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Czechoslovak Mathematical Journal, Vol. 59 (2009), No. 1, 81–94

Persistent URL: <http://dml.cz/dmlcz/140465>

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LOWER BOUNDS FOR MATRICES ON BLOCK
WEIGHTED SEQUENCE SPACES I

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(Received October 15, 2006)

Abstract. In this paper we consider some matrix operators on block weighted sequence spaces $l_p(w, F)$. The problem is to find the lower bound of some matrix operators such as Hausdorff and Hilbert matrices on $l_p(w, F)$. This study is an extension of papers by G. Bennett, G.J.O. Jameson and R. Lashkaripour.

Keywords: block weighted sequence spaces, lower bound, inequality, Hausdorff matrix, Hilbert matrix

MSC 2010: 47B37, 46B45, 26D15

1. INTRODUCTION

Suppose $p \geq 1$ and $w = (w_n)$ is a decreasing non-negative sequence. We define the weighted sequence space $l_p(w)$ as

$$l_p(w) := \left\{ x = (x_n) : \sum_{n=1}^{\infty} w_n |x_n|^p \text{ is finite} \right\},$$

with a norm $\|\cdot\|_{p,w}$ which is defined in the following way:

$$\|x\|_{p,w} = \left(\sum_{n=1}^{\infty} w_n |x_n|^p \right)^{1/p}.$$

Assume that F is a partition of positive integers. If $F = (F_n)$, where each F_n is a finite interval of positive integers and

$$\max F_n < \min F_{n+1} \quad (n = 1, 2, \dots),$$

we define the block weighted sequence space $l_p(w, F)$ as

$$l_p(w, F) := \left\{ x = (x_n) : \sum_{n=1}^{\infty} w_n |\langle x, F_n \rangle|^p \text{ is finite} \right\},$$

where $\langle x, F_n \rangle = \sum_{j \in F_n} x_j$. The norm on $l_p(w, F)$, denoted by $\| \cdot \|_{p,w,F}$, is defined as follows:

$$\|x\|_{p,w,F} = \left(\sum_{n=1}^{\infty} w_n |\langle x, F_n \rangle|^p \right)^{1/p}.$$

For a certain I_n such as $I_n = \{n\}$, $I = (I_n)$ is a partition of positive integers, $l_p(w, I) = l_p(w)$ and also $\|x\|_{p,w,I} = \|x\|_{p,w}$.

We write $\|A\|_{p,w,F}$ for the norm of A as an operator from $l_p(w, I)$ into $l_p(w, F)$. The problem of the norm of matrix operators on $l_p(w)$ and $l_p(w, F)$ is considered in [5], [6], [7] and [8].

We consider lower bounds L of the form

$$\|Ax\|_{p,w,F} \geq L \|x\|_{p,w,I},$$

for all decreasing non-negative sequences x . The constant L is independent of x . We seek the largest possible value of L , and denote the best lower bound by $L_{p,w,F}(A)$ for matrix operators from $l_p(w, I)$ into $l_p(w, F)$. Also, if A is an operator from $l_p(w, I)$ into itself, we denote the best lower bound by $L_{p,w,I}(A)$. We shall use all the above notation when $p < 1$.

In Section 2, we generalize two techniques obtained by Bennett in Section 7 of [1] and deduce the lower bound for the Hausdorff matrix. In Section 3, we also generalize Theorem 1 of [4] for matrix operators from $l_p(w, I)$ into $l_p(w, F)$ and study the lower bound problem for the Hilbert and Copson matrices.

Throughout this paper, we denote the transpose matrix of A by A^t , and the conjugate exponent of p by p^* , so that $p^* = p/(p-1)$.

2. HAUSDORFF MATRIX OPERATOR

In this part we consider the Hausdorff matrix operator $H(\mu) = (h_{j,k})$ with entries of the form

$$h_{j,k} = \begin{cases} \binom{j-1}{k-1} \Delta^{j-k} a_k & \text{if } 1 \leq k \leq j, \\ 0 & \text{if } k > j, \end{cases}$$

where Δ is the difference operator, that is

$$\Delta a_k = a_k - a_{k+1},$$

and (a_k) is a sequence of real numbers, normalized so that $a_1 = 1$. If

$$a_k = \int_0^1 \theta^{k-1} d\mu(\theta) \quad (k = 1, 2, \dots),$$

where μ is a probability measure on $[0, 1]$, then for all $j, k = 1, 2, \dots$,

$$h_{j,k} = \begin{cases} \binom{j-1}{k-1} \int_0^1 \theta^{k-1} (1-\theta)^{j-k} d\mu(\theta) & \text{if } 1 \leq k \leq j, \\ 0 & \text{if } k > j. \end{cases}$$

The Hausdorff matrix contains some famous classes of matrices. These classes are as follows:

- i) Choice $d\mu(\theta) = \alpha(1-\theta)^{\alpha-1} d\theta$ gives the Cesàro matrix of order α ;
- ii) Choice $d\mu(\theta) = \textit{point evaluation at } \theta = \alpha$ gives the Euler matrix of order α ;
- iii) Choice $d\mu(\theta) = |\log \theta|^{\alpha-1} / \Gamma(\alpha) d\theta$ gives the Hölder matrix of order α ;
- iv) Choice $d\mu(\theta) = \alpha \theta^{\alpha-1} d\theta$ gives the Gamma matrix of order α .

The Cesàro, Hölder and Gamma matrices have non-negative entries whenever $\alpha > 0$, and also the Euler matrix is non-negative when $0 \leq \alpha \leq 1$.

In this section, we are considering the lower bound problem for the Hausdorff matrix (general form), and also for the Cesàro, Hölder and Gamma matrices.

Proposition 2.1. *Let $A = (a_{n,k})$ be an upper-triangle matrix with non-negative entries and let $0 < p \leq 1$. If*

$$\sup_n \sum_{k=n}^{\infty} a_{n,k} = k_1 > 0, \quad \inf_k \sum_{n=1}^k a_{n,k} = k_2,$$

then

$$L_{p,w,I}(A) \geq k_1^{(p-1)/p} k_2^{1/p}.$$

Proof. Suppose x is a non-negative sequence. Applying Hölder's inequality we have

$$\begin{aligned} \sum_{k=n}^{\infty} a_{n,k} w_k x_k^p &= \sum_{k=n}^{\infty} a_{n,k}^{1-p} (a_{n,k} w_k^{1/p} x_k)^p \\ &\leq \left(\sum_{k=n}^{\infty} a_{n,k} \right)^{1-p} \left(\sum_{k=n}^{\infty} a_{n,k} w_k^{1/p} x_k \right)^p \\ &\leq k_1^{1-p} \left(\sum_{k=n}^{\infty} a_{n,k} w_k^{1/p} x_k \right)^p. \end{aligned}$$

Since A is an upper-triangle matrix with non-negative entries and w is decreasing, we have

$$\begin{aligned}
k_1^{1-p} \sum_{n=1}^{\infty} w_n \left(\sum_{k=1}^{\infty} a_{n,k} x_k \right)^p &= k_1^{1-p} \sum_{n=1}^{\infty} w_n \left(\sum_{k=n}^{\infty} a_{n,k} x_k \right)^p \\
&\geq k_1^{1-p} \sum_{n=1}^{\infty} \left(\sum_{k=n}^{\infty} a_{n,k} w_k^{1/p} x_k \right)^p \geq \sum_{n=1}^{\infty} \left(\sum_{k=n}^{\infty} a_{n,k} w_k x_k^p \right) \\
&= \sum_{k=1}^{\infty} w_k x_k^p \left(\sum_{n=1}^k a_{n,k} \right) \geq k_2 \sum_{k=1}^{\infty} w_k x_k^p.
\end{aligned}$$

Hence $\|Ax\|_{p,w,I}^p \geq k_1^{p-1} k_2 \|x\|_{p,w,I}^p$ and so we have the desired conclusion. \square

In the next statement, we seek a lower bound for the quasi-Hausdorff matrix when the sequences are non-negative. We recall the transpose of the Hausdorff matrix which is called the quasi-Hausdorff matrix.

Theorem 2.1. *Suppose that $H(\mu)$ is the Hausdorff matrix and $0 < p \leq 1$. Then*

$$\|H^t(\mu)x\|_{p,w,I} \geq \left(\int_0^1 \theta^{(1-p)/p} d\mu(\theta) \right) \|x\|_{p,w,I}$$

for every non-negative sequence x . The constant is the best possible.

Proof. Let $E(\alpha)$ be the Euler matrix of order α . Since all row sums of $E^t(\alpha)$ are $1/\alpha$ and all column sums are 1, applying Proposition 2.1 we obtain

$$L_{p,w,I}(E^t(\alpha)) \geq \alpha^{(1-p)/p}.$$

Let $F = (F_n)$ be defined as above. We now apply Minkowski's inequality to show

$$\begin{aligned}
\|H^t(\mu)x\|_{p,w,I} &= \left(\sum_{n=1}^{\infty} w_n \left(\sum_{j \in F_n} (H^t(\mu)x)_j \right)^p \right)^{1/p} = \left(\sum_{n=1}^{\infty} w_n \left(\sum_{k=1}^{\infty} h_{k,n} x_k \right)^p \right)^{1/p} \\
&= \left(\sum_{n=1}^{\infty} w_n \left(\int_0^1 \sum_{k=1}^{\infty} e_{k,n} x_k d\mu(\alpha) \right)^p \right)^{1/p} \\
&\geq \int_0^1 \left(\sum_{n=1}^{\infty} w_n \left(\sum_{k=1}^{\infty} e_{k,n} x_k \right)^p \right)^{1/p} d\mu(\alpha) \\
&= \int_0^1 \|E^t(\alpha)x\|_{p,w} d\mu(\alpha) \geq \left(\int_0^1 \alpha^{(1-p)/p} d\mu(\alpha) \right) \|x\|_{p,w,I}.
\end{aligned}$$

This completes the proof of the above inequality. Therefore for any real number $\alpha > 0$ we have

$$(I) \quad \|H^t(\mu)x\|_{p,w+\alpha,I} \geq \left(\int_0^1 \theta^{(1-p)/p} d\mu(\theta) \right) \|x\|_{p,w+\alpha,I}$$

for all non-negative sequences x in $l_p(w, I)$. We will show that the above constant is the best possible.

Let $\varrho > 1/p$ and let n be a fixed integer such that $n \geq \varrho$. We define x by

$$x_k = \begin{cases} 0 & \text{if } k < n, \\ \binom{k-\varrho}{k-n} / \binom{k}{n} & \text{if } k \geq n. \end{cases}$$

Since

$$x_k = \frac{(k-\varrho) \dots (n+1-\varrho)}{k \dots (n+1)} \sim k^{-\varrho}$$

when $k \rightarrow \infty$, it follows that $\|x\|_p < \infty$ and $\|x\|_p \rightarrow \infty$ when $\varrho \rightarrow 1/p$. Since w is decreasing and also $w_k + \alpha \geq \alpha$ for all k , we have

$$\alpha^{1/p} \|x\|_p \leq \|x\|_{p,w+\alpha,I} \leq (w_1 + \alpha)^{1/p} \|x\|_p,$$

so $\|x\|_{p,w+\alpha,I} < \infty$ and $\|x\|_{p,w+\alpha,I} \rightarrow \infty$ when $\varrho \rightarrow 1/p$. Moreover, for all $m > n$ we have

$$(H^t(\mu)x)_m = x_m \int_0^1 \theta^{\varrho-1} d\mu(\theta),$$

hence

$$\begin{aligned} \|H^t(\mu)x\|_{p,w+\alpha,I}^p &= \sum_{m=1}^n (w_m + \alpha) \left(\sum_{k=m}^{\infty} h_{k,m} x_k \right)^p + \sum_{m=n+1}^{\infty} (w_m + \alpha) (H^t(\mu)x)_m^p \\ &\leq n(w_1 + \alpha) \sup_{k,m} |h_{k,m}|^p \|x\|_1^p + \left(\int_0^1 \theta^{\varrho-1} d\mu(\theta) \right)^p \|x\|_{p,w+\alpha,I}^p \end{aligned}$$

and also

$$(L_{p,w+\alpha,I}(H^t(\mu)))^p \leq \frac{n(w_1 + \alpha) \sup_{k,m} |h_{k,m}|^p \|x\|_1^p}{\|x\|_{p,w+\alpha,I}^p} + \left(\int_0^1 \theta^{\varrho-1} d\mu(\theta) \right)^p.$$

If $\varrho \rightarrow 1/p$, then

$$L_{p,w+\alpha,I}(H^t(\mu)) \leq \int_0^1 \theta^{(1-p)/p} d\mu(\theta).$$

Therefore

$$L_{p,w+\alpha,I}(H^t(\mu)) = \int_0^1 \theta^{(1-p)/p} d\mu(\theta),$$

and the constant in (I) is the best possible. Hence for all m there is a non-negative sequence $y_m \in l_p(w, I)$ such that

$$\frac{\|H^t(\mu)y_m\|_{p,w+\alpha,I}}{\|y_m\|_{p,w+\alpha,I}} < \int_0^1 \theta^{(1-p)/p} d\mu(\theta) + \frac{1}{m}.$$

Since $\|H^t(\mu)y_m\|_{p,w,I} \leq \|H^t(\mu)y_m\|_{p,w+\alpha,I}$, we have

$$\begin{aligned} \frac{\|H^t(\mu)y_m\|_{p,w+\alpha,I}}{\|y_m\|_{p,w+\alpha,I}} &\geq \frac{\|H^t(\mu)y_m\|_{p,w,I}}{\|y_m\|_{p,w+\alpha,I}} \\ &= \frac{\|y_m\|_{p,w,I}}{\|y_m\|_{p,w+\alpha,I}} \cdot \frac{\|H^t(\mu)y_m\|_{p,w,I}}{\|y_m\|_{p,w,I}} \\ &\geq \frac{\|y_m\|_{p,w,I}}{\|y_m\|_{p,w+\alpha,I}} L_{p,w,I}(H^t(\mu)), \end{aligned}$$

and so

$$\frac{\|y_m\|_{p,w,I}}{\|y_m\|_{p,w+\alpha,I}} L_{p,w,I}(H^t(\mu)) \leq \int_0^1 \theta^{(1-p)/p} d\mu(\theta) + \frac{1}{m}.$$

If $\alpha \rightarrow 0$, since $\|x\|_{p,w+\alpha,I} < \infty$, we have $\|x\|_{p,w+\alpha,I} \rightarrow \|x\|_{p,w,I}$ and so

$$L_{p,w,I}(H^t(\mu)) \leq \int_0^1 \theta^{(1-p)/p} d\mu(\theta) + \frac{1}{m}.$$

Now, if $m \rightarrow \infty$, we have

$$L_{p,w,I}(H^t(\mu)) \leq \int_0^1 \theta^{(1-p)/p} d\mu(\theta).$$

Therefore

$$L_{p,w,I}(H^t(\mu)) = \int_0^1 \theta^{(1-p)/p} d\mu(\theta).$$

This completes the proof of the theorem. \square

Example. We denote the Gamma matrix of order 2 by $\Gamma(2)$. If $\Gamma^t(2) = (b_{i,j})$ is the transpose of the Gamma matrix, we have

$$b_{i,j} = \begin{cases} \frac{i}{\frac{1}{2}j(j+1)} & \text{if } j \geq i, \\ 0 & \text{if } j < i \end{cases}$$

and

$$L_{p,w,I}(\Gamma^t(2)) = \frac{2p}{p+1}.$$

We now give a lower bound for the quasi-Hausdorff matrix when the sequences are non-negative. We recall the transpose of the Hausdorff matrix which is called the quasi-Hausdorff matrix.

Proposition 2.2. *Let $0 < p, q < 1$, and let A be a matrix with non-negative entries. Then*

$$\|Ax\|_{q,w,I} \geq L\|x\|_{p,w,I}$$

for all non-negative x if and only if

$$\|A^t y\|_{p^*,w,I} \geq L\|y\|_{q^*,w,I}$$

for all non-negative y , where p^*, q^* are the conjugate exponents of p and q , respectively.

Proof. Suppose u is a sequence with non-negative entries. First we show that

$$(I) \quad \|u\|_{t,w,I} = \inf\{\langle u, v \rangle : v \text{ is a non-negative sequence and } \|v\|_{t^*,w,I} \geq 1\}$$

for $0 < t < 1$ or $t < 0$, where $\langle u, v \rangle = \sum_{k=1}^{\infty} w_k u_k v_k$.

Let v be a non-negative sequence such that $\|v\|_{t^*,w,I} \geq 1$. Then applying Hölder's inequality, we deduce that

$$\begin{aligned} \langle u, v \rangle &= \sum_{k=1}^{\infty} w_k u_k v_k = \sum_{k=1}^{\infty} w_k^{1/t+1/t^*} u_k v_k \geq \left(\sum_{k=1}^{\infty} w_k u_k^t \right)^{1/t} \left(\sum_{k=1}^{\infty} w_k v_k^{t^*} \right)^{1/t^*} \\ &= \|u\|_{t,w,I} \|v\|_{t^*,w,I} \geq \|u\|_{t,w,I}. \end{aligned}$$

Hence $\inf \langle u, v \rangle \geq \|u\|_{t,w,I}$.

We divide the proof of the converse inequality in two cases as follows:

Case 1. If $u > 0$, we take

$$\tilde{v}_k = u_k^{t-1}, \quad v_k = \frac{\tilde{v}_k}{\|\tilde{v}\|_{t^*,w,I}}.$$

Hence $\|\tilde{v}\|_{t^*,w,I} = \|u\|_{t,w,I}^{t-1}$ and $\langle u, v \rangle = \|u\|_{t,w,I}$, so that

$$\inf \langle u, v \rangle \leq \|u\|_{t,w,I}.$$

Case 2. If some $u_k = 0$, we consider (i), (ii):

(i) For $t < 0$, $\|u\|_{w,t,I} = 0$ and set

$$v_n = \begin{cases} 0 & \text{for } n \neq k, \\ \frac{1}{w_k^{1/t^*}} & \text{for } n = k. \end{cases}$$

(ii) For $0 < t < 1$, we set

$$\tilde{v}_k = \begin{cases} u_k^{t-1} & \text{for } u_k > 0, \\ \left(\frac{\varepsilon}{w_k 2^k}\right)^{1/t^*} & \text{for } u_k = 0 \end{cases}$$

and $v_k = \tilde{v}_k / \|\tilde{v}\|_{t^*,w,I}$, where ε is positive.

Hence $\|v\|_{t^*,w,I} = 1$, $\|\tilde{v}\|_{t^*,w,I} \geq 1/(\varepsilon + \|u\|_{t,w,I}^t)^{-1/t^*}$ and also

$$\langle u, v \rangle \leq \|u\|_{t,w,I}^t (\varepsilon + \|u\|_{t,w,I}^t)^{-1/t^*},$$

so that

$$\inf \langle u, v \rangle \leq \|u\|_{t,w,I}^t (\varepsilon + \|u\|_{t,w,I}^t)^{-1/t^*}.$$

If ε tends to zero, we have

$$\inf \langle u, v \rangle \leq \|u\|_{t,w,I}.$$

This completes the proof of (I).

Applying (I) twice, we deduce that

$$\begin{aligned} \inf_{\|x\|_{p,w,I} \geq 1} \|Ax\|_{q,w,I} &= \inf_{\|x\|_{p,w,I} \geq 1} \inf_{\|y\|_{q^*,w,I} \geq 1} \langle Ax, y \rangle \\ &= \inf_{\|x\|_{p,w,I} \geq 1} \inf_{\|y\|_{q^*,w,I} \geq 1} \langle x, A^t y \rangle \\ &= \inf_{\|y\|_{q^*,w,I} \geq 1} \inf_{\|x\|_{p,w,I} \geq 1} \langle x, A^t y \rangle \\ &= \inf_{\|y\|_{q^*,w,I} \geq 1} \|A^t y\|_{p^*,w,I} \end{aligned}$$

and so we have the statement. \square

In the next statement, we are seeking a lower bound of the Hausdorff matrix when the sequences are non-negative.

Corollary 2.2. *Suppose that $p < 0$, and let $H(\mu)$ be the Hausdorff matrix. Then*

$$\|H^t(\mu)x\|_{p,w,I} \geq \left(\int_0^1 \theta^{-1/p} d\mu(\theta) \right) \|x\|_{p,w,I}$$

for every non-negative sequence x . The constant is the best possible.

Proof. Since $0 < p^* < 1$, applying Theorem 2.1 and Proposition 2.1 we obtain the statement. \square

Corollary 2.3. Assume $0 < p \leq 1$, and let $H(\mu)$ be the Hausdorff matrix. Then

$$\|H^t(\mu)x\|_p \geq \left(\int_0^1 \theta^{(1-p)/p} d\mu(\theta) \right) \|x\|_p$$

for every non-negative sequence x . The constant is the best possible.

Proof. By taking $w_n = 1$ for all n in the previous corollary, we have the above inequality. \square

Corollary 2.4. If $p > 0$ and $H(\mu)$ is the Hausdorff matrix, then

$$\sum_{n=1}^{\infty} w_n \left(\sum_{k=1}^n \frac{h_{n,k}}{|x_k|} \right)^{-p} \leq \left(\int_0^1 \theta^{1/p} d\mu(\theta) \right)^{-p} \sum_{k=1}^{\infty} w_k |x_k|^p$$

for every non-negative sequence, and the constant is the best possible.

Proof. Suppose that y is a sequence with non-negative entries. Since $-p < 0$, applying Corollary 2.2, we arrive at

$$\|H^t(\mu)y\|_{-p,w,I} \geq \left(\int_0^1 \theta^{1/p} d\mu(\theta) \right) \|y\|_{-p,w,I}.$$

Hence

$$\sum_{n=1}^{\infty} w_n \left(\sum_{k=1}^n h_{n,k} y_k \right)^{-p} \leq \left(\int_0^1 \theta^{1/p} d\mu(\theta) \right)^{-p} \sum_{k=1}^{\infty} w_k |y_k|^p.$$

By replacing y_k with $1/|x_k|$ for $k = 1, 2, \dots$, we have the result. \square

3. LOWER BOUND OF MATRIX OPERATORS

In this section, we deal with the problem of finding a lower bound of certain matrix operators from $l_p(w, I)$ into $l_p(w, F)$, which is considered for some matrix operators such as Cesàro, Copson and Hilbert operators in [2], [4] and [9] on $l_p(w)$ and on Lorentz sequence spaces $d(w, p)$. We generalize Theorem 1 from [4] to certain matrix operators from $l_p(w, I)$ into $l_p(w, F)$.

Lemma 3.1 ([4], Lemma 2). Let $p \geq 1$. Suppose that (a_j) , (x_j) are non-negative sequences and (x_j) is decreasing and tends to 0. Write $A_n = \sum_{j=1}^n a_j$ (with $A_0 = 0$),

and $B_n = \sum_{j=1}^n a_j x_j$. Then

(i) $B_n^p - B_{n-1}^p \geq (A_n^p - A_{n-1}^p)x_n^p$ for all n ;

(ii) if $\sum_{j=1}^{\infty} a_j x_j$ is convergent, then

$$\left(\sum_{j=1}^{\infty} a_j x_j \right)^p \geq \sum_{n=1}^{\infty} A_n^p (x_n^p - x_{n+1}^p).$$

Corollary 3.1. If (x_j) is a non-negative decreasing sequence and $X_n = x_1 + \dots + x_n$, then for each n

$$X_n^p - X_{n-1}^p \geq [n^p - (n-1)^p]x_n^p.$$

Proof is elementary. □

Theorem 3.1. Suppose $p \geq 1$ and let $A = (a_{i,j})$ be a matrix operator from $l_p(w, I)$ into $l_p(w, F)$ with non-negative entries. Write $r_{j,i} = \sum_{k=1}^i a_{j,k}$ and

$$S_i = \sum_{n=1}^{\infty} w_n \left(\sum_{j \in F_n} r_{j,i} \right)^p.$$

Then

$$L_{p,w,F}^p(A) = \inf_n \frac{S_n}{V_n}.$$

Proof. Let x be in $l_p(w, I)$ such that $x_1 \geq x_2 \geq \dots \geq 0$ and $m = \inf(S_n/V_n)$. Since $\sum_{i \in F_n} y_i = \sum_{j=1}^{\infty} c_{n,j} x_j$, where $c_{n,j} = \sum_{i \in F_n} a_{i,j}$, hence $\sum_{j=1}^k c_{n,j} = \sum_{i \in F_n} r_{i,k}$. Applying Lemma 3.1, we obtain

$$\begin{aligned} \|Ax\|_{p,w,F}^p &= \sum_{n=1}^{\infty} w_n \left(\sum_{j \in F_n} y_j \right)^p \geq \sum_{n=1}^{\infty} w_n \sum_{i=1}^{\infty} \left(\sum_{j \in F_n} r_{j,i} \right)^p (x_i^p - x_{i+1}^p) \\ &= \sum_{i=1}^{\infty} S_i (x_i^p - x_{i+1}^p). \end{aligned}$$

Since

$$\|x\|_{p,w,I}^p = \sum_{n=1}^{\infty} W_n (x_n^p - x_{n+1}^p),$$

we deduce that

$$\|Ax\|_{p,w,F}^p \geq m \|x\|_{p,w,I}^p.$$

Therefore

$$L_{p,w,F}^p(A) \geq m.$$

Further, if we take $x_1 = x_2 = \dots = x_n = 1$ and $x_k = 0$ for all $k \geq n + 1$, then

$$\|x\|_{p,w,I}^p = W_n, \quad \|Ax\|_{p,w,F}^p = S_n.$$

Hence

$$\overline{L}_{p,w,F}^p(A) \leq m.$$

This completes the proof of the theorem. \square

Corollary 3.2. *Suppose $A = (a_{i,j})$ is a matrix operator from $l_p(w, I)$ into itself with non-negative entries. We write $r_{i,n} = \sum_{j=1}^n a_{i,j}$ and $S_n = \sum_{i=1}^{\infty} w_i r_{i,n}^p$ and $W_n = w_1 + \dots + w_n$. Then*

$$L_{p,w,I}(A)^p = \inf_n \frac{S_n}{W_n}.$$

Note 3.1. For $p > 1$, the last part of Theorem 3.1 shows that $\|A\|_{p,w,F}^p \geq \sup_n (S_n/W_n)$, but $l_p(w, F) = l_p(w)$ when $F_i = \{i\}$ and equality does not hold (see [5]).

Write

$$t_n = \sum_{i=1}^{\infty} w_i \left(\sum_{j \in F_i} a_{j,n} \right)^p$$

and

$$s_n = S_n - S_{n-1} = \sum_{i=1}^{\infty} w_i \left[\left(\sum_{j \in F_i} r_{j,n} \right)^p - \left(\sum_{j \in F_i} r_{j,n-1} \right)^p \right],$$

where $S_n = s_1 + \dots + s_n$. For $p = 1$ we have $t_n = s_n$. It is elementary that $\inf_n (S_n/W_n) \geq \inf_n (s_n/w_n)$. We now apply Lemma 3.1 to deduce the following result.

Proposition 3.1. *If A satisfies all conditions mentioned in Theorem 3.1 and $(a_{i,j})$ decreases with j for each i , then*

$$L_{p,w,F}(A)^p \geq \inf_n [n^p - (n-1)^p] \frac{t_n}{w_n}.$$

Proof. Corollary 3.2 yields that

$$\left(\sum_{j \in F_i} r_{j,n} \right)^p - \left(\sum_{j \in F_i} r_{j,n-1} \right)^p \geq [n^p - (n-1)^p] \left(\sum_{j \in F_i} a_{j,n} \right)^p.$$

Thus

$$s_n \geq [n^p - (n-1)^p] \sum_{i=1}^{\infty} v_i \left(\sum_{j \in F_i} a_{j,n} \right)^p = [n^p - (n-1)^p] t_n$$

and so we have the statement. \square

Let $p \geq 1$ and let A be a matrix operator with non-negative entries. If $y = Ax$ and w is a decreasing sequence, then for each non-negative sequence x we have

$$\|Ax\|_{p,w,F}^p = \sum_{i=1}^{\infty} w_i \left(\sum_{j \in F_i} y_j \right)^p \geq \sum_{i=1}^{\infty} w_i \sum_{j \in F_i} y_j^p \geq \sum_{i=1}^{\infty} w_i y_i^p = \|Ax\|_{p,w}^p.$$

It follows that

$$L_{p,w,F}(A) \geq L_{p,w}(A).$$

Corollary 3.3. *Suppose that A is the Cesàro operator and $p \geq 1$. If $w_n = 1/n$, then*

$$L_{p,w,F}(A) \geq 1.$$

Proof. If we apply Theorem 4 from [4], we deduce that $L_{p,w}(A) = 1$ and so we have the statement. \square

The Copson matrix is an upper triangular matrix. We will solve the lower bound problem for this operator by the next statement. In fact, we characterize a class of operators for which the lower bound constant is equal to one.

Theorem 3.2. Assume that A is an upper triangular matrix, i.e. $a_{n,k} = 0$ for $n > k$, and $\sum_{n=1}^k a_{n,k} = 1$ for all k (in other words, A is a quasi-summability matrix). Let $p \geq 1$ and let $w = (w_n)$ be a non-negative decreasing sequence. Then

$$L_{p,w,F}(A) = 1.$$

Proof. If we apply Proposition 2 from [4], we have $L_{p,w}(A) = 1$. Hence $L_{p,w,F}(A) \geq L_{p,v}(A) = 1$. Since $1 \in F_1$ and $Ae_1 = e_1$, we deduce that

$$\|Ae_1\|_{p,w,F} = \|e_1\|_{p,w,I} = w_1.$$

This completes the proof of the theorem. \square

In the next statement, we consider the lower bound constant for the Hilbert matrix operator H .

Theorem 3.3. Suppose that H is the Hilbert matrix operator and $p \geq 1$. Let $F_i = \{2i - 1, 2i\}$ and $w_n = 1/n^\alpha$, where $0 < \alpha < 1$. Then

$$L_{p,w,F}(H)^p \geq \sum_{k=1}^{\infty} \frac{1}{k^\alpha(k+1/2)^p}.$$

Proof. Let $E_k = \{i \in Z : (k-1)n < i \leq kn\}$, where $k \geq 1$. If $i \in E_k$, then

$$\left(\frac{i}{n}\right)^\alpha (2i+n)^p \leq k^\alpha(2kn+n)^p.$$

Since E_k has n members,

$$n^{p+\alpha-1} \sum_{i \in E_k} \frac{1}{i^\alpha(2i+n)^p} \geq \frac{n^p}{k^\alpha(2kn+n)^p} = \frac{1}{k^\alpha(2k+1)^p}.$$

Hence

$$n^{p+\alpha-1} \sum_{k=1}^{\infty} \frac{1}{k^\alpha(2k+n)^p} \geq \sum_{k=1}^{\infty} \frac{1}{k^\alpha(2k+1)^p},$$

and also

$$\inf_n n^{p+\alpha-1} \sum_{k=1}^{\infty} \frac{1}{k^\alpha(2k+n)^p} = \sum_{k=1}^{\infty} \frac{1}{k^\alpha(2k+1)^p}.$$

We now apply Proposition 3.1, and with the above notation we have

$$t_n = \sum_{i=1}^{\infty} \frac{1}{i^\alpha} \left(\frac{1}{2i-1+n} + \frac{1}{2i+n} \right)^p,$$

and so

$$L_{p,w,F}(H)^p \geq \inf_n [n^p - (n-1)^p] n^\alpha t_n.$$

Since $n^p - (n-1)^p \geq n^{p-1}$, we deduce that

$$\begin{aligned} L_{p,w,F}(H)^p &\geq \inf_n n^{p+\alpha-1} \sum_{i=1}^{\infty} \frac{1}{i^\alpha} \left(\frac{1}{2i-1+n} + \frac{1}{2i+n} \right)^p \\ &\geq 2^p \inf_n n^{p+\alpha-1} \sum_{k=1}^{\infty} \frac{1}{k^\alpha (2k+n)^p} \\ &= 2^p \sum_{k=1}^{\infty} \frac{1}{k^\alpha (2k+1)^p}. \end{aligned}$$

This completes the proof of the theorem. □

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