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RESULTS ON F-CONTINUOUS GRAPHS

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Abstract. For any nontrivial connected graph F and any graph G, the *F-degree* of a vertex v in G is the number of copies of F in G containing v. G is called *F-continuous* if and only if the F-degrees of any two adjacent vertices in G differ by at most 1; G is *F-regular* if the F-degrees of all vertices in G are the same. This paper classifies all P_4-continuous graphs with girth greater than 3. We show that for any nontrivial connected graph F other than the star $K_{1,k}$, $k \geqslant 1$, there exists a regular graph that is not F-continuous. If F is 2-connected, then there exists a regular F-continuous graph that is not F-regular.

Keywords: continuous, F-continuous, F-regular, regular graph

MSC 2010: 05C12, 05C78

1. Introduction

Chartrand et al. in [1] consider the general case of integer-valued functions f defined on a metric space of objects associated with a particular graph G. Such a function is *continuous* if and only if $|f(x) - f(y)| \leqslant 1$ for every two adjacent elements x and y in the metric space. When the metric space is the vertex set of G, a continuous function defined on $V(G)$ is, in fact, a labeling of the vertices of G with nonnegative integers such that the labels of any two vertices v and u connected with an edge differ by at most 1. Such a labeling is referred to as a *continuous labeling.* Degree-continuous graphs provide an example of graphs with a certain type of a continuous labeling. A graph G is called *degree-continuous* if $|\deg(v) - \deg(v')| \leqslant 1$ for every pair $\{v, v'\}$ of adjacent vertices of G. For more information on degree-continuous graphs see [5].

This paper is concerned with graphs $G = (V, E)$ together with a different continuous labeling. Given any nontrivial connected graph F, and any vertex $v \in V(G)$, the *F-degree of v in G*, denoted F-$\deg_G(v)$, is the number of copies (not necessarily induced) of F in G containing v. Thus, the degree of v, denoted $\deg_G(v)$, and the

P_2-degree of v are the same where P_n denotes the path on n vertices. When no confusion is possible, we write F-deg(v) instead of F-deg$_G(v)$, and deg(v) instead of deg$_G(v)$. A graph G is *F-continuous* (or *F-degree continuous*) if and only if the F-degrees of any two adjacent vertices in $V(G)$ differ by at most 1. If, in addition, F-deg$(v) = r$ for all $v \in V(G)$, then, G is *F-regular* of degree r.

Without loss of generality we can assume that G, as well as F, is nontrivial and connected; we do not allow loops or multiple edges. If no copy of F can be found in G, then F-deg$(v) = 0$ for all $v \in V(G)$, and trivially, G is F-continuous and even F-regular. The *girth* $g(G)$ of a graph G is the minimum among all cycle lengths taken over all cycles in G; the *circumference* $c(G)$ of G is the length of the largest cycle appearing in G. If G has no cycles, by default $g(G) = \infty$. The *distance* between any two vertices of G is the length of the shortest path between them; the *diameter* $d(G)$ of G is the largest over all distances between pairs of vertices in G.

The concept of F-degree was introduced by Chartrand et al. [2] in 1987; results on F-continuous graphs can be found in [3]. In addition to determining all P_3-continuous graphs, Chartrand, Jarrett et al. [3] show that if G is F-continuous for all nontrivial connected graphs F, then, $G = P_n$ or G is regular. However, there are nontrivial connected graphs F such that there exists a regular graph G that is not F-continuous. Certainly, if $F = K_{1,k}$, $k \geqslant 2$, and G is an r-regular graph, then $K_{1,k}$-deg$(v) = (k+1)\binom{r}{k}$ for every $v \in V(G)$. Thus, there is no regular graph which is not $K_{1,k}$-continuous. In the case when F is a 2-connected graph, however, Chartrand et al. construct a regular graph that is not F-continuous [3].

In Section 3, we extend the above result from 2-connected graphs F to all nontrivial connected graphs other than $K_{1,k}$, $k \geqslant 2$, confirming a conjecture in [3]. Furthermore, we show that for every 2-connected graph F, there exists a regular F-continuous graph that is not F-regular. We begin, in Section 2, by classifying all P_4-continuous graphs that contain no triangles.

2. P_4-continuous graphs

This section is entirely devoted to the case of $F = P_4$. All P_2-continuous graphs have been studied in [5], and all P_3-continuous graphs have been classified in [3]. We determine all P_4-continuous graphs with girth greater than 3.

Let H and K denote the graphs on five and four vertices, respectively, shown in Figure 1. Our main result is given below.

Theorem 2.1. *Let G be a connected P_4-continuous graph with girth $g(G) > 3$ and minimum degree δ. Then, G is isomorphic to one of H, P_n, $K_{1,n}$, for some integer $n \geqslant 1$, or G is δ-regular.*

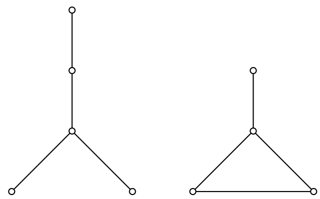

Figure 1. Two P_4-continuous graphs: acyclic graph H on 5 vertices and graph K on 4 vertices

Before we prove Theorem 2.1, we consider some special cases.

Lemma 2.2. *Let G be a connected P_4-continuous graph, and let C_3 denote the cycle on 3 vertices.*

(i) *If P_4-$\deg(v) = 0$ for some vertex v of G, then $G \cong C_3$ or $K_{1,n}$ for some integer $n \geqslant 1$.*

(ii) *If P_4-$\deg(v) = 1$ for some vertex v of G, then $G \cong H$ or P_n for some integer $n \geqslant 4$.*

(iii) *If $\deg(v) = 1$ for some vertex v of G and G contains a copy of P_4, then $G \cong H$, K or P_n for some integer $n \geqslant 4$.*

Proof. (i) The distance between any two vertices x and y of G is less than or equal to the length of a path from x to y passing through v. Since G is connected, such a path always exists; it must be that $d(G) \leqslant 2$ and the result follows.

(ii) Since v is contained in a copy of P_4, there exists a vertex u adjacent to v with $\deg(u) > 1$; i.e. $\{u, w\} \in E(G)$ for some vertex w other than v. For any vertex x adjacent to v other than u, $\langle w, u, v, x\rangle$ is a copy of P_4. Therefore, $\deg(v) = 1$ or $\deg(v) = 2$. If $\deg(v) = 2$, no new edges or vertices can be added without contradicting P_4-$\deg(v) = 1$. It must be that $G \cong P_4$. Suppose that $\deg(v) = 1$ and let $\langle v, u, w, y_1\rangle$ be the copy of P_4 containing v. Now, no new edges adjacent to w can be added; u can only be adjacent to a new vertex y_2 in which case $G \cong H$ and no additional edges are present. Otherwise, it must be that $G \cong P_4$ or a new vertex y_2 is adjacent to y_1. Again, either $G \cong P_5$ or there is a new vertex y_3 that can only be adjacent to y_2. Repeating the same procedure we see that $G \cong P_n$ for some integer $n \geqslant 4$.

(iii) Let u be the only vertex adjacent to v. Denote by G' the graph with vertex set $V(G) - v$ and edge set $E(G) - \{v, u\}$. A copy of P_4 in G that contains v must necessarily contain u as well. Any copy of P_4 that contains u but does not contain v must lie entirely in G'. Therefore, P_4-$\deg_{G'}(u) = 0$ or 1. If P_4-$\deg_{G'}(u) = 0$, then, by (i), either $G' \cong C_3$ and $G \cong K$, or $G' \cong K_{1,n}$ and G is P_4-continuous and contains a copy of P_4 only if $G \cong P_4$ or $G \cong H$. If P_4-$\deg_{G'}(u) = 1$, then, by (ii), $G' \cong H$ or

$G' \cong P_n$, $n \geqslant 4$. It is easy to see that the only way for G to be P_4-continuous in this case is if $G \cong H$ or $G \cong P_n$, $n \geqslant 5$. $\square$

Proof of Theorem 2.1. By Lemma 2.2, we may assume that $\delta \geqslant 2$ and that G contains a copy of P_4. Let v be a vertex of G of degree δ, and let $u_1, \ldots, u_\delta \in V(G)$ denote the neighbours of v where $\deg(u_i) := d_i + 1$. For each i, let $u_{i,1}, \ldots, u_{i,d_i} \in V(G)$ denote the d_i neighbours of u_i other than v with $\deg(u_{i,j}) := d_{i,j} + 1$ for $j = 1, \ldots, d_i$. Certainly not all $u_{i,j}$ have to be distinct. Define

$$c_i := \sum_{j=1}^{d_i} d_{i,j}$$

and without loss of generality assume that $c_1 \geqslant c_i$ for $i = 2, \ldots, \delta$.

Since G contains no triangles, the P_4-degree of a vertex in G depends only on the degrees of all vertices of distance two or less from the given vertex. If A denotes the number of copies of P_4 in G that contain both v and u_1, then

$$\begin{aligned} P_4\text{-deg}(v) &= A + \sum_{i=2}^{\delta} c_i + (\delta - 2) \sum_{i=2}^{\delta} d_i, \\ P_4\text{-deg}(u_1) &\geqslant A + c_1(d_1 - 1) + c_1(\delta - 1) = A + c_1(d_1 + \delta - 2) \end{aligned}$$

since each neighbour of $u_{1,j}$, $j = 1, \ldots, d_1$, must have degree at least δ. For each $i = 2, \ldots, \delta$, $c_1 \geqslant c_i \geqslant d_i(\delta - 1)$, leading to

$$c_1 \geqslant \sum_{i=2}^{\delta} d_i.$$

It must be that

$$P_4\text{-deg}(v) \leqslant A + c_1(\delta - 1) + c_1(\delta - 2) = A + c_1(2\delta - 3)$$

and since $d_1 \geqslant \delta - 1$,

$$1 \geqslant P_4\text{-deg}(u_1) - P_4\text{-deg}(v) \geqslant c_1(d_1 - \delta + 1) \geqslant d_1(\delta - 1)(d_1 - \delta + 1).$$

The above inequality does not hold when $d_1 > \delta - 1$ since $\delta \geqslant 2$; we must have $d_1 = \delta - 1$ and $\deg(u_1) = \delta$. Then,

$$P_4\text{-deg}(v) \leqslant A + c_1(2\delta - 3) \leqslant P_4\text{-deg}(u_1). \tag{1}$$

If equality holds in the first part of (1), then $c_1 = c_i$ for all $i = 2, \dots, \delta$, and by the same argument as above applied to c_i, $\deg(u_i) = \delta$. All neighbours of the arbitrary vertex v of degree δ must also have degree δ, showing that G is δ-regular.

Otherwise, equality must hold in the second part of (1). Assume that $c_k < c_1$ for some k, $2 \leqslant k \leqslant j$. Since G is P_4-continuous, $P_4\text{-}\deg(v) = A + c_1(2\delta - 3) - 1$, and then,

$$\begin{aligned} A + \sum_{i=2}^{\delta} c_i + (\delta - 2) \sum_{i=2}^{\delta} d_i &= A + c_1(2\delta - 3) - 1, \\ c_k + c_1(\delta - 2) + c_1(\delta - 2) &\geqslant c_1(2\delta - 3) - 1, \\ c_k &\geqslant c_1 - 1. \end{aligned}$$

Thus, $c_k = c_1 - 1$ and $c_i = c_1$ for all $i \neq k$. Our argument, then, applies to all c_i, $i \neq k$, and shows that $\deg(u_i) = \delta$ for all $i \neq k$. Then, $d_k \geqslant d_i$ for all $i = 1, \dots, \delta$ since $d_i = \delta - 1$ is the smallest possible when $i \neq k$. We get,

$$\begin{aligned} A + c_1(2\delta - 3) - 1 &= A + c_1 - 1 + c_1(\delta - 2) + (\delta - 2) \sum_{i=2}^{\delta} d_i, \\ c_1 &= \sum_{i=2}^{\delta} d_i, \\ c_1 &\leqslant d_k(\delta - 1). \end{aligned}$$

But then, $c_k \geqslant d_k(\delta - 1) \geqslant c_1$ which contradicts the fact that $c_k = c_1 - 1$. Therefore, it must be that $c_1 = c_i$ for all $i = 2, \dots, \delta$ and as before G is δ-regular. □

To complete the classification of P_4-continuous graphs of girth other than three, we conclude this section with a closer look at regular graphs.

Lemma 2.3. *Let $n \geqslant 4$ be a positive integer. Let G be an r-regular connected graph with $g(G) \geqslant n - 1$. Then, for every $v \in V(G)$,*

$$P_n\text{-}\deg(v) = \frac{nr(r-1)^{n-2}}{2} - n\, C_{n-1}\text{-}\deg(v)$$

where C_{n-1} is the cycle on $n - 1$ vertices.

Proof. Fix $v \in V(G)$. If $g(G) \geqslant n$, then $C_{n-1}\text{-}\deg(v) = 0$, and for v to be at position i of the path, $1 \leqslant i \leqslant n$, we have r choices for the first edge incident to v and $r - 1$ choices for each additional edge of P_n. Finally, since there are n possible positions for v and since P_n is symmetric, the result follows. If $g(G) = n - 1$, we are counting illegitimate copies of P_n whenever v lies on a copy of C_{n-1}. Moreover, every such false copy of P_n is counted exactly n times. □

Corollary 2.4. *Let $n \geqslant 4$ be a positive integer. A regular connected graph G with $g(G) \geqslant n-1$ is P_n-continuous if and only if it is C_{n-1}-regular.*

Corollary 2.5. *An r-regular graph G with girth $g(G) > 3$ is always P_4-continuous and, in fact, P_4-regular of degree $2r(r-1)^2$. An r-regular graph with girth equal to 3 is P_4-continuous if and only if it is C_3-regular. There does not exist a regular P_4-continuous graph that is not P_4-regular.*

Open Problem 2.6. Determine all P_4-continuous graphs with girth 3 and minimum degree at least 2.

3. F-continuous graphs and regular graphs

In this section we examine F-continuous and F-regular graphs for a general graph F. Using a counting argument similar to the one used in the proof of Lemma 2.3 we can consider the case when F is any tree.

Lemma 3.1. *Let T be a tree with diameter $d(T) = d \geqslant 3$ and let G be an r-regular connected graph with $g(G) \geqslant d+1$. Then, G is T-regular.*

Proof. Fix $v_0 \in V(G)$. When v_0 is contained in a copy T' of T in G, v_0 is identified with a vertex t' of T'. Think of T' as a rooted tree with root t' and say that v_0 *lies in a copy of T in position T'*. There exists a set of rooted trees $T_1, T_2, \ldots, T_a$ that satisfy

1. T_i is isomorphic to T as undirected graphs for $i = 1, 2, \ldots, a$, and
2. For any graph H, and any vertex $v \in V(H)$,

$$T\text{-}\deg_H(v) = \sum_{i=1}^{a} n_i(H, v)$$

where $n_i(H, v)$ denotes the number of times v lies in a copy of T in H in position T_i.

The integer a depends only on the structure of T. In the case of the tree P_4, for example, $a = 3$ and Figure 2 shows the set of three rooted trees.

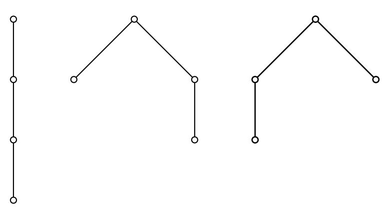

Figure 2. Set of three rooted trees for P_4

We want to show that $T\text{-deg}_G(v_0)$ is constant. For each $i = 1, 2, \ldots, a$,

$$n_i(G, v_0) = \binom{r}{\deg_{T_i}(t_i)} \prod_{u \in V(T_i),\ u \neq t_i} \binom{r-1}{\deg_{T_i}(u) - 1}$$

where $t_i \in V(T_i)$ denotes the root of T_i. The correctness of the counting argument is guaranteed by $g(G) \geqslant d + 1$ which is large enough to never mistake a cyclic graph in G for a copy of T. Therefore, $n_i(G, v_0)$, $i = 1, 2, \ldots, a$, is a function of r and the structure of T showing that $T\text{-deg}_G(v_0)$ will remain the same irrespective of the choice of vertex v_0 of G. □

We make use of the following result of Erdös and Sachs, the proof of which can be found in [4].

Lemma 3.2 [4]**.** *For every two integers $r \geqslant 2$ and $g \geqslant 3$, there exists an r-regular graph G with $g(G) = g$.*

The next theorem solves an open problem posed in [3].

Theorem 3.3. *Given any nontrivial connected graph F other than the star $K_{1,k}$, $k \geqslant 1$, there exists a regular graph that is not F-continuous.*

P r o o f. Chartrand et al. in [3] have resolved the case of 2-connected graphs F. It will suffice, then, to construct a regular graph with the desired property for any other possible F, falling into two categories.

Case 1: F is a tree.

Let $d(F) = d$, $d \geqslant 3$, and $|V(F)| = n$. Note that $d < 3$ implies that F is a star graph. As hinted by Lemma 3.1, the idea is to construct a regular graph of girth d which contains exactly one copy of C_d. We start with a copy of F to avoid designing a regular graph that is trivially F-continuous because all of its F-degrees are zero. Pick two vertices x and y of F distance d apart and let the path P, passing through vertices $x, v_1, v_2, \ldots, v_{d-1}, y$ in that order, be a path of length d. Denote by Δ the highest degree of a vertex in F and set $r = 4\Delta$. We will construct an r-regular graph that is not F-continuous.

Attach a single cycle C_d to the vertex x of a copy of F by identifying x with a vertex on the cycle. Each vertex of this new graph, that we will call H, has a degree less than or equal to $\Delta + 2 < r$. Collectively, for the vertices in the copy of F we need additional

$$nr - 2 - \sum_{v \in V(F)} \deg_F(v)$$

edges, in order to make all of them have degree r in the new graph we are creating. For the vertices in the cycle C_d, excluding x, we need $(d-1)(r-2)$ more edges. Note that

$$(d-1)(r-2)+nr-2-\sum_{v\in V(F)}\deg_F(v)=r(n+d-1)-2d-\sum_{v\in V(F)}\deg_F(v)=2q$$

is even since r is even. By Lemma 3.2, there exists an r-regular graph J with $g(J)=d+1$. Take q distinct copies of J and remove the same edge $\{s,t\}$ in each copy. Then, glue each of those graphs to H by adding the edges $\{s,u\}$ and $\{t,w\}$, where u and w are vertices of H in such a way that will complete the degree of each vertex to r. Denote the new r-regular graph by G. Certainly G contains no cycles of length less than $d+1$ except the single cycle C_d we started with. If G' is any r-regular graph with $g(G')\geqslant d+1$, Lemma 3.1 will imply that $F\text{-deg}_{G'}(v)=A$ for all $v\in V(G')$ and some positive constant A. Since we have the cycle C_d in G, however, the F-degree of some vertices of G will be less than A since A would count some cyclic graphs as copies of F.

Consider the adjacent vertices x and v_1 of G. Despite the edges we added to H, v_1 does not lie directly on the cycle C_d, and therefore, no double counting will occur and $F\text{-deg}_G(v_1)=A$. However, the same counting procedure applied to $F\text{-deg}_G(x)$ will consider the cycle C_d as an acyclic path of length d at least twice, once in either direction. Then, $F\text{-deg}_G(x)\leqslant A-2$, making the F-degrees of x and v_1 differ by more than 1; we have shown that G is not F-continuous.

Case 2: F is not a tree.

Let $c(F)=c$ and say that F has m cycles C_c. For each $v\in V(F)$, define the *proximity of v in F*, denoted $\text{prox}_F(v)$, to be the length of a shortest path from v to a vertex on any of the m cycles C_c in F. If v lies on one of the m cycles, then $\text{prox}_F(v)=0$. Also, let

$$p=\max\{\text{prox}_F(v)\colon\ v\in V(F)\}.$$

Identify two copies F_1 and F_2 of F at the same vertex x, where $\text{prox}_F(x)=p$. Add an additional vertex y and the edge $\{x,y\}$, and denote the resulting graph by H. Let r be the largest degree of a vertex in H. Using H, we will construct an r-regular graph G that is not F-continuous. In particular, our goal is to make $F\text{-deg}_G(y)=0$ while $F\text{-deg}_G(x)\geqslant 2$.

Using Lemma 3.2, there exists an r-regular graph J of girth $g=\max\{c+1,p\}$. Note that

$$\sum_{u\in V(H),u\neq x,y}(r-\deg_H(u))=2\sum_{u\in V(F),u\neq x}(r-\deg_F(u))=2q$$

is even. Let $J_1, J_2, \ldots, J_q$ be q disjoint copies of J. Remove the same edge, say $\{s,t\}$ from each copy. Then, glue each copy $J_i - \{s,t\}$ to H by adding the edges $\{s, v_1\}$ and $\{t, v_2\}$ where v_1 is a vertex of F_1, v_2 is the corresponding vertex of F_2, and $v_1, v_2 \neq x$. Continue to glue the copies of J until all vertices of H, except possibly x and y, have degree r.

Next, we deal with the vertices x and y. Let $b = \deg_F(x)$. Note that $\deg_H(x) = 2b+1$ and $\deg_H(y) = 1$. Take $r - (2b+1)$ more copies of J, remove the same edge $\{s,t\}$, and attach each copy to H by adding the edges $\{x,s\}$ and $\{y,t\}$. In the graph we have constructed so far, all vertices will have degree r, except possibly the vertex y that will have degree $r - 2b$. So, finally, take b copies of J, remove the same edge $\{s,t\}$, and glue each copy to our graph by the edges $\{y,s\}$ and $\{y,t\}$. Denote the final graph by G. Certainly G is r-regular and the only cycles C_c in G are the $2m$ such cycles in F_1 and F_2. Furthermore, since F_1 and F_2 contain the vertex x, it is clear that $F\text{-}\deg_G(x) \geqslant 2$. We are left to show that $F\text{-}\deg_G(y) = 0$.

Assume on the contrary that y is contained in a copy F' of F, where F' is a subgraph of G. Then, F' must contain m of the $2m$ cycles C_c in G. By our definition of p, $\text{prox}_{F'}(y) \leqslant p$.

However, if we remove the vertex x from G, G is no longer connected, and all of the cycles of type C_c will lie in a different component than the vertex y. Also, since $g(J) \geqslant p$, the shortest distance from x to a cycle C_c in G remains p. That is, any shortest path from y to a cycle C_c must start with the edge $\{y,x\}$ and continue with a path from x to a cycle C_c. Thus,

$$\text{prox}_G(y) \geqslant 1 + p$$

which is impossible because $\text{prox}_{F'}(y) \geqslant \text{prox}_G(y)$. Therefore, x and y are adjacent vertices of G whose F-degrees differ by more than 1; G is not F-continuous. $\square$

Chartrand et al. in [3] pose yet another open problem concerning regular graphs. They question whether for every nontrivial connected graph F, $F \neq K_{1,k}$ for $k \geqslant 1$, there exists a regular F-continuous graph which is not F-regular. In [3] they answer this question in the affirmative if F is any nontrivial complete graph K_n. Here, we show that the answer is still affirmative if F is any 2-connected graph - a graph which remains connected after removing any two of its vertices and their adjacent edges.

Theorem 3.4. *For every nontrivial 2-connected graph F, there exists a regular F-continuous graph that is not F-regular.*

Proof. Let $c(F) = c$ and take two disjoint copies F_1 and F_2 of F. Add a new vertex y and two new edges $\{y, x_1\}$ and $\{y, x_2\}$, where x_1 is a vertex in F_1 and x_2 is the corresponding vertex in F_2. Denote the graph constructed so far by H. If $\Delta(H)$ is the largest degree of a vertex in H, let $r = 4\Delta(H)$. We will add edges and vertices to H to convert it to an r-regular graph. Observe that

$$r - 2 + 2\bigg(\sum_{v\in V(F)} r - \deg_F(v)\bigg) - 2 = 2q$$

is even since r is even. Using q disjoint copies of an r-regular graph J with $g(J) = c+1$ we can transform H into an r-regular graph G with girth c using the same approach as in the proof

of Theorem 3.3. The only cycles of length c in G would be the ones in F_1 and F_2. This and the fact that F is 2-connected guarantees that F_1 and F_2 are the only copies of F in G. Then, F-$\deg_G(y) = 0$ while F-$\deg_G(x_1) = F$-$\deg_G(x_2) = 1$ and there is no vertex in G that is contained in both F_1 and F_2. Therefore, G is not F-regular but it is F-continuous. □

When F is not 2-connected, however, the same result does not necessarily hold. In particular, when $F = P_4$ there does not exist a regular P_4-continuous graph that is not P_4-regular as seen in Corollary 2.5.

Open Problem 3.5. For every integer $n \geqslant 5$, does there exist a regular P_n-continuous graph that is not P_n-regular?

Open Problem 3.6. Given any nontrivial connected graph F that is not 2-connected, does there exist a regular F-continuous graph that is not F-regular?

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