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DECOMPOSITION OF BIPARTITE GRAPHS INTO CLOSED TRAILS

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Abstract. Let $\mathrm{Lct}(G)$ denote the set of all lengths of closed trails that exist in an even graph G. A sequence $(t_1,\ldots,t_p)$ of elements of $\mathrm{Lct}(G)$ adding up to $|E(G)|$ is G-realisable provided there is a sequence $(T_1,\ldots,T_p)$ of pairwise edge-disjoint closed trails in G such that T_i is of length t_i for $i=1,\ldots,p$. The graph G is arbitrarily decomposable into closed trails if all possible sequences are G-realisable. In the paper it is proved that if $a\geqslant 1$ is an odd integer and $M_{a,a}$ is a perfect matching in $K_{a,a}$, then the graph $K_{a,a}-M_{a,a}$ is arbitrarily decomposable into closed trails.

Keywords: even graph, closed trail, graph arbitrarily decomposable into closed trails, bipartite graph

MSC 2010: 05C70

All graphs we are dealing with in this paper are simple, finite and nonoriented. We use the standard terminology and notation of graph theory.

For $p,q\in\mathbb{Z}$ let $[p,q]$ denote the *integer interval* bounded by p and q, i.e. $[p,q]:=\{z\in\mathbb{Z}\colon p\leqslant z\leqslant q\}$; similarly, let $[p,\infty):=\{z\in\mathbb{Z}\colon p\leqslant z\}$. The *concatenation* of finite sequences $A=(a_1,\ldots,a_m)$ and $B=(b_1,\ldots,b_n)$ is the sequence $AB:=(a_1,\ldots,a_m,b_1,\ldots,b_n)$. The concatenation is an associative operation on finite sequences; we use this fact in the notation $\prod_{i=1}^{k}A_i$ representing the concatenation of finite sequences A_i, $i\in[1,k]$, in the order given by the sequence $(A_1,\ldots,A_k)$. As usual, A^k denotes $\prod_{i=1}^{k}A_i$ with $A_i=A$ for any $i\in[1,k]$, and A^0 is the empty sequence $(\ )$. A finite sequence $A=(a_1,\ldots,a_m)$ is *changeable* to a finite sequence $A'=(a'_1,\ldots,a'_m)$ of the same length (in symbols $A\sim A'$) if there is a bijection

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$\pi \subseteq [1,m] \times [1,m]$ such that $a'_i = a_{\pi(i)}$ for any $i \in [1,m]$. If $I \subseteq [1,m]$, we denote by $A\langle I\rangle$ the subsequence of A formed by all a_i's with $i \in I$ (ordered in compliance with the natural ordering of I).

A *closed trail* of length $n \in [3,\infty)$ (an *n-trail* for short) is a sequence $\prod_{i=1}^{n+1}(x_i)$ of vertices of G such that $x_1 = x_{n+1}$ and if $i,j \in [1,n]$, $i \neq j$, then $\{x_i, x_{i+1}\} \in E(G)$ and $\{x_i, x_{i+1}\} \neq \{x_j, x_{j+1}\}$. A graph G is Eulerian if it has a closed trail of length $|E(G)|$. It is well known that a graph of order at least three is Eulerian if and only if it is connected and *even* (all its vertices are of even degrees). Thus, we may identify the notions of a closed trail in a graph G and a nontrivial connected even subgraph of G. Let $\mathrm{Lct}(G)$ be the set of all lengths of closed trails existing in G and let $\mathrm{Sct}(G)$ be the set of all finite sequences consisting of elements of $\mathrm{Lct}(G)$ that add up to $|E(G)|\}$. Deleting a closed trail from an even graph G yields an even subgraph of G. Continuing this process until all edges of G are exhausted leads to a sequence $\widetilde{\mathcal{T}} := (\widetilde{T}_1, \ldots, \widetilde{T}_p)$ of pairwise edge-disjoint closed trails in G such that, for any $i \in [1,p]$, $\tilde{t}_i := |E(\widetilde{T}_i)| \in \mathrm{Lct}(G)$, and $\tilde{\tau} := (\tilde{t}_1, \ldots, \tilde{t}_p) \in \mathrm{Sct}(G)$; the sequence $\tilde{\tau}$ is said to be *G-realisable* and the sequence $\widetilde{\mathcal{T}}$ is a *G-realisation* of the sequence $\tilde{\tau}$. An even graph G is *arbitrarily decomposable into closed trails* (ADCT) provided all sequences of $\mathrm{Sct}(G)$ are G-realisable.

There are some classes of even graphs that are known to be ADCT. Among these are complete graphs K_n for n odd, the graphs $K_n - M_n$, where M_n is a perfect matching in K_n, for n even (Balister [1]) and complete bipartite graphs $K_{a,b}$ for a, b even (Horňák and Woźniak [8]). An even graph that is large and dense enough is necessarily ADCT. Namely, according to Balister [2], there are positive constants n and ε such that an even graph G is ADCT whenever $|V(G)| \geqslant n$ and $\delta(G) \geqslant (1-\varepsilon)|V(G)|$. Horňák and Kocková [7] proved that if an even complete tripartite graph $K_{p,q,r}$ with $p \leqslant q \leqslant r$, is ADCT, then either $(p,q,r) \in \{(1,1,3),(1,1,5)\}$ or $p = q = r$; moreover, the graphs $K_{1,1,3}$, $K_{1,1,5}$ and $K_{p,p,p}$ with $p = 5 \cdot 2^l$, $l \in [0,\infty)$, are ADCT. The notion of an ADCT graph can be generalized in a natural way to oriented graphs (see Balister [3] and Cichacz [5]) and to pseudographs (Cichacz et al. [6]).

It may happen that an even graph is not ADCT though all its connected components are. For example, both C_8 (an 8-vertex cycle) and $K_{2,4}$ are ADCT, but $C_8 \cup K_{2,4}$ is not since the sequence $(4)^4 \in \mathrm{Sct}(C_8 \cup K_{2,4})$ is not $(C_8 \cup K_{2,4})$-realisable. On the other hand, if the graphs G^1, G^2 are ADCT and $E(G^1) \cap E(G^2) = \emptyset$, but $V(G^1) \cap V(G^2) \neq \emptyset$, when trying to prove that a sequence $\tau \in \mathrm{Sct}(G^1 \cup G^2)$ is $(G^1 \cup G^2)$-realisable, we have at our disposal not only closed trails of G^1 and G^2, but also closed trails $T^1 \cup T^2$, where T^i is a closed trail of G^i, $i = 1,2$, and $V(T^1) \cap V(T^2) \neq \emptyset$. Therefore, a potential general strategy for proving that a graph G is ADCT can be described as follows: Write G as an edge-disjoint, but not vertex-disjoint, union of ADCT graphs G^1 and G^2, and require from G^i-realisations, $i = 1, 2$, to have an additional property that some of their chosen trails contain common vertices of $V(G^1) \cap V(G^2)$.

Clearly, when analyzing whether a nontrivial connected even graph G is ADCT, it is sufficient to show that any sequence $(t_1, \ldots, t_p) \in \mathrm{Sct}(G)$ of length $p \geqslant 2$ is G-realisable; indeed, the graph G is Eulerian, and so the unique sequence $(|E(G)|)$ of length 1 in $\mathrm{Sct}(G)$ is trivially G-realisable. We have also the following evident statement:

Lemma 1. *If G is an even graph, $\tau_1, \tau_2 \in \mathrm{Sct}(G)$ and $\tau_1 \sim \tau_2$, then the sequence τ_1 is G-realisable if and only if τ_2 is.*

Pick disjoint sets $X^j = \{x_i^j \colon i \in [1, \infty)\}$, $j = 1, 2$, and let $X_{p,q}^j := \{x_i^j \colon i \in [p, q]\}$ for $p, q \in [1, \infty)$. In this paper the complete bipartite graph $K_{a,b}$ will have the bipartition $\{X_{1,a}^1, X_{1,b}^2\}$ and $M_{a,a}$ will be the perfect matching in $K_{a,a}$ consisting of $\{x_i^1, x_i^2\}$ for $i \in [1, a]$. If a is odd, then $K'_{a,a} := K_{a,a} - M_{a,a}$ is an even graph. The main aim of our paper is to show that the graph $K'_{a,a}$ is ADCT for any odd $a \in [1, \infty)$. We proceed by induction on a and we use the above general strategy. For odd $a \geqslant 7$ consider the even subgraph $F_a \cong K'_{a-4,a-4}$ of $K'_{a,a}$ induced on the set $X_{5,a}^1 \cup X_{5,a}^2$. The even graph $H_a := K'_{a,a} - F_a$ is an edge-disjoint union of the even graph $K'_{5,5}$ and two even subgraphs $G_a^1 \cong G_a^2 \cong K_{4,a-5}$ of $K'_{a,a}$ where G_a^i is induced on the set $X_{1,4}^i \cup X_{6,a}^{3-i}$, $i = 1, 2$. Thus putting $G_a := K'_{5,5} \cup G_a^1$ we obtain $H_a = G_a \cup G_a^2$. We shall show subsequently that the graphs $K'_{5,5}$ and G_a, H_a are ADCT; furthermore, G_a-realisations and H_a-realisations can be chosen to have appropriate additional properties. Note that all the graphs mentioned are bipartite. The following assertion shows the maximal extent of the set $\mathrm{Lct}(G)$ for an even bipartite graph G.

Proposition 2. *If G is an even bipartite graph, then* $\mathrm{Lct}(G) \subseteq \{2k \colon k \in [2, |E(G)|/2 - 2]\} \cup \{|E(G)|\}$.

P r o o f. All subgraphs of G are bipartite, hence all closed trails in G (as edge-disjoint unions of cycles) are of even lengths. A subgraph T of G with $|E(T)| = |E(G)| - 2$ is not even (and therefore not a closed trail) for $G - T$ has at least two vertices of degree one. □

When proving that an even bipartite graph G is ADCT we do not exhibit the structure of $\mathrm{Lct}(G)$ explicitly, but we show implicitly that $\mathrm{Lct}(G)$ is of maximal extent by finding all G-realisations that are theoretically possible from the point of view of Proposition 2.

Recall again the result on complete bipartite graphs:

Theorem 3. *If a, b are even integers with $2 \leqslant a \leqslant b$, then the graph $K_{a,b}$ is* ADCT.

We know due to Chou et al. [4] that sequences of $\mathrm{Sct}(K_{a,b})$ with small terms have $K_{a,b}$-realisations consisting of cycles:

Theorem 4. *If a, b are even integers with $a \geqslant 4$, $b \geqslant 6$ and $\tau = (t_1, \dots, t_p) \in \mathrm{Sct}(K_{a,b})$ with $t_i \in \{4, 6, 8\}$ for any $i \in [1, p]$, then there is a $K_{a,b}$-realisation $(T_1, \dots, T_p)$ of the sequence τ such that T_i is a cycle for any $i \in [1, p]$.*

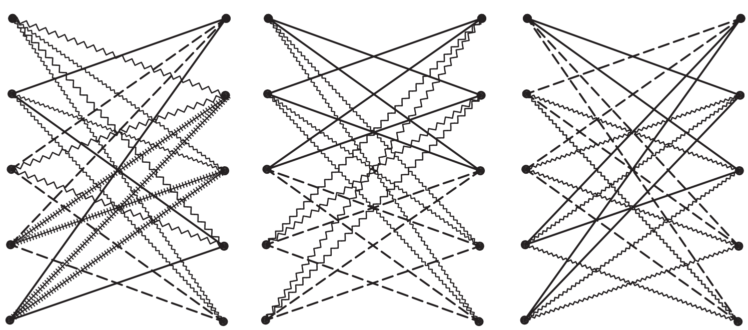
We start our analysis by dealing with $a \leqslant 5$.

Proposition 5. *The graph $K'_{a,a}$ with $a \in \{1, 3, 5\}$ is* ADCT.

Proof. We have $K'_{1,1} \cong 2K_1$, and so for $a = 1$ the result follows from $\mathrm{Sct}(K'_{1,1}) = \mathrm{Lct}(K'_{1,1}) = \emptyset$.

Since $K'_{3,3} \cong C_6$, the unique sequence $(6) \in \mathrm{Sct}(K'_{3,3})$ is trivially $K'_{3,3}$-realisable.

The sequences $(4)^5$, $(4)^2(6)^2$ and $(6)^2(8)$ are $K'_{5,5}$-realisable, see Figure 1. Observe that any two 4-trails of the left $K'_{5,5}$-realisation have a common vertex, hence every sequence in $\mathrm{Sct}(K'_{5,5})$, whose all terms are divisible by 4, is $K'_{5,5}$-realisable. Moreover, in the middle $K'_{5,5}$-realisation any 4-trail has a common vertex with any 6-trail. Therefore, the remaining sequences $(4, 6, 10), (6, 14), (10)^2 \in \mathrm{Sct}(K'_{5,5})$ are $K'_{5,5}$-realisable, too. □



$(4)^5$ $(4)^2(6)^2$ $(6)^2(8)$

Figure 1. $K'_{5,5}$-realisations of three sequences

We shall need also the following three simple statements:

Proposition 6. *If G is a complete bipartite graph with bipartition $\{X, Y\}$ and $\pi \subseteq X \times X$, $\varrho \subseteq Y \times Y$ are bijections, then the mapping $\alpha \subseteq V(G) \times V(G)$ with $\alpha|X = \pi$ and $\alpha|Y = \varrho$ is an automorphism of G.*

Proposition 7. *If $a \in [1, \infty)$ and $\pi \subseteq [1, a] \times [1, a]$ is a bijection, then the mappings $\overline{\pi}, \tilde{\pi} \subseteq V(K'_{a,a}) \times V(K'_{a,a})$, determined by $\overline{\pi}(x_i^j) = x_{\pi(i)}^j$ and $\tilde{\pi}(x_i^j) = x_{\pi(i)}^{3-j}$ for any $i \in [1, a]$ and $j \in [1, 2]$, are automorphisms of $K'_{a,a}$.*

Lemma 8. *If T_1, T_2 are edge-disjoint closed trails in $K'_{5,5}$ and $k \in [1, 2]$, then $|(V(T_1) \cup V(T_2)) \cap X_{1,5}^k| \geqslant 3$.*

Proof. If $|E(T_1) \cup E(T_2)| \geqslant 10$, then the edges of $E(T_1) \cup E(T_2)$ must cover at least $\lceil \frac{10}{4} \rceil = 3$ vertices of $X_{1,5}^k$ (note that $\Delta(K'_{5,5}) = 4$). The same is true if both T_1 and T_2 are 4-trails, since then the subgraph of $K'_{5,5}$ that is induced by the eight edges incident with x_i^k or x_j^k, $i, j \in [1, 5]$, $i \neq j$, has two vertices of degree 1 (namely x_i^{3-k} and x_j^{3-k}), and so it cannot be equal to $T_1 \cup T_2$. □

Theorem 9. *The graph G_a is* ADCT *for any odd integer $a \geqslant 7$. Moreover, given $s \in [4, 5]$, any sequence $\tau = (t_1, \ldots, t_p) \in \mathrm{Sct}(G_a)$ of length $p \geqslant 2$ has a G_a-realisation $(T_1, \ldots, T_p)$ such that T_1 contains as a subgraph a 3-vertex path with endvertices x_1^2 and x_s^2 and T_2 contains the vertex x_2^2.*

Proof. We use the general strategy with ADCT graphs $G^1 := K'_{5,5}$ (Proposition 5) and $G^2 := G_a^1$ (Theorem 3); the structure of the graph G_a is presented in Figure 2.

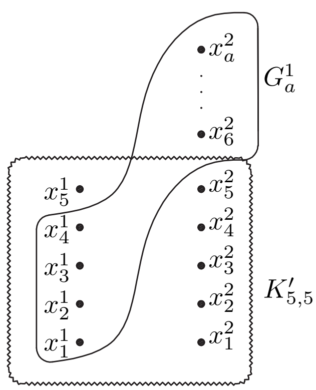


Figure 2. The graph G_a

First we show how to proceed provided three special conditions are fulfilled.

(C1) If there is I^1 with $[1,2] \subseteq I^1 \subseteq [1,p]$ and $\sum_{i\in I_1} t_i = |E(G^1)| = 20$, put $I^2 := [1,p] - I^1$ and $\tau^l := \tau\langle I^l\rangle$, $l = 1,2$. There is a G^1-realisation $(T_1, T_2)\mathcal{T}^1$ of the sequence τ^1 and a G^2-realisation $\mathcal{T}^2$ of the sequence τ^2. Then $\mathcal{T} := (T_1, T_2)\mathcal{T}^1\mathcal{T}^2$ is a G_a-realisation of the sequence $\tau^1\tau^2 \sim \tau$. Any closed trail in a bipartite graph with bipartition $\{U, V\}$ is an alternating sequence of vertices of U and V. Therefore, by Proposition 7 and Lemma 8, we may suppose without loss of generality that the trails T_1 and T_2 have the required properties.

(C2) If there are I^1 and $j \in [1,p] - I^1$ such that $[1,2] \subseteq I^1 \cup \{j\}$, $\sum_{i\in I^1} t_i \leqslant 16$ and $\sum_{i\in I_1} t_i + t_j \geqslant 24$, put $I^2 := [1,p] - I^1 - \{j\}$, $t_j^1 := 20 - \sum_{i\in I_1} t_i$ and $t_j^2 := \sum_{i\in I^1} t_i + t_j - 20$. There is a G^l-realisation $(T_j^l)\mathcal{T}^l$ of the sequence $(t_j^l)\tau\langle I^l\rangle \in \mathrm{Sct}(G^l)$, $l = 1,2$; for $i \in [1,2] - \{j\} \subseteq I^1$ let T_i be a t_i-trail of $\mathcal{T}^1$. Using Propositions 6, 7 and Lemma 8 we may suppose without loss of generality that T_1 (or T_1^1 if $j = 1$) contains as a subgraph a 3-vertex path with endvertices x_1^2 and x_s^2, T_2 (or T_2^1 if $j = 2$) contains the vertex x_2^2 and $V(T_j^1) \cap V(T_j^2) \cap X_{1,4}^1 \neq \emptyset$. Then $T_j := T_j^1 \cup T_j^2$ is a t_j-trail and $(T_j)\mathcal{T}^1\mathcal{T}^2$ is an appropriate G_a-realisation of the sequence $(t_j)\tau\langle I^1\rangle\tau\langle I^2\rangle \sim \tau$.

(C3) If there are I^1 and $\{j,k\} \subseteq [1,p] - I^1$ such that $[1,2] \subseteq I^1 \cup \{j,k\}$, $\min\{t_j, t_k\} \geqslant 8$, $\sum_{i\in I^1} t_i \leqslant 12$ and $\sum_{i\in I_1} t_i + t_j + t_k \geqslant 28$, put $I^2 := [1,p] - I^1 - \{j,k\}$, $t_j^1 := \min\Big\{16 - \sum_{i\in I^1} t_i, t_j - 4\Big\}$, $t_k^1 := \max\Big\{4, 24 - \sum_{i\in I^1} t_i - t_j\Big\}$, $t_j^2 := t_j - t_j^1$ and $t_k^2 := t_k - t_k^1$. Then $t_j^l + t_k^l + \sum_{i\in I^l} t_i = |E(G^l)|$ and there is a G^l-realisation $(T_j^l, T_k^l)\mathcal{T}^l$ of the sequence $(t_j^l, t_k^l)\tau\langle I^l\rangle$, $l = 1,2$; for $i \in [1,2] - \{j,k\} \subseteq I^1$ let T_i be a t_i-trail of $\mathcal{T}^1$. By Propositions 6, 7 and Lemma 8 we may suppose without loss of generality that T_1 (or T_1^1 if $1 \in \{j,k\}$) contains as a subgraph a 3-vertex path with endvertices x_1^2 and x_s^2, T_2 (or T_2^1 if $2 \in \{j,k\}$) contains the vertex x_2^2 and $V(T_m^1) \cap V(T_m^2) \cap X_{1,4}^1 \neq \emptyset$ for any $m \in \{j,k\}$. Then $T_m := T_m^1 \cup T_m^2$ is a t_m-trail, $m = j,k$ and $(T_j, T_k)\mathcal{T}^1\mathcal{T}^2$ is a required G_a-realisation of the sequence $(t_j, t_k)\tau\langle I^1\rangle\tau\langle I^2\rangle \sim \tau$.

Let $i_1, i_2 \in [1,2]$ be such that $i_1 \neq i_2$ and $t_{i_1} \leqslant t_{i_2}$. Since there are no additional requirements on T_i with $i \in [3,p]$, having in mind Lemma 1, in our analysis we may suppose without loss of generality that $t_i \leqslant t_{i+1}$ for any $i \in [3, p-1]$.

(1) $t_1 + t_2 \geqslant 24$.

(11) If $t_{i_1} \geqslant 18$, then $I^1 := \emptyset$, $j := 1$, $k := 2 \to$ (C3), i.e. the condition (C3) is satisfied with the presented values of I^1, j and k.

(12) If $t_{i_1} \leqslant 16$, then $I^1 := \{i_1\}$, $j := i_2 \to$ (C2).

(2) If $t_1 + t_2 = 22$, then $t_{i_1} \leqslant 10$, $t_{i_2} \geqslant 12$ and $\sum_{i=3}^{p} t_i = 4a - 22 \equiv 2 \pmod 4$, hence there is $l \in [3,p]$ with $t_l \equiv 2 \pmod 4$.

(21) If $t_p \geqslant 8$, then $I^1 := \{i_1\}$, $j := i_2$, $k := p \to$ (C3).

(22) If $t_p(=t_l)=6$, then $I^1:=\{i_1,p\}$, $j:=i_2\to$ (C2).

(3) If $t_1+t_2=20$, then $I^1:=[1,2]\to$ (C1).

(4) If $t_1+t_2=18$, then $t_{i_1}\leqslant 8$, $t_{i_2}\geqslant 10$ and there is $l\in[3,p]$ with $t_l\equiv 2 \pmod 4$.

(41) If $t_l\geqslant 10$, then $I^1:=\{i_1\}$, $j:=i_2$, $k:=l\to$ (C3).

(42) If $t_l=6$, then $I^1:=\{i_1,l\}$, $j:=i_2\to$ (C2).

(5) If $t_1+t_2\leqslant 16$, let $q\in[2,p-1]$ be determined by the inequalities $\sum\limits_{i=1}^{q} t_i\leqslant 22$ and $\sum\limits_{i=1}^{q+1} t_i\geqslant 24$.

(51) If $\sum\limits_{i=1}^{q} t_i=22$, then $q\geqslant 3$ and there is $l\in[q+1,p]$ with $t_l\equiv 2 \pmod 4$.

(511) $t_q\geqslant 6$.

(5111) If $t_p\geqslant t_q+2$, then $I^1:=[1,q-1]$, $j:=p\to$ (C2).

(5112) If $t_i=t_q$ for any $i\in[q+1,p]$, then $t_q=t_l\equiv 2 \pmod 4$.

(51121) If $t_q\geqslant 10$, then $I^1:=[1,q-1]$, $j:=q$, $k:=q+1\to$ (C3).

(51122) If $t_q=6$, put $\tau^1:=(4)\prod\limits_{i=1}^{q-1}(t_i)\in \mathrm{Sct}(G^1)$, $\tau^2:=(8)(6)^{p-1-q}\in \mathrm{Sct}(G^2)$ and consider a G^1-realisation $(T_q^1)\prod\limits_{i=1}^{q-1}(T_i)$ of the sequence τ^1 and a G^2-realisation $(T_{q+1}^2)\prod\limits_{i=q+2}^{p}(T_i)$ of the sequence τ^2 yielded by Theorem 4. Let $T_q^1=\prod\limits_{i=1}^{5}(b_i)$ with $b_1=b_5\in X_{1,5}^1$ and let $T_{q+1}^2=\prod\limits_{i=1}^{9}(c_i)$ with $c_1=c_9\in X_{1,4}^1$. Since T_{q+1}^2 is a cycle, we have $V(T_{q+1}^2)\cap X_{1,4}^1=X_{1,4}^1$. By Proposition 7 we may suppose without loss of generality that $b_1=c_1$ and $b_3=c_5$. With $T_q:=(c_1,b_2)\prod\limits_{i=5}^{9}(c_i)$ and $T_{q+1}:=(c_1,b_4)\prod\limits_{i=1}^{5}(c_{6-i})$ then $(T_1,\dots,T_p)$ is a G_a-realisation of the sequence τ. Since $q\geqslant 3$, by Proposition 7 and Lemma 8 we may suppose without loss of generality that the additional requirements on T_1 and T_2 are fulfilled.

(512) If $t_q=4$, then $t_1+t_2\equiv 2 \pmod 4$, and so $q\geqslant 4$ and $\sum\limits_{i=1}^{q-2} t_i=14$.

(5121) If $t_p\geqslant 10$, then $I^1:=[1,q-2]$, $j:=p\to$ (C2).

(5122) If $t_p\leqslant 8$, then $t_l=6$ and $I^1:=[1,q-2]\cup\{l\}\to$ (C1).

(52) If $\sum\limits_{i=1}^{q} t_i=20$, then $I^1:=[1,q]\to$ (C1).

(53) If $\sum\limits_{i=1}^{q} t_i=18$, then $q\geqslant 3$ and there is $l\in[q+1,p]$ with $t_l\equiv 2 \pmod 4$.

(531) If $t_q\geqslant 6$, then $\sum\limits_{i=1}^{q-1} t_i\leqslant 12$.

(5311) If $t_p\geqslant t_q+6$, then $I^1:=[1,q-1]$, $j:=p\to$ (C2).

(5312) If there is $m \in [q+1, p]$ with $t_m = t_q+2$, then $I^1 := [1, q-1] \cup \{m\} \to$ (C1).

(5313) If $t_i \in \{t_q, t_q+4\}$ for any $i \in [q+1, p]$, then $t_q \equiv t_l \equiv 2 \pmod 4$, hence $t_q \leqslant 10$.

(53131) If $t_q = 10$, then $q = 3$, $I^1 := [1, q-1]$, $j := q$, $k := q+1 \to$ (C3).

(53132) If $t_q = 6$, put $\tau^1 := (8) \prod_{i=1}^{q-1} (t_i) \in \mathrm{Sct}(G^1)$ and $\tau^2 := (t_p - 2) \prod_{i=q+1}^{p-1} (t_i) \in \mathrm{Sct}(G^2)$. Consider a G^1-realisation $(T^1_q) \prod_{i=1}^{q-1} (T_i)$ of the sequence τ^1 and a G^2-realisation $(T^2_p) \prod_{i=q+1}^{p-1} (T_i)$ of the sequence τ^2. Let $T^1_q = \prod_{i=1}^{9} (b_i)$ with $b_1 = b_9 \in X^1_{1,5}$ and let $T^2_p = \prod_{i=1}^{t_p-1} (c_i)$ with $c_1 = c_{t_p-1} \in X^1_{1,4}$. We have $|V(T^1_q) \cap X^1_{1,5}| \geqslant 3$ (if T^1_q is not a cycle, it is a union of two edge-disjoint 4-trails and then it suffices to use Lemma 8). Therefore, we may suppose without loss of generality that $b_5 \neq b_1$. Moreover, by Proposition 6, the assumption $c_1 = b_1$ and $c_3 = b_5$ also does not cause a loss of generality. With $T_q := (b_1, c_2) \prod_{i=1}^{5} (b_{6-i})$ and $T_p := (c_1, b_8, b_7, b_6) \prod_{i=3}^{t_p-1} (c_i)$ then, using Proposition 7 and Lemma 8, we may suppose without loss of generality that $(T_1, \ldots, T_p)$ is an appropriate G_a-realisation of the sequence τ.

(532) $t_q = 4$.

(5321) If $t_l \geqslant 10$, then $I^1 := [1, q-1]$, $j := l \to$ (C2).

(5322) If $t_l = 6$, then $I^1 := [1, q-1] \cup \{l\} \to$ (C1).

(54) If $\sum_{i=1}^{q} t_i \leqslant 16$, then $I^1 := [1, q]$, $j := q+1 \to$ (C2). □

Theorem 10. *The graph H_a is* ADCT *for any odd integer $a \geqslant 7$. Moreover, any sequence $\tau = (t_1, \ldots, t_p) \in \mathrm{Sct}(H_a)$ of length $p \geqslant 2$ has an H_a-realisation $(T_1, \ldots, T_p)$ such that there are $(i_r, j_r) \in [5, a] \times [1, 2]$ with $x^{j_r}_{i_r} \in V(T_r)$, $r = 1, 2$, and $i_1 \neq i_2$.*

P r o o f. We proceed similarly as in the proof of Theorem 9 with ADCT graphs $G^1 := G^2_a$ (Theorem 3) and $G^2 := G_a$ (Theorem 9). The graph H_a is depicted in Figure 3.

(C4) If there is $I^1 \subseteq [1, p]$ such that $|[1, 2] \cap I^1| \geqslant 1$ and $\sum_{i \in I_1} t_i = |E(G^1)| = 4a - 20$, put $I^2 := [1, p] - I^1$ and $\tau^l := \tau\langle I^l \rangle$, $l = 1, 2$. Let $\mathcal{T}^l$ be a G^l-realisation of the sequence τ^l, $l = 1, 2$, and let T_i be a t_i-trail of $\mathcal{T}^1\mathcal{T}^2$, $i = 1, 2$. If $[1, 2] \subseteq I^1$, by Proposition 6 we may suppose without loss of generality that $x^1_{5+i} \in V(T_i)$, $i = 1, 2$; in such a case we are done with $(i_1, j_1) := (6, 1)$ and $(i_2, j_2) := (7, 1)$. If there is $m \in [1, 2]$ such that $m \in I^1$ and $3 - m \in I^2$, then, by Proposition 6 and Theorem 9, we may suppose without loss of generality that $(i_m, j_m) := (6, 1)$ and $(i_{3-m}, j_{3-m}) := (5, 2)$ are appropriate pairs.

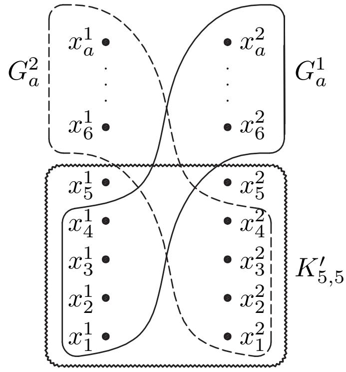


Figure 3. The graph H_a

(C5) If there are I^1 and $j \in [1,p] - I^1$ such that $|[1,2] \cap (I^1 \cup \{j\})| \geqslant 1$, $\sum\limits_{i\in I^1} t_i \leqslant 4a-24$ and $\sum\limits_{i\in I_1} t_i + t_j \geqslant 4a-16$, put $I^2 := [1,p] - I^1 - \{j\}$, $t_j^1 := 4a - 20 - \sum\limits_{i\in I_1} t_i$, $t_j^2 := \sum\limits_{i\in I^1} t_i + t_j + 20 - 4a$ and $m := \min(\{0\} \cup I^2)$. Consider a G^1-realisation $(T_j^1)\mathcal{T}^1$ of the sequence $(t_j^1)\tau\langle I^1\rangle \in \mathrm{Sct}(G^1)$ and let T_i be a t_i-trail of $\mathcal{T}^1$ with $i \in ([1,2]-\{j\})\cap I^1$. By Proposition 6 we may suppose without loss of generality that $x_2^2 \in V(T_j^1)$, $j \in [1,2] \Rightarrow x_{5+j}^1 \in V(T_j^1)$ and $x_{5+i}^1 \in V(T_i)$ for any $i \in ([1,2]-\{j\})\cap I^1$.

If $I^2 \neq \emptyset$ (so that $m \geqslant 1$), by Theorem 9 there is a G^2-realisation $(T_m, T_j^2)\mathcal{T}_2$ of the sequence $(t_m, t_j^2)\tau\langle I^2 - \{m\}\rangle \in \mathrm{Sct}(G^2)$ such that $\{x_1^2, x_5^2\} \subseteq V(T_m)$ and $x_2^2 \in V(T_j^2)$. Then $T_j := T_j^1 \cup T_j^2$ is a t_j-trail and $(T_j, T_m)\mathcal{T}^1\mathcal{T}^2$ is a required H_a-realisation of the sequence $(t_j, t_m)\tau\langle I^1\rangle\tau\langle I^2 - \{m\}\rangle \sim \tau$. Appropriate pairs are as follows: if $m \in [1,2]$, then $(i_m, j_m) := (5,2)$ and $(i_{3-m}, j_{3-m}) := (8-m, 1)$; if $m \notin [1,2]$, then $(i_r, j_r) := (5+r, 1)$, $r = 1,2$.

If $I^2 = \emptyset$ (and $m = 0$), then $T_j := T_j^1 \cup G^2$ is a t_j-trail and $(T_j^1)\mathcal{T}_1$ is an appropriate H_a-realisation of the sequence $(t_j)\tau\langle I^1\rangle \sim \tau$.

(C6) If there are I^1 and $\{j,k\} \subseteq [1,p] - I^1$ such that $[1,2] \subseteq I^1 \cup \{j,k\}$, $\min\{t_j, t_k\} \geqslant 8$, $\sum\limits_{i\in I^1} t_i \leqslant 4a - 28$ and $\sum\limits_{i\in I_1} t_i + t_j + t_k \geqslant 4a - 12$ (we may suppose without loss of generality that $j < k$), then with $I^2 := [1,p] - I^1 - \{j,k\}$, $t_j^1 := \min\{4a - 24 - \sum\limits_{i\in I^1} t_i, t_j - 4\}$, $t_k^1 := \max\{4, 4a - 16 - \sum\limits_{i\in I^1} t_i - t_j\}$, $t_j^2 := t_j - t_j^1$ and $t_k^2 := t_k - t_k^1$ we have $t_j^l + t_k^l + \sum\limits_{i\in I^l} t_i = |E(G^l)|$ and $\tau^l := (t_j^l, t_k^l)\tau\langle I^l\rangle \in \mathrm{Sct}(G^l)$, $l = 1,2$. Consider a G^1-realisation $(T_j^1, T_k^1)\mathcal{T}^1$ of the sequence τ^1 and let T_i be a t_i-trail of $\mathcal{T}^1$ with $i \in [1,2] - \{j,k\} \subseteq I^1$. Because of Proposition 6 we may suppose without loss of generality that $x_1^2 \in V(T_j^1)$, $x_2^2 \in V(T_k^1)$, $m \in [1,2] \cap \{j,k\} \Rightarrow x_{5+m}^1 \in V(T_m^1)$ and $x_{5+i}^1 \in V(T_i)$ for any $i \in [1,2] - \{j,k\}$. By Theorem 9 there is a G^2-realisation $(T_j^2, T_k^2)\mathcal{T}^2$ of the sequence τ^2 such that $x_1^2 \in V(T_j^2)$ and $x_2^2 \in V(T_k^2)$.

Then $T_m := T_m^1 \cup T_m^2$ is a t_m-trail, $m = j, k$ and $(T_j, T_k)\mathcal{T}^1\mathcal{T}^2$ is an H_a-realisation of the sequence $(t_j, t_k)\tau\langle I^1\rangle\tau\langle I^2\rangle \sim \tau$ with required properties; appropriate pairs are $(i_r, j_r) := (5 + r, 1)$, $r = 1, 2$.

The additional requirements on T_1 and T_2 are symmetrical and there are no additional requirements on T_i with $i \in [3, p]$; therefore, in our analysis we may suppose without loss of generality that $t_1 \leqslant t_2$ and $t_i \leqslant t_{i+1}$ for any $i \in [3, p-1]$.

(1) $t_1 + t_2 \geqslant 4a - 16$.

(11) If $t_1 \leqslant 4a - 24$, then $I^1 := \{1\}$, $j := 2 \to$ (C5).

(12) If $t_1 \geqslant 4a - 22$, then $t_1 \geqslant 6$.

(121) If $a \geqslant 9$, then $t_1 + t_2 \geqslant 8a - 44 \geqslant 4a - 12$, $t_1 \geqslant 14$ and $I^1 := \emptyset$, $j := 1$, $k := 2 \to$ (C6).

(122) If $a = 7$, then $|E(G^1)| = 8$.

(1221) If $t_1 \geqslant 8$, then $t_1 + t_2 \geqslant 4a - 12$ and $I^1 := \emptyset$, $j := 1$, $k := 2 \to$ (C6).

(1222) If $t_1 = 6$, by Theorem 9 there is a G^2-realisation $(T_2^2)\prod_{i=3}^{p}(T_i)$ of the sequence $(t_2 - 2)\prod_{i=3}^{p}(t_i) \in \mathrm{Sct}(G^2)$ such that T_2^2 contains as a subgraph a 3-vertex path with endvertices x_1^2 and x_4^2. Thus, we may suppose without loss of generality that $T_2^2 = \prod_{i=1}^{t_2-1}(c_i)$ where $c_1 = c_{t_2-1} = x_1^2$ and $c_3 = x_4^2$. With $T_1 := (x_1^2, c_2, x_4^2, x_7^1, x_3^2, x_6^1, x_1^2)$ and $T_2 := (c_1, x_7^1, x_2^2, x_6^1)\prod_{i=3}^{t_2-1}(c_i)$ then $(T_1, \ldots, T_p)$ is a required H_a-realisation of the sequence τ; appropriate pairs are $(i_r, j_r) := (5 + r, 1)$, $r = 1, 2$.

(2) If $t_1 + t_2 = 4a - 18$, then $\sum_{i=3}^{p} t_i = 4a - 2 \equiv 2 \pmod 4$ and there is $l \in [3, p]$ satisfying $t_l \equiv 2 \pmod 4$.

(21) If $t_1 \leqslant 4a - 28$, then $t_2 \geqslant 10$.

(211) If $t_p \geqslant 8$, then $I^1 := \{1\}$, $j := 2$, $k := p \to$ (C6).

(212) $t_p (= t_l) = 6$.

(2121) If $t_1 \leqslant 4a - 30$, then $I^1 := \{1, p\}$, $j := 2 \to$ (C5).

(2122) If $t_1 = 4a - 28$, then $t_2 = 10$, $a \leqslant 9$, $t_1 = 8$, $a = 9$ and $I^1 := \{2, p\} \to$ (C4).

(22) If $t_1 \geqslant 4a - 26$, then $t_2 \leqslant 8$, $a = 7$, $t_1 = 4$ and $t_2 = 6$.

(221) If $t_p \geqslant 8$, then $I^1 := \{1\}$, $j := p \to$ (C5).

(222) If $t_p = 6$, then from $\sum_{i=3}^{p} t_i = 26$ it follows that $t_3 = 4$, and so $I^1 := \{1, 3\} \to$ (C4).

(3) If $t_1 + t_2 = 4a - 20$, then $I^1 := [1, 2] \to$ (C4).

(4) If $t_1 + t_2 = 4a - 22$, then $a \geqslant 9$, $t_2 \geqslant 8$ and there is $l \in [3, p]$ with $t_l \equiv 2 \pmod 4$.

(41) If $t_1 \leqslant 4a - 34$, then $t_2 \geqslant 12$.

(411) If $t_l \geqslant 10$, then $I^1 := \{1\}$, $j := 2$, $k := l \to$ (C6).

(412) If $t_l = 6$, then $I^1 := \{1, l\}$, $j := 2 \to$ (C5).

(42) If $t_1 \geqslant 4a - 32$, then $a = 9$ and $t_2 \in \{8, 10\}$.

(421) If $t_l \geqslant 10$, then $I^1 := \{1\}$, $j := 2$, $k := l \to$ (C6).

(422) If $t_l = 6$, then $t_i \in \{4, 6\}$ for any $i \in [3, p]$, $\sum\limits_{i=3}^{p} t_i = 38$ and the sequence $\prod\limits_{i=3}^{p} (t_i)$ contains at least two 4's and at least one 6. Thus, there is $I^1 \subseteq [2, p]$ such that $2 \in I^1$, $\sum\limits_{i \in I^1} t_i = 16$ and the condition (C4) is satisfied.

(5) If $t_1 + t_2 \leqslant 4a - 24$, let $q \in [2, p-1]$ be determined by the inequalities $\sum\limits_{i=1}^{q} t_i \leqslant 4a - 18$ and $\sum\limits_{i=1}^{q+1} t_i \geqslant 4a - 16$.

(51) If $\sum\limits_{i=1}^{q} t_i = 4a - 18$, then $q \geqslant 3$ and there is $l \in [q+1, p]$ with $t_l \equiv 2 \pmod 4$.

(511) $t_q \geqslant 6$.

(5111) If $t_p \geqslant t_q + 2$, then $I^1 := [1, q-1]$, $j := p \to$ (C5).

(5112) If $t_i = t_q$ for any $i \in [q+1, p]$, then $t_q = t_l \equiv 2 \pmod 4$.

(51121) If $t_q \geqslant 10$, then $I^1 := [1, q-1]$, $j := q$, $k := q + 1 \to$ (C6).

(51122) If $t_q = 6$, then $6 | 4a - 2 = 6(p - q)$, hence $a \equiv 5 \pmod 6$ and $p - q \geqslant 7$.

(511221) If $t_2 \geqslant 12$, then $I^1 := \{1\} \cup [3, q+1]$, $j := 2 \to$ (C5).

(511222) $t_2 \leqslant 10$.

(5112221) If $t_2 = 10$, then $I^1 := [q+5, p]$, $j := 2 \to$ (C5).

(5112222) If $t_2 = 8$, then $I^1 := \{1\} \cup [3, q+1] \to$ (C4).

(5112223) If $t_2 = 6$, then $I^1 := \{2\} \cup [q+5, p] \to$ (C4).

(5112224) $t_2 = 4$.

(51122241) If $t_3 = 4$, then $I^1 := [1, 3] \cup [q+6, p] \to$ (C4).

(51122242) If $t_3 = 6$, then $\tau = (4)^2(6)^{p-2}$, $6p - 4 = |E(H_a)| = 8a - 20$ and $p \equiv 0 \pmod 2$. Put $\tau_1 := (8)(6)^2$, $\tau_2 := (6)^{\frac{p-4}{2}} =: \tau_3$ and consider a $K'_{5,5}$-realisation $(T_{1,2}, T_3, T_4)$ of the sequence τ_1 presented in Figure 1, a G_a^1-realisation $(\widetilde{T}_5) \prod\limits_{i=6}^{\frac{p+4}{2}} (T_i)$ of the sequence τ_2 and a G_a^2-realisation $\prod\limits_{i=\frac{p+6}{2}}^{p} (T_i)$ of the sequence τ_3. The closed trail $T_{1,2}$ is an 8-cycle, hence by Proposition 7 we may suppose without loss of generality that $V(T_{1,2}) \cap X_{1,5}^1 = X_{1,4}^1$ and $T_{1,2} = \prod\limits_{i=1}^{9} (b_i)$ with $b_1 = b_9 \in X_{1,4}^1$. By Proposition 6 we may suppose without loss of generality that $\widetilde{T}_5 = \prod\limits_{i=1}^{7} (c_i)$ with $c_1 = c_7 = b_1$, $c_3 = b_3$, $c_5 = b_7$, $c_2 = x_6^2$ and $c_6 = x_7^2$. Then $(T_1, \dots, T_p)$ with $T_1 := (b_1, b_2, b_3, c_2, b_1)$, $T_2 := (b_9, b_8, b_7, c_6, b_9)$ and $T_5 := (b_3, c_4, b_7, b_6, b_5, b_4, b_3)$ is a required H_a-realisation of the sequence τ; appropriate pairs are $(i_r, j_r) := (5 + r, 2)$, $r = 1, 2$.

(512) If $t_q = 4$, then $q \geqslant 4$ and $\sum_{i=1}^{q-2} t_i = 4a - 26$.

(5121) If $t_p \geqslant 10$, then $I^1 := [1, q-2]$, $j := p \to$ (C5).

(5122) If $t_p \leqslant 8$, then $t_l = 6$ and $I^1 := [1, q-2] \cup \{l\} \to$ (C4).

(52) If $\sum_{i=1}^{q} t_i = 4a - 20$, then $I^1 := [1, q] \to$ (C4).

(53) If $\sum_{i=1}^{q} t_i = 4a - 22$, then $q \geqslant 3$.

(531) $t_q \geqslant 6$.

(5311) If $t_p \geqslant t_q + 6$, then $I^1 := [1, q-1]$, $j := p \to$ (C5).

(5312) If there is $m \in [q+1, p]$ such that $t_m = t_q + 2$, then $I^1 := [1, q-1] \cup \{m\} \to$ (C4).

(5313) If $t_i \in \{t_q, t_q + 4\}$ for any $i \in [q+1, p]$, then $t_q \equiv t_l \equiv 2 \pmod 4$, $(p-q)(t_q + 4) \geqslant 4a + 2 = \sum_{i=1}^{q} t_i + 24 \geqslant t_q + 24$, $p - q \geqslant \frac{t_q + 24}{t_q + 4} > 1$ and $p - q \geqslant 2$.

(53131) If $t_{p-1} \geqslant 10$, then $I^1 := [1, q-1]$, $j := p-1$, $k := p \to$ (C6).

(53132) If $t_{p-1} = 6$, then $t_q = 6$.

(531321) If $t_2 \geqslant 8$, then $I^1 := \{1\} \cup [3, q+1]$, $j := 2 \to$ (C5).

(531322) If $t_2 \leqslant 6$, then by Theorem 4 there exists a G^1-realisation $\mathcal{T}^1 := (T_q^1) \prod_{i=1}^{q-1} (T_i)$ of the sequence $(8) \prod_{i=1}^{q-1} (t_i)$ such that all trails of $\mathcal{T}^1$ are cycles. Therefore, by Proposition 6 we may suppose without loss of generality that $x_{5+i}^1 \in V(T_i)$, $i = 1, 2$, and $T_q^1 = \prod_{i=1}^{9} (b_i)$ with $b_1 = b_9 = x_1^2$ and $b_5 = x_4^2$. By Theorem 9 there is a G^2-realisation $(T_{q+1}^2) \prod_{i=q+2}^{p} (T_i)$ of the sequence $(4) \prod_{i=q+2}^{p} (t_i)$ such that T_{q+1}^2 contains as a subgraph a 3-vertex path with endvertices x_1^2 and x_4^2. Thus, we may suppose without loss of generality that $T_{q+1}^2 = \prod_{i=1}^{5} (c_i)$ where $c_1 = c_5 = x_1^2$ and $c_3 = x_4^2$. Then $(T_1, \ldots, T_p)$ with $T_{q+1} := (b_5, c_4) \prod_{i=1}^{5} (b_i)$ and $T_{q+2} := (b_9, c_2) \prod_{i=5}^{9} (b_i)$ is a required H_a-realisation of the sequence τ; appropriate pairs are $(i_r, j_r) := (5 + r, 1)$, $r = 1, 2$.

(532) $t_q = 4$.

(5321) If $t_p \geqslant 10$, then $I^1 := [1, q-1]$, $j := p \to$ (C5).

(5322) If $t_p \leqslant 8$, then $t_l = 6$ and $I^1 := [1, q-1] \cup \{l\} \to$ (C4).

(54) If $\sum_{i=1}^{q} t_i \leqslant 4a - 24$, then $I^1 := [1, q]$, $j := q + 1 \to$ (C5). □

Theorem 11. *If a is an odd integer, $a \geqslant 3$, then the graph $K'_{a,a}$ is* ADCT. *Moreover, if $r = \frac{1}{6}(a(a-1) - 2) \in \mathbb{Z}$, there is a $K'_{a,a}$-realisation $(T_1, \ldots, T_r)$ of the sequence $(6)^{r-1}(8) \in \mathrm{Sct}(K'_{a,a})$ such that T_r has as a subgraph a 5-vertex path.*

Proof. We proceed by induction on a. The graphs $K'_{a,a}$ with $a \leqslant 5$ are ADCT by Proposition 5. Further, the 8-trail of the $K'_{5,5}$-realisation of the sequence $(6)^2(8) \in \mathrm{Sct}(K'_{5,5})$ presented in Figure 1 is a cycle, and so trivially it has as a subgraph a 5-vertex path.

So, suppose that $a \geqslant 7$, the graph $K'_{a-4,a-4}$ is ADCT and, provided $s := \frac{1}{6}((a-4)(a-5)-2) \in \mathbb{Z}$, there is a G^1-realisation $\prod_{i=1}^{s}(T_i^1)$ of the sequence $(6)^{s-1}(8) \in \mathrm{Sct}(G^1)$ such that T_s^1 has as a subgraph a 5-vertex path. We can use again the general strategy, since the graph $K'_{a,a}$ (see Figure 4) is an edge-disjoint union of ADCT graphs $G^1 := F_a$ (the induction hypothesis) and $G^2 := H_a$ (Theorem 10). Consider a sequence $\tau = (t_1, \ldots, t_p) \in \mathrm{Sct}(K'_{a,a})$.

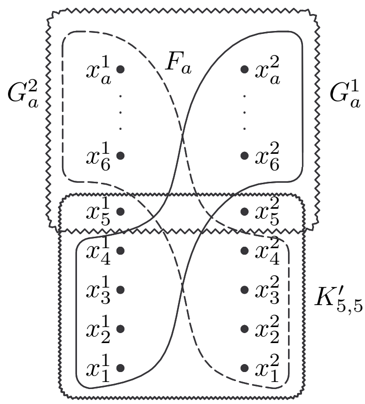


Figure 4. The graph $K'_{a,a}$

(C7) If there is $I^1 \subseteq [1,p]$ such that $\sum_{i\in I^1} t_i = a^2 - 9a + 20 = |E(G^1)|$, put $I^2 := [1,p] - I^1$, $\tau^l := \tau\langle I^l\rangle \in \mathrm{Sct}(G^l)$ and consider a G^l-realisation $\mathcal{T}^l$ of the sequence τ^l, $l = 1,2$. Then $\mathcal{T}^1\mathcal{T}^2$ is a $K'_{a,a}$-realisation of the sequence $\tau^1\tau^2 \sim \tau$.

(C8) If there are I^1 and $j \in [1,p] - I^1$ such that $\sum_{i\in I^1} t_i \leqslant a^2 - 9a + 16$ and $\sum_{i\in I_1} t_i + t_j \geqslant a^2 - 9a + 24$, put $I^2 := [1,p] - I^1 - \{j\}$, $t_j^1 := a^2 - 9a + 20 - \sum_{i\in I_1} t_i$, $t_j^2 := \sum_{i\in I^1} t_i + t_j - a^2 + 9a - 20$. Then $\tau^l := (t_j^l)\tau\langle I^l\rangle \in \mathrm{Sct}(G^l)$, $l = 1,2$. By Theorem 10 there is a G^2-realisation $(T_j^2)\mathcal{T}^2$ of the sequence τ^2 such that there is $(i_1, j_1) \in [5,a] \times [1,2]$ with $x_{i_1}^{j_1} \in V(T_j^2)$. By the induction hypothesis there is a G^1-realisation $(T_j^1)\mathcal{T}^1$ of the sequence τ^1; by Proposition 7 we may suppose without loss of generality that $x_{i_1}^{j_1} \in V(T_j^1)$. Then $T_j := T_j^1 \cup T_j^2$ is a t_j-trail and $(T_j)\mathcal{T}^1\mathcal{T}^2$ is a $K'_{a,a}$-realisation of the sequence $(t_j)\tau\langle I^1\rangle\tau\langle I^2\rangle \sim \tau$.

(C9) If there are I^1 and $\{j,k\} \subseteq [1,p] - I^1$ such that $\min\{t_j, t_k\} \geqslant 8$, $\sum_{i \in I^1} t_i \leqslant a^2 - 9a + 12$ and $\sum_{i \in I_1} t_i + t_j + t_k \geqslant a^2 - 9a + 28$, then with $I^2 := [1,p] - I^1 - \{j,k\}$, $t_j^1 := \min\Big\{a^2 - 9a + 16 - \sum_{i \in I^1} t_i, t_j - 4\Big\}$, $t_k^1 := \max\Big\{4, a^2 - 9a + 24 - \sum_{i \in I^1} t_i - t_j\Big\}$, $t_j^2 := t_j - t_j^1$ and $t_k^2 := t_k - t_k^1$ we have $t_j^l + t_k^l + \sum_{i \in I^l} t_i = |E(G^l)|$ and $\tau^l := (t_j^l, t_k^l)\tau\langle I^l \rangle \in \mathrm{Sct}(G^l)$, $l = 1, 2$. Theorem 10 yields a G^2-realisation $(T_j^2, T_k^2)\mathcal{T}^2$ of the sequence τ^2 such that there are $(i_r, j_r) \in [5,a] \times [1,2]$, $r = 1, 2$, with $x_{i_1}^{j_1} \in V(T_j^2)$, $x_{i_2}^{j_2} \in V(T_k^2)$ and $i_1 \neq i_2$. By the induction hypothesis there is a G^1-realisation $(T_j^1, T_k^1)\mathcal{T}^1$ of the sequence τ^1; by Proposition 7 we may suppose without loss of generality that $x_{i_1}^{j_1} \in V(T_j^1)$ and $x_{i_2}^{j_2} \in V(T_k^1)$ (note that both T_j^1 and T_k^1 have at least two vertices in both $X_{5,a}^1$ and $X_{5,a}^2$). Then $T_m := T_m^1 \cup T_m^2$ is a t_m-trail, $m = j, k$, and $(T_j, T_k)\mathcal{T}^1\mathcal{T}^2$ is a $K'_{a,a}$-realisation of the sequence $(t_j, t_k)\tau\langle I^1 \rangle \tau \langle I^2 \rangle \sim \tau$.

Because of Lemma 1 we may suppose without loss of generality that τ is a non-decreasing sequence. Let $q \in [0, p-1]$ be determined by the inequalities $\sum_{i=1}^{q} t_i \leqslant a^2 - 9a + 22$ and $\sum_{i=1}^{q+1} t_i \geqslant a^2 - 9a + 24$.

(1) If $\sum_{i=1}^{q} t_i = a^2 - 9a + 22$, then $\sum_{i=q+1}^{p} t_i = 8a - 22$ and there is $l \in [q+1, p]$ such that $t_l \equiv 2 \pmod 4$.

(11) $t_q \geqslant 6$.

(111) If $t_p \geqslant t_q + 2$, then $I^1 := [1, q-1]$, $j := p \to$ (C8).

(112) If $t_i = t_q$ for any $i \in [q+1, p]$, then $t_q = t_l \equiv 2 \pmod 4$.

(1121) If $t_q \geqslant 10$, then $I^1 := [1, q-1]$, $j := q$, $k := q+1 \to$ (C9).

(1122) If $t_q = 6$, then $6q \geqslant \sum_{i=1}^{q} t_i \geqslant 8$, $q \geqslant 2$, $8a - 22 = 6(p-q)$, $4a - 11 \equiv 0 \pmod 3$, $a \equiv 5 \pmod 6$, $a(a-1) \equiv 2 \pmod 6$, the sequence τ must contain at least two 4's and $I^1 := [3, q+1] \to$ (C7).

(12) If $t_q = 4$, then $4q \geqslant 8$ and $q \geqslant 2$.

(121) If $t_l \geqslant 10$, then $I^1 := [1, q-2]$, $j := l \to$ (C8).

(122) If $t_l = 6$, then $I^1 := [1, q-2] \cup \{l\} \to$ (C7).

(2) If $\sum_{i=1}^{q} t_i = a^2 - 9a + 20$, then $I^1 := [1, q] \to$ (C7). Note that if the r defined in the statement of our Theorem is an integer, then $a(a-1) \equiv 2 \pmod 6$, $a \equiv 5 \pmod 6$, $a^2 - 9a + 20 \equiv 0 \pmod 6$, $4a - 20 \equiv 0 \pmod 6$, and so $\tau = (6)^{p-1}(8)$ yields $8a - 20 = 6(p-q-1) + 8$, $6(p-q-1) \geqslant 60$, $p - q - 1 \geqslant 10$, $6(p-q-1) \equiv 0 \pmod 4$ and $p - q - 1 \equiv 0 \pmod 2$. The graph G^2 is an edge-disjoint union of ADCT graphs $G_1^2 := G_a^1$, $G_2^2 := G_a^2$ and $G_3^2 := K'_{5,5}$. Put $\tau^1 := (6)^q$, $\tau_1^2 := (6)^{\frac{p-q-3}{2}} =: \tau_2^2$, $\tau_3^2 := (6)^2(8)$ and let $\mathcal{T}^1$ be a G^1-realisation of the sequence τ^1 and let $\mathcal{T}_m^2$ be a G^2_m-realisation of the sequence τ^2_m, $m = 1, 2, 3$, where $\mathcal{T}^2_3 = (T_{p-2}, T_{p-1}, T_p)$ is that presented in Figure 1. Then $\mathcal{T}^1\mathcal{T}^2_1\mathcal{T}^2_2\mathcal{T}^2_3$ is a $K'_{a,a}$-realisation of the sequence $(6)^{p-1}(8)$ and the 8-trail T_p (which is a cycle) has trivially as a subgraph a 5-vertex path.

(3) If $\sum_{i=1}^{q} t_i = a^2 - 9a + 18$, there is $l \in [q+1, p]$ such that $t_l \equiv 2 \pmod 4$.

(31) $t_q \geqslant 6$.

(311) If $t_p \geqslant t_q + 6$, then $I^1 := [1, q-1]$, $j := p \to$ (C8).

(312) If there is $m \in [q+1, p]$ such that $t_m = t_q + 2$, then $I^1 := [1, q-1] \cup \{m\} \to$ (C7).

(313) If $t_i \in \{t_q, t_q + 4\}$ for any $i \in [q+1, p]$, then $t_q \equiv t_l \equiv 2 \pmod 4$.

(3131) $p \geqslant q + 2$.

(31311) If $t_{p-1} \geqslant 10$, then $I^1 := [1, q-1]$, $j := p-1$, $k := p \to$ (C9).

(31312) $t_{p-1} = 6$.

(313121) If $t_1 = 4$, then $I^1 := [2, q+1] \to$ (C7).

(313122) If $t_1 = 6$, then $a^2 - 9a + 18 = 6q$, $a \equiv 3 \pmod 6$, $\sum_{i=q+1}^{p} t_i = 8a - 18 \equiv 0 \pmod 6$, $t_p = 6$, $\tau = (6)^p$, $8a - 18 = 6(p-q)$, $p - q \geqslant 9$, $6(p-q) \equiv 6 \pmod{48}$ and $p - q - 1 \equiv 0 \pmod 8$. The graph G^2 is an edge-disjoint union of ADCT graphs $G^2_1 := G_a$ and $G^2_2 := G^2_a$. Put $\tau^1 := (8)(6)^{q-1}$, $\tau^2_1 := (6)^{\frac{p-q+3}{2}}$ and $\tau^2_2 := (4)(6)^{\frac{p-q-5}{2}}$. By the induction hypothesis and by Lemma 1 there is a G^1-realisation $(T^1_q)\mathcal{T}^1$ of the sequence τ^1 such that T^1_q has as a subgraph a 5-vertex path. By Proposition 7 we may suppose without loss of generality that $T^1_q = \prod_{i=1}^{9}(b_i)$ where $b_1 = b_9 \in X^1_{5,a}$ and $\prod_{i=1}^{5}(b_i)$ is a path. By Theorem 10 there is a G^2_1-realisation $\mathcal{T}^2_1$ of the sequence τ^2_1. Further, by Theorem 3 there is a G^2_2-realisation $(T^2_{q+1})\mathcal{T}^2_2$ of the sequence τ^2_2; by Proposition 6 we may suppose without loss of generality that $T^2_{q+1} = \prod_{i=1}^{5}(c_i)$ where $c_1 = c_5 = b_1$ and $c_3 = b_5$. With $T_q := (b_5, c_2)\prod_{i=1}^{5}(b_i)$ and $T_{q+1} := (b_9, c_4)\prod_{i=5}^{9}(b_i)$ then $(T_q, T_{q+1})\mathcal{T}^1\mathcal{T}^2_1\mathcal{T}^2_2$ is a $K'_{a,a}$-realisation of the sequence $\tau = (6)^p$.

(3132) If $p = q + 1$, then $t_p = 8a - 18$, $t_q \geqslant 8a - 22$ and $I^1 := [1, q-1]$, $j := q$, $k := p \to$ (C9).

(32) $t_q = 4$.

(321) If $t_l \geqslant 10$, then $I^1 := [1, q-1]$, $j := l \to$ (C8).

(322) If $t_l = 6$, then $I^1 := [1, q-1] \cup \{l\} \to$ (C7).

(4) If $\sum_{i=1}^{q} \leqslant a^2 - 9a + 16$, then $I^1 := [1, q]$, $j := q + 1 \to$ (C8). □

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