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# WEAKLY CONNECTED DOMINATION STABLE TREES

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*Abstract.* A dominating set $D \subseteq V(G)$ is a *weakly connected dominating set* in $G$ if the subgraph $G[D]_w = (N_G[D], E_w)$ weakly induced by $D$ is connected, where $E_w$ is the set of all edges having at least one vertex in $D$. *Weakly connected domination number* $\gamma_w(G)$ of a graph $G$ is the minimum cardinality among all weakly connected dominating sets in $G$. A graph $G$ is said to be *weakly connected domination stable* or just $\gamma_w$*-stable* if $\gamma_w(G) = \gamma_w(G+e)$ for every edge $e$ belonging to the complement $\overline{G}$ of $G$. We provide a constructive characterization of weakly connected domination stable trees.

*Keywords*: weakly connected domination number, tree, stable graphs

*MSC 2010*: 05C05, 05C69

## 1. Introduction

Let $G = (V, E)$ be a connected undirected simple graph. The *neighbourhood* $N_G(v)$ of a vertex $v \in V(G)$ is the set of all vertices adjacent to $v$. For a set $X \subseteq V(G)$, the *neighbourhood* $N_G(X)$ is defined to be $\bigcup_{v \in X} N_G(v)$ and the *closed neighbourhood* $N_G[X]$ is $N_G(X) \cup X$. The degree of a vertex $v$ is $d_G(v) = |N_G(v)|$.

A subset $D$ of $V(G)$ is *dominating* in $G$ if every vertex of $V(G) - D$ has at least one neighbour in $D$. Let $\gamma(G)$ be the minimum cardinality among all dominating sets in $G$.

Subgraph weakly induced by a set $D \subseteq V(G)$ is the graph $G[D]_w = (N_G[D], E_w)$, where $E_w$ is the set of all edges having at least one vertex in $D$. A dominating set $D \subseteq V(G)$ is a *weakly connected dominating set* in $G$ if the subgraph weakly induced by $D$ is connected. Dunbar et al. [2] have defined the *weakly connected domination number* $\gamma_w(G)$ of a graph $G$ to be the minimum cardinality among all weakly connected dominating sets in $G$.

Let $n = n(G)$ be the order of a graph $G$ and let $n_1 = n_1(G)$ denote the number of leaves of $G$, that is the number of vertices of degree one. A vertex $v$ is called a *support vertex* if it is adjacent to a leaf.

It is easy to observe that for any graph $G$ we have $\gamma(G) - 1 \leqslant \gamma(G + e) \leqslant \gamma(G)$ for every edge $e \in E(\overline{G})$. Sumner and Blitch [1] have defined domination critical graphs. A graph $G$ is said to be *domination critical*, or just *$\gamma$-critical*, if $\gamma(G) = \gamma$ and $\gamma(G + e) = \gamma - 1$ for every edge $e$ in the complement $\overline{G}$ of $G$.

A graph is said to be *domination stable*, or just *$\gamma$-stable*, if $\gamma(G) = \gamma(G + e)$ for every edge $e$ in the complement $\overline{G}$ of $G$.

A subset $D$ of $V(G)$ is *connected dominating* in $G$ if $D$ is dominating and a subgraph $G[D]$ induced by $D$ is connected. Let $\gamma_c(G)$ be the minimum cardinality among all connected dominating sets in $G$.

In [4] X. Chen et al. defined the connected domination critical graphs. A graph $G$ is said to be *connected domination critical* in the following sense: $\gamma_c(G+vu) < \gamma_c(G)$ for each $u, v \in V(G)$ with $v$ not adjacent to $u$.

We define the graph $G$ to be *weakly connected domination stable* (*$\gamma_w$-stable*) if $\gamma_w(G + vu) = \gamma_w(G)$ for each $u, v \in V(G)$ with $v$ not adjacent to $u$.

In this paper we characterize all weakly connected domination stable trees.

## 2. Results

We begin with the following lemma.

**Lemma 1.** *If $T$ is a tree and $D \subseteq V(T)$, then $D$ is a weakly connected dominating set of $T$ if and only if the set $V(T) - D$ is independent.*

P r o o f. Let $D$ be a weakly connected dominating set of $T$ and suppose there is an edge $uv \in E(T)$ such that $u, v \in V(T) - D$. Since $D$ is dominating, $N_T(u) \cap D \neq \emptyset$, $N_T(v) \cap D \neq \emptyset$ and, since $T$ is a tree, $N_T(u) \cap N_T(v) = \emptyset$. Let $u' \in N_T(u) \cap D$ and $v' \in N_T(v) \cap D$. Since $D$ is weakly connected, there is an $(u' - v')$-path $P$ such that $u, v \notin P$, what produces a cycle and gives contradiction.

Now let $D$ be a subset of $V(T)$ such that $V(T) - D$ is independent. Suppose $D$ is not weakly connected dominating set of $T$. If $D$ is not weakly connected, then $T[D]_w$ is not connected and there is an edge $uv$ such that $u, v \notin D$. Then $u, v \in V(T) - D$ and $V(T) - D$ is not independent. If $D$ is not dominating, then there is a vertex $x \in V(T) - D$ which has no neighbour in $D$. Since $G$ is connected, $x$ has a neighbour in $V(T) - D$ and again $V(T) - D$ is not independent. □

In [5] the following theorem was proved:

**Theorem 2.** *If $G$ is a connected graph, then $\gamma_w(G) - 1 \leqslant \gamma_w(G+e) \leqslant \gamma_w(G)$ for every edge $e \in E(\overline{G})$.*

**Corollary 3.** *If $G$ is $\gamma_w$-critical, then $\gamma_w(G) = \gamma_w(G + e) + 1$ for every edge $e \in E(\overline{G})$.*

We are now in position to constructively characterize all $\gamma_w$-stable trees. To this aim we define some operations and a family of trees, similarly to [3].

If $T$ is a tree, then we define the status of a vertex $v \in V(T)$, denoted $\mathrm{sta}(v)$, to be $A$ or $B$. Let $\mathscr{T}^*$ be a family of trees with a status coloring that can be obtained from a sequence $T_1, \ldots, T_j$ $(j > 1)$ of trees with a status coloring such that $T_1$ is a star $K_{1,s}$ for $s \geqslant 2$, where initially $\mathrm{sta}(v) = A$ for the central vertex $v$ of $T_1$, $\mathrm{sta}(u) = B$ for every leaf $u$ of $T_1$ and $T = T_j$, and, if $j \geqslant 2$, then $T_{i+1}$ can be obtained from $T_i$ by one of two operations $\mathscr{X}$ and $\mathscr{Y}$ listed below. Once a vertex is assigned a status, this status remains unchanged as the tree is recursively constructed.

Intuitively, if a vertex $v$ has status $A$ or $B$ in a $\gamma_w$-stable tree with a status coloring, then using we construct a new $\gamma_w$-stable tree with a status coloring by adding certain stars using one of the operations $\mathscr{X}$ and $\mathscr{Y}$.

- **Operation** $\mathscr{X}$: The tree $T_{i+1}$ is obtained from $T_i$ by adding a star $K_{1,r}$ for $r \geqslant 2$ and an edge $uv$, where $u$ is a vertex of $T_i$ such that $\mathrm{sta}(u) = A$ and $v$ is the center of $K_{1,r}$, and letting $\mathrm{sta}(v) = A$ and $\mathrm{sta}(x) = B$ for each leaf $x$ from $K_{1,r}$.
- **Operation** $\mathscr{Y}$: The tree $T_{i+1}$ is obtained from $T_i$ by adding a star $K_{1,r}$ for $r \geqslant 1$ and an edge $uv$, where $u$ is a vertex of $T_i$ such that $\mathrm{sta}(u) = B$ and $v$ is the center of $K_{1,r}$, and letting $\mathrm{sta}(v) = A$ and $\mathrm{sta}(x) = B$ for each leaf $x$ from $K_{1,r}$. If $r = 1$, then we take one vertex of $K_{1,1}$ to be a center and the other one to be a leaf.

Let $\mathscr{T}$ be a family of all trees $T$ for which there exists a status coloring of $T$ such that $T$ with this status coloring belongs to $\mathscr{T}^*$.

**Lemma 4.** *If $T$ is a tree belonging to the family $\mathscr{T}$, then there is the unique minimum weakly connected dominating set of $T$.*

P r o o f. Let $T$ be a tree belonging to the family $\mathscr{T}$. Then there exists a status coloring of $T$ such that $T$ with this status coloring belongs to $\mathscr{T}^*$. Assume there are $k$ vertices with status $A$ in $T$. Then $D = \{a_1, \ldots, a_k\}$, where $\mathrm{sta}(a_i) = A$ for $i = 1, \ldots, k$ is the unique minimum weakly connected dominating set of $T$. □

**Lemma 5.** *If $T$ is a tree with at least three vertices and $D$ is the unique minimum $\gamma_w$-set in $T$, then $D$ contains no leaves.*

P r o o f. Suppose there is a leaf $v \in D$, where $D$ is the unique minimum weakly connected dominating set of $T$. Then $(D - \{v\}) \cup \{u\}$, where $u$ is the only neighbour of $v$, is a weakly connected dominating set of $T$, a contradiction. $\square$

**Theorem 6.** *If $T$ is a tree with at least three vertices, then $T$ belongs to the family $\mathscr{T}$ if and only if there is a unique minimum weakly connected dominating set in $T$.*

P r o o f. Denote by $T^*$ the tree $T$ with a status coloring such that $T^* \in \mathscr{T}^*$. If $T$ belongs to the family $\mathscr{T}$, then the result follows from Lemma 1. Let $T$ be a tree with at least three vertices and assume there is the unique minimum weakly connected dominating set in $T$. We use induction on $\gamma_w(T)$, the weakly connected domination number of $T$.

If $\gamma_w(T) = 1$, then $T$ is a star with at least two leaves and of course $T \in \mathscr{T}$. Assume $\gamma_w(T) > 1$ and let $P = (v_0, \ldots, v_l)$ be a longest path in $T$. Since $\gamma_w(T) > 1$, we have $l \geqslant 3$. Let $D$ be the minimum weakly connected dominating set in $T$. From Lemma 5 we have $v_0 \notin D$. Thus $v_1 \in D$.

We now consider three possibilities depending on the degree of $v_2$. Let $T_1$ be the tree obtained from $T$ by removing $v_1$ and all of its neighbours except for $v_2$ and denote by $T_1^*$ the tree $T_1$ with a status coloring such that $T_1^* \in \mathscr{T}^*$. It is possible to observe that there is a unique minimum weakly connected dominating set in $T_1$ and $\gamma_w(T_1) < \gamma_w(T)$. Thus by the induction hypothesis, $T_1$ belongs to the family $\mathscr{T}$.

*Case* 1. If $v_2$ is a support vertex, then $v_2 \in D$. Moreover, if $d_T(v_1) = 2$, then $D - \{v_1\} \cup \{v_0\}$ would be another $\gamma_w(T)$-set, which gives a contradiction. Hence $d_T(v_1) > 2$ and $T^*$ may be obtained from $T_1^*$ by Operation $\mathscr{X}$.

*Case* 2. If $d_T(v_2) > 2$ and $v_2$ is not a support vertex, then $\mathrm{sta}(v_2) = B$ and $T^*$ may be obtained from $T_1^*$ by Operation $\mathscr{Y}$.

*Case* 3. If $d_T(v_2) = 2$, then $v_2$ is a leaf of $T_1$ and $T^*$ may be obtained from $T_1^*$ by Operation $\mathscr{Y}$. $\square$

**Theorem 7.** *A tree $T$ is $\gamma_w$-stable if and only if there is a unique minimum weakly connected dominating set in $T$.*

P r o o f. Let $T$ be a tree. Suppose there is a unique minimum weakly connected dominating set in $T$ and $T$ is not $\gamma_w$-stable. Then there is an edge $uv \in E(\overline{T})$ such that $\gamma_w(T') < \gamma_w(T)$, where $T' = T + uv$. Observe that by Corollary 1 $\gamma_w(T') + 1 = \gamma_w(T)$. Let $D'$ be a minimum weakly connected dominating set of $T'$. We consider three cases.

*Case* 1. If $u, v \notin D'$, then $D'$ is a weakly connected dominating set in $T$ and $\gamma_w(T) \leqslant |D'| = \gamma_w(T')$, which gives a contradiction.

*Case* 2. If $u, v \in D'$, then if $D'$ is weakly connected in $T$, then similarly to Case 1 we obtain a contradiction. If $D'$ is not weakly connected in $T$, then there is exactly one $(u - v)$-path in $T'[D']_w$. Hence there exists an edge $xy$ in $T'$ such that neither of the vertices $x, y$ belongs to $D'$ and $x, y$ belong to the unique $(u - v)$-path in $T$.

Thus $D_1 = D' \cup \{x\}$ and $D_2 = D' \cup \{y\}$ are weakly connected dominating sets in $T$ and $|D_1| = |D_2| = \gamma_w(T)$, which gives a contradiction with the fact that there exists exactly one minimum weakly connected dominating set in $T$.

*Case* 3. Exactly one of the vertices of $\{u, v\}$ does not belong to $D'$, assume $u \in D', v \notin D'$. If $D'$ is a weakly connected dominating set of $T$, then similarly to *Case* 1 we obtain a contradiction. If $D'$ is dominating, but not weakly connected in $T$, we again obtain a contradiction, similarly to *Case* 2. Thus assume that $D'$ is not dominating in $T$. Then $u$ is the unique neighbour of $v$ in $T'$ belonging to $D'$. Since $T$ is a tree, $T'$ is a unicyclic graph and for this reason at most one edge of $T'$ is not incident with a vertex of $D'$. In this way we conclude that $v$ is a leaf of $T$ and $D'$ is a weakly connected set in $T$. Hence $D' \cup \{v\}$ and $D' \cup \{z\}$, where $z$ is the neighbour of $v$ in $T$, are two distinct weakly connected dominating sets of cardinality $\gamma_w(T)$ in $T$, a contradiction.

Now we show that if $T$ is $\gamma_w$-stable, then there exists exactly one minimum weakly connected dominating set in $T$. Suppose to the contrary that there are at least two $\gamma_w(T)$-sets, say $D_1$ and $D_2$. Then $|D_1 \oplus D_2| \geqslant 2$, where $D_1 \oplus D_2 = (D_1 - D_2) \cup (D_2 - D_1)$.

**Claim 1.** *Every vertex belonging to $D_1 - D_2$ has a neighbour in $D_2 - D_1$ and every vertex belonging to $D_2 - D_1$ has a neighbour in $D_1 - D_2$.*

Suppose this is not true, let $u \in D_1 - D_2$ and $N_T(u) \cap (D_2 - D_1) = \emptyset$. Then of course $u \notin D_2$, but from Observation 1, every neighbour of $u$ belongs to $D_2$. Since $N_T(u) \cap (D_2 - D_1) = \emptyset$, we have $N_T(u) \subseteq D_1$. But then $D_1 - \{u\}$ is a smaller weakly connected dominating set of $T$, which gives a contradiction. □

Since $T$ is a tree, Claim 1 implies that $T[D_1 \oplus D_2]$ is a non-trivial forest. Let $u$ be a leaf of $T[D_1 \oplus D_2]$. Without loss of generality let $u \in D_1 - D_2$ and let $v$ be the neighbour of $u$ such that $v \in D_2 - D_1$. Let us choose $v$ such that $v$ is not a leaf of $T$ (if $v$ is a leaf of $T$, then we can take $u$ instead of $v$ and $v$ instead of $u$).

Let $x$ be a neighbour of $v$ such that $x \neq u$. Since $D_1$ is weakly connected, $x \in D_1$ and since $T$ is a tree, $ux \notin E(T)$. For this reason, $D = D_1 - \{u\}$ is a weakly connected dominating set of $T + ux$ and $\gamma_w(T + ux) \leqslant |D| < \gamma_w(T)$, which contradicts the fact that $T$ is $\gamma_w$-stable.

**Lemma 8.** *If there is the unique maximum independent set in $T$, then also there is the exactly one minimum weakly connected dominating set in $T$.*

P r o o f. Let $D \subseteq V(T)$ such that $V(T) - D$ is the unique maximum independent set of $T$. Since $V(T) - D$ is independent, from Lemma 1 $D$ is weakly connected. Since $V(T) - D$ is maximal, $D$ is a minimum weakly connected dominating set of $T$. If $D$ is not the unique minimum weakly connected dominating set of $T$, $V(T) - D$ is not the unique maximum independent set of $T$, what gives a contradiction. Hence $D$ is exactly one minimum weakly connected dominating set in $T$. □

**Corollary 9.** *Let $T$ be a tree of order at least three. Then the following conditions are equivalent:*

- *$T$ belongs to the family $\mathscr{T}$;*
- *$T$ is $\gamma_w$-stable;*
- *there is exactly one minimum weakly connected dominating set in $T$;*
- *there is a unique maximum independent set in $T$.*

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