{2k-2}, and a_{4k-2}, b_{4k-2} respectively. There are three pendant edges at each of v_{2k} and v_{4k}. Let the other ends of these pendant edges be c_{2k}, d_{2k}, e_{2k}, and c_{4k}, d_{4k}, e_{4k} respectively.

The vertices are labeled as follows.

Vertices	Labels
$v_1, v_2, \ldots, v_{2k-1}$	$1, 2, \ldots 2k-1$
v_{2k}	$2k+1$
$v_{2k+1}, v_{2k+2}, \ldots, v_{4k-1}$	$-1, -2, \ldots, -(2k-1)$
v_{4k}	$-(2k+1)$
$u_2, u_3, \ldots, u_k$	$-4k, -(4k-1), \ldots, -(3k+2)$
$u_{k+3}, u_{k+4}, \ldots, u_{2k-3}$	$-(3k+1), -(3k+2), \ldots, -(2k+7)$
u_{2k-1}	$2k$
$u_{2k+2}, u_{2k+3}, \ldots, u_{3k}$	$4k, (4k-1), \ldots, (3k+2)$

$u_{3k+3}, u_{3k+4}, \ldots, u_{4k-3}$	$(3k+1), (3k+2), \ldots, (2k+7)$
u_{4k-1}	$-2k$
a_{2k-2}	$-(2k+6)$
b_{2k-2}	$-(2k+4)$
a_{4k-2}	$2k+6$
b_{4k-2}	$2k+4$
c_{2k}	$-(2k+2)$
d_{2k}	$-(2k+3)$
e_{2k}	$-(2k+5)$
c_{4k}	$(2k+2)$
d_{4k}	$(2k+3)$
e_{4k}	$(2k+5)$

Thus the vertex labels are $\pm 1, \pm 2, \ldots, \pm 4k$.

A direct calculation shows that the edges on the spine, (v_1, v_2), (v_2, v_3), $\ldots$, (v_{4k}, v_1) have labels $3, 5, 7, \ldots, 4k-3, 4k, 2k, -3, -5, -7, \ldots, -(4k-3), -4k, -2k$. The following table gives the labels of the pendant edges.

Pendant edges	Labels
(u_2, v_2)	$-(4k-2)$
(u_3, v_3)	$-(4k-4)$
$\ldots$	
(u_k, v_k)	$-(2k+2)$
(u_{k+3}, v_{k+3})	$-(2k-2)$
(u_{k+4}, v_{k+4})	$-(2k-4)$
$\ldots$	
(u_{2k-3}, v_{2k-3})	-10
(u_{2k-1}, v_{2k-1})	$4k-1$
(u_{2k+2}, v_{2k+2})	$4k-2$
(u_{2k+3}, v_{2k+3})	$4k-4$
$\ldots$	
(u_{3k}, v_{3k})	$2k+2$
(u_{3k+3}, v_{3k+3})	$2k-2$
(u_{3k+4}, v_{3k+4})	$2k-4$
$\ldots$	
(u_{4k-3}, v_{4k-3})	10
(u_{4k-1}, v_{4k-1})	$-(4k-1)$
(a_{2k-2}, v_{2k-2})	-8
(b_{2k-2}, v_{2k-2})	-6
(a_{4k-2}, v_{4k-2})	8
(b_{4k-2}, v_{4k-2})	6

(c_{2k}, v_{2k})	-1
(d_{2k}, v_{2k})	-2
(e_{2k}, v_{2k})	-4
(c_{4k}, v_{4k})	1
(d_{4k}, v_{4k})	2
(e_{4k}, v_{4k})	4

Thus the edge labels are $\pm1, \pm2, \ldots, \pm4k$. □

7. A general construction of SVG graphs and some unsolved problems

In this last section we consider a general construction of SVG graphs and apply this construction to unicyclic graphs.

Theorem 7.1. *Let i, j, and k be any positive integers. Consider a graph G with any vertex labeling (not necessarily SVG) with $-i$ and j as two vertex labels. Then we can add $2(i+j)$ vertices and $2(i+j)$ edges to G, so that both the new vertex labels and the new edge labels are $\pm(k+1), \pm(k+2), \ldots, \pm(k+i+j)$. Moreover, if G is a tree (respectively unicyclic), then the new graph is also a tree (respectively unicyclic).*

Proof. First assume that $j \geqslant i$. To the vertex with label $-i$, append $2j$ edges. The new vertices are labeled $-(k+1), -(k+2), \ldots, -(k+i), \pm(k+i+1), \pm(k+i+2), \ldots, \pm(k+j), k+j+1, k+j+2, \ldots, k+i+j$. The new edge labels are $-(k+i+1), -(k+i+2), \ldots, -(k+2i), -(k+2i+1), -(k+2i+2), \ldots, -(k+i+j), k+1, k+2, \ldots, k-i+j, k-i+j+1, k-i+j+2, \ldots, k+j$. To the vertex with label j, append $2i$ edges. The new vertices are labeled $k+1, k+2, \ldots, k+i, -(k+j+1), -(k+j+2), \ldots, -(k+i+j)$. The new edge labels are $k+j+1, k+j+2, \ldots, k+i+j, -(k+1), -(k+2), \ldots, -(k+i)$.

Now assume that $j \leqslant i$. To the vertex labeled $-i$, append $2j$ edges. The new vertices are labeled $-(k+1), -(k+2), \ldots, -(k+j), k+i+1, k+i+2, \ldots, k+i+j$. The new edge labels are $-(k+i+1), -(k+i+2), \ldots, -(k+i+j), k+1, k+2, \ldots, k+j$. To the vertex with label j, append $2i$ edges. The new vertices are labeled $k+1, k+2, \ldots, k+j, \pm(k+j+1), \pm(k+j+2), \ldots \pm(k+i), -(k+i+1), -(k+i+2), \ldots, -(k+i+j)$. The new edge labels are $k+j+1, k+j+2, \ldots, k+2j, k+2j+1, k+2j+2, \ldots, k+i+j, -(k+1), -(k+2), \ldots, -(k+i-j), -(k+i-j+1), -(k+i-j+2), \ldots, -(k+i)$. □

This construction can be applied to build SVG unicyclic graphs from any SVG unicyclic graph, and to build SVG trees from any SVG tree with an odd number of vertices. In the former case, the vertex and edge labels are the same. In the latter case, the vertex and edge labels are the same, except that there is an additional vertex with label 0.

Corollary 7.2. *$U_3(2k+1, 2k+1, 0)$ is SVG, for any $k \geqslant 0$.*

Proof. From the diagram below, $U_3(1, 1, 0)$ is SVG.

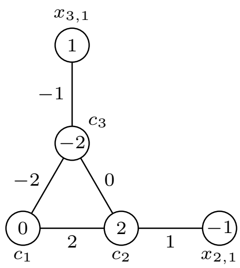


Figure 20

Repeatedly apply the construction to add a pair of edges to the vertex labeled -1 and a pair of edges to the vertex labeled 1. □

Corollary 7.3. *$U_3(2k+1, 2k+1, 2m)$ is SVG, for any $k, m \geqslant 0$.*

Proof. To the graph in Corollary 7.2, repeatedly add pairs of edges to the vertex labeled 0, using $\pm$ (the next greater integer from the current labeling) as the new labels. □

Corollary 7.4. *$U_3(1, 4k+1, 2k)$ is SVG, for any $k \geqslant 0$.*

Proof. From the diagram below, $U_3(1, 1, 0)$ is SVG.

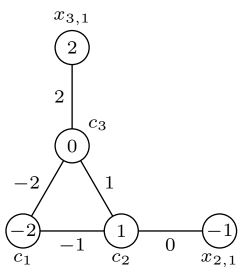


Figure 21

Repeatedly apply the construction to add two edges to the vertex labeled -2 and four edges to the vertex labeled 1. □

Corollary 7.5. *$U_3(2m+1, 4k+1, 2k)$ is SVG, for any $k, m \geqslant 0$.*

Proof. To the graph in Corollary 7.4, repeatedly add pairs of edges to the vertex labeled 0. $\square$

Corollary 7.6. *$U_6(1+x_1, x_2, 1+x_3, 1+y_1, y_2, 1+y_3)$ is SVG, for any non-negative even integers x_2, x_3, y_2, y_3, if $x_1 = 2y_2 + 3y_3$, and $y_1 = 2x_2 + 3x_3$.*

Proof. By Theorem 6.3, $U_6(1, 0, 1, 1, 0, 1)$ is SVG. Apply the construction to the vertex labeled 1 and the vertices labeled -2 and -3 on the spine, and then apply the construction again to the vertex labeled -1 and the vertices labeled 2 and 3 on the spine. $\square$

Corollary 7.7. *If $k \geqslant 2, U_{4k+2}(1+x_1, 1+x_2, 1+x_3, \ldots, 1+x_k, x_{k+1}, x_{k+2}, 1+x_{k+3}, 1+x_{k+4}, \ldots, 1+x_{2k}, 2+x_{2k+1}, 1+y_1, 1+y_2, 1+y_3, \ldots, 1+y_k, y_{k+1}, y_{k+2}, 1+y_{k+3}, 1+y_{k+4}, \ldots, 1+y_{2k}, 2+y_{2k+1})$ is SVG for any non-negative even integers $x_2, x_3, \ldots, x_{2k+1}, y_2, y_3, \ldots, y_{2k+1}$, if $x_1 = 2y_2 + 3y_3 + \ldots + (2k+1)y_{2k+1}$, and $y_1 = 2x_2 + 3x_3 + \ldots + (2k+1)x_{2k+1}$.*

Proof. By Theorem 6.4, $U_{4k+2}(1^k, 0^2, 1^{k-2}, 2, 1^k, 0^2, 1^{k-2}, 2)$ is SVG for any $k \geqslant 2$. Apply the construction to the vertex labeled 1 and each of the negatively labeled vertices (other than -1) on the spine, and then apply the construction again to the vertex labeled -1 and each of the positively labeled vertices (other than 1) on the spine. $\square$

Corollary 7.8.

(1) *$U_4(1+x_1, x_2, 1+y_1, y_2)$ is SVG for any non-negative even integers x_2 and y_2, if $x_1 = 2y_2$, and $y_1 = 2x_2$.*

(2) *$U_8(x_1, 1+x_2, 1+x_3, 2+x_4, y_1, 1+y_2, 1+y_3, 2+y_4)$ is SVG for any non-negative even integers $x_2, x_3, x_4, y_2, y_3, y_4$, if $x_1 = 2y_2+3y_3+5y_4$, and $y_1 = 2x_2+3x_3+5x_4$.*

(3) *$U_{12}(x_1, 2+x_2, x_3, x_4, 1+x_5, 3+x_6, y_1, 2+y_2, y_3, y_4, 1+y_5, 3+y_6)$ is SVG for any non-negative even integers $x_2, x_3, x_4, x_5, x_6, y_2, y_3, y_4, y_5, y_6$, if $x_1 = 2y_2 + 3y_3 + 4y_4 + 5y_5 + 7y_6$, and $y_1 = 2x_2 + 3x_3 + 4x_4 + 5x_5 + 7x_6$.*

(4) *$U_{16}(x_1, 1+x_2, 1+x_3, 1+x_4, x_5, 1+x_6, 1+x_7, 3+x_8, y_1, 1+y_2, 1+y_3, 1+y_4, y_5, 1+y_6, 1+y_7, 3+y_8)$ is SVG for any non-negative even integers $x_2, x_3, x_4, x_5, x_6, x_7, x_8, y_2, y_3, y_4, y_5, y_6, y_7, y_8$, if $x_1 = 2y_2 + 3y_3 + 4y_4 + 5y_5 + 6y_6 + 7y_7 + 9y_8$, and $y_1 = 2x_2 + 3x_3 + 4x_4 + 5x_5 + 6x_6 + 7x_7 + 9x_8$.*

Proof. Use the labelings in Theorem 6.5, apply the construction to the vertex labeled 1 and each of the negatively labeled vertices (other than -1) on the spine, and then apply the construction again to the vertex labeled -1 and each of the positively labeled vertices (other than 1) on the spine. $\square$

Corollary 7.9. *If* $k \geqslant 5, U_{4k}(x_1, 1+x_2, 1+x_3, \ldots, 1+x_k, x_{k+1}, x_{k+2}, 1+x_{k+3}, 1+x_{k+4}, \ldots, 1+x_{2k-3}, 2+x_{2k-2}, 1+x_{2k-1}, 3+x_{2k}, y_1, 1+y_2, 1+y_3, \ldots, 1+y_k, y_{k+1}, y_{k+2}, 1+y_{k+3}, 1+y_{k+4}, \ldots, 1+y_{2k-3}, 2+y_{2k-2}, 1+y_{2k-1}, 3+y_{2k})$ *is SVG for any non-negative even integers* $x_2, x_3, \ldots, x_{2k}, y_2, y_3, \ldots, y_{2k}$ *if* $x_1 = 2y_2 + 3y_3 + \ldots + (2k-1)y_{2k-1} + (2k+1)y_{2k}$*, and* $y_1 = 2x_2 + 3x_3 + \ldots + (2k-1)x_{2k-1} + (2k+1)x_{2k}$.

Proof. By Theorem 6.6, $U_{4k}(0, 1^{k-1}, 0^2, 1^{k-5}, 2, 1, 3, 0, 1^{k-1}, 0^2, 1^{k-5}, 2, 1, 3)$ is SVG for any $k \geqslant 5$. Apply the construction to the vertex labeled 1 and each of the negatively labeled vertices (other than -1) on the spine, and then apply the construction again to the vertex labeled -1 and each of the positively labeled vertices (other than 1) on the spine. $\square$

Theorem 7.10. *Any ring-worm is an induced subgraph of an SVG ring-worm.*

Proof. Consider any ring-worm with spine C_n. If n is odd, use Theorem 6.1. If n is divisible by 4, use Corollary 7.8 or 7.9. If n is divisible by 2 but not by 4, use Corollary 7.6 or 7.7. In each case, simply add sufficiently many pendant edges to the given ring-worm to use the Theorem or Corollary. $\square$

We list here an unsolved problem and a conjecture for future research.

Problem. Characterize ring-worms which are non-SVG.

Conjecture. *Any unicyclic graph is an induced subgraph of an SVG unicyclic graph.*

References

[1] *S. Cabannis, J. Mitchem and R. Low*: On edge-graceful regular graphs and trees. Ars Combin. *34* (1992), 129–142.
[2] *J. A. Gallian*: A dynamic survey of graph labeling. Electronic J. Combin. *6* (2001), 1–144.
[3] *J. Keene and A. Simoson*: Balanced strands for asymmetric, edge-graceful spiders. Ars Combin. *42* (1996), 49–64.
[4] *Q. Kuan, S.-M. Lee, J. Mitchem and A. K. Wang*: On edge-graceful unicyclic graphs. Congress. Numer. *61* (1988), 65–74.
[5] *L. M. Lee, S.-M. Lee and G. Murty*: On edge-graceful labelings of complete graphs-solutions of Lo's conjecture. Congress. Numer. *62* (1988), 225–233.
[6] *S.-M. Lee*: A conjecture on edge-graceful trees. Scientia, Ser. A *3* (1989), 45–57.
[7] *S.-M. Lee*: New directions in the theory of edge-graceful graphs. Proceedings of the 6th Caribbean Conference on Combinatorics & Computing (1991), 216–231.
[8] *S.-M. Lee*: On strongly indexable graphs and super vertex-graceful graphs, manuscript.
[9] *S.-M. Lee and E. Leung*: On super vertex-graceful trees. Congress. Numer. *167* (2004), 3–26.
[10] *S.-M. Lee, P. Ma, L. Valdes and S.-M. Tong*: On the edge-graceful grids. Congress. Numer. *154* (2002), 61–77.

[11] *S.-M. Lee and E. Seah*: Edge-graceful labelings of regular complete k-partite graphs. Congress. Numer. *75* (1990), 41–50.
[12] *S.-M. Lee and E. Seah*: On edge-gracefulness of the composition of step graphs with null graphs. Combinatorics, Algorithms, and Applications in Society for Industrial and Applied Mathematics (1991), 326–330.
[13] *S.-M. Lee and E. Seah*: On the edge-graceful (n, kn)-multigraphs conjecture. J. Comb. Math. and Comb. Computing *9* (1991), 141–147.
[14] *S.-M. Lee, E. Seah and S. P. Lo*: On edge-graceful 2-regular graphs. J. Comb. Math. and Comb. Computing *12* (1992), 109–117.
[15] *S.-M. Lee, E. Seah and S.-M. Tong*: On the edge-magic and edge-graceful total graphs conjecture. Congress. Numer. *141* (1999), 37–48.
[16] *S.-M. Lee, E. Seah and P. C. Wang*: On edge-gracefulness of the kth power graphs. Bull. Inst. Math. Academia Sinica *18* (1990), 1–11.
[17] *S. P. Lo*: On edge-graceful labelings of graphs. Congress. Numer. *50* (1985), 231–241.
[18] *J. Peng and W. Li*: Edge-gracefulness of $Cm \times Cn$. Proceedings of the Sixth Conference of Operations Research Society of China. Hong Kong: Global-Link Publishing Company, Changsha, 2000, pp. 942–948.
[19] *J. Mitchem and A. Simoson*: On edge-graceful and super-edge-graceful graphs. Ars Combin. *37* (1994), 97–111.
[20] *A. Riskin and S. Wilson*: Edge graceful labellings of disjoint unions of cycles. Bulletin of the Institute of Combinatorics and its Applications *22* (1998), 53–58.
[21] *K. Schaffer and S.-M. Lee*: Edge-graceful and edge-magic labelings of Cartesian products of graphs. Congress. Numer. *141* (1999), 119–134.
[22] *W. C. Shiu, S.-M. Lee and K. Schaffer*: Some k-fold edge-graceful labelings of $(p, p-1)$-graphs. J. Comb. Math. and Comb. Computing *38* (2001), 81–95.
[23] *S. Wilson and A. Riskin*: Edge-graceful labellings of odd cycles and their products. Bulletin of the ICA *24* (1998), 57–64.

Authors' addresses: Sin-Min Lee, Elo Leung, Department of Computer Science, San Jose State University, San Jose, California 95192, U.S.A.; Ho Kuen Ng, Department of Mathematics, San Jose State University, San Jose, California 95192, U.S.A.