

Huanyin Chen

Clean matrices over commutative rings

Czechoslovak Mathematical Journal, Vol. 59 (2009), No. 1, 145–158

Persistent URL: <http://dml.cz/dmlcz/140469>

Terms of use:

© Institute of Mathematics AS CR, 2009

Institute of Mathematics of the Czech Academy of Sciences provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these *Terms of use*.



This document has been digitized, optimized for electronic delivery and stamped with digital signature within the project *DML-CZ: The Czech Digital Mathematics Library* <http://dml.cz>

CLEAN MATRICES OVER COMMUTATIVE RINGS

HUANYIN CHEN, Hangzhou

(Received March 12, 2007)

Abstract. A matrix $A \in M_n(R)$ is e -clean provided there exists an idempotent $E \in M_n(R)$ such that $A - E \in \text{GL}_n(R)$ and $\det E = e$. We get a general criterion of e -cleanness for the matrix $[[a_1, a_2, \dots, a_{n+1}]]$. Under the n -stable range condition, it is shown that $[[a_1, a_2, \dots, a_{n+1}]]$ is 0-clean iff $(a_1, a_2, \dots, a_{n+1}) = 1$. As an application, we prove that the 0-cleanness and unit-regularity for such $n \times n$ matrix over a Dedekind domain coincide for all $n \geq 3$. The analogous for $(s, 2)$ property is also obtained.

Keywords: matrix, clean element, unit-regularity

MSC 2010: 15A23, 16E50

1. INTRODUCTION

An element in a ring is clean (unit-regular) provided it is the sum (product) of an idempotent and an invertible element. A ring R is unit-regular provided every element in R is unit-regular. In [1, Theorem 1], Camillo and Khurana proved that every element in a unit-regular ring is clean. In [9, Theorem], Nicholson and Varadarajan proved that every countable linear transformation over a division ring is clean. This shows that clean elements may not be unit-regular even in a regular ring. In fact, the relationship between cleanness and unit-regularity is rather subtle (cf. [4] and [10]).

Recall that $A \in M_n(R)$ is e -clean provided there exists an idempotent $E \in M_n(R)$ such that $A - E \in \text{GL}_n(R)$ and $\det E = e$. We get a general criterion of e -cleanness for the matrix $[[a_1, a_2, \dots, a_{n+1}]]$. We use $(a_1, \dots, a_n) = 1$ to stand for the condition $a_1R + \dots + a_nR = R$. A ring R is said to satisfy the n -stable range condition provided $(a_1, \dots, a_n, a_{n+1}) = 1$ in R implies that there exist $c_1, \dots, c_n \in R$ such that $(a_1 + a_{n+1}c_1, \dots, a_n + a_{n+1}c_n) = 1$ in R (see [8]). Let $a_1, a_2, \dots, a_{n+1} \in R$ ($n \in \mathbb{N}$). If R satisfies the n -stable range condition, we will prove that $[[a_1, a_2, \dots, a_{n+1}]]$ is

0-clean iff $(a_1, a_2, \dots, a_{n+1}) = 1$. In [7], Khurana and Lam proved that there are many matrices $[[a, b]] \in M_2(\mathbb{Z})$ which are unit-regular while they are not 0-clean, e.g., $[[12, 5]]$, $[[13, 5]]$, $[[12, 7]]$, etc. As an application, we prove that the 0-cleanness and unit-regularity for such $n \times n$ matrix over a Dedekind domain coincide for all $n \geq 3$. We say that $a \in R$ is $(s, 2)$ provided a is the sum of two units. An analog of the $(s, 2)$ property is also obtained.

Throughout the paper, all rings are commutative rings with an identity. $M_n(R)$ denotes the set of all $n \times n$ matrices over R , $\text{GL}_n(R)$ denotes the n -dimensional general linear group of R and $U(R) = \text{GL}_1(R)$. \mathbb{N} stands for the set of all natural numbers. We write $[[a_1, a_2, \dots, a_n]]$ for the matrix whose first row is (a_1, a_2, \dots, a_n) and other entries are zeros.

2. CLEANNESS

In this section we get a general criterion for an $n \times n$ matrix $[[a_1, a_2, \dots, a_n]]$ over a commutative ring to be e -clean. This gives a generalization of [7, Theorem 3.2] as well.

Theorem 2.1. *Let $a_1, \dots, a_n \in R$, and let $e \in R$ be an idempotent. Then $[[a_1, a_2, \dots, a_n]]$ is e -clean if and only if the following conditions hold:*

- (1) *There exist $x_1, \dots, x_n \in R$ such that $a_1x_1 + \dots + a_nx_n \in R$ is e -clean.*
- (2) *$ex_2 = \dots = ex_n = 0$.*
- (3) *$x_1 \equiv 1 \pmod{x_2R + \dots + x_nR}$.*

Proof. Suppose that $[[a_1, a_2, \dots, a_n]]$ is e -clean. Then we have an idempotent matrix $E = (e_{ij}) \in M_n(R)$ and a $U = (u_{ij}) \in \text{GL}_n(R)$ such that $[[a_1, a_2, \dots, a_n]] = E + U$ and $\det E = e$. Thus,

$$\begin{pmatrix} a_1 - e_{11} & a_2 - e_{12} & \dots & a_n - e_{1n} \\ -e_{21} & -e_{22} & \dots & -e_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -e_{n1} & -e_{n2} & \dots & -e_{nn} \end{pmatrix} = U.$$

This implies that

$$\begin{vmatrix} a_1 & a_2 & \dots & a_n \\ -e_{21} & -e_{22} & \dots & -e_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -e_{n1} & -e_{n2} & \dots & -e_{nn} \end{vmatrix} + (-1)^n \det E = \det U.$$

Hence, $a_1A_{11} + a_2A_{12} + \dots + a_nA_{1n} = (-1)^{n+1}e + u$, where $u = \det U \in U(R)$ and each A_{1i} is the algebraic complement corresponding to a_i ($1 \leq i \leq n$). Let $x_1 = (-1)^{n+1}A_{11}, x_2 = (-1)^{n+1}A_{12}, \dots, x_n = (-1)^{n+1}A_{1n}$. Then $a_1x_1 + a_2x_2 + \dots + a_nx_n = e + (-1)^{n+1}u$ is e -clean. As $E \in M_n(R)$ is an idempotent with $\det E = e$, in view of [7, Proposition 2.7] we get $ee_{ii} = e, ee_{ij} = 0$ ($1 \leq i \neq j \leq n$). This implies that $eA_{12} = \dots = eA_{1n} = 0$; hence, $ex_2 = \dots = ex_n = 0$.

Clearly, we have $(-e_{21})A_{11} + (-e_{22})A_{12} + \dots + (-e_{2n})A_{1n} = 0$, and thus,

$$(-e_{21})A_{11} \equiv 0 \pmod{x_2R + \dots + x_nR}.$$

On the other hand, $u_{11}A_{11} + u_{12}A_{12} + \dots + u_{1n}A_{1n} = u$, and thus,

$$u_{11}A_{11} \equiv u \pmod{x_2R + \dots + x_nR}.$$

As $u \in U(R)$, we deduce that

$$-e_{21} \equiv 0 \pmod{x_2R + \dots + x_nR}.$$

Similarly, we show that

$$-e_{31}, \dots, -e_{n1} \equiv 0 \pmod{x_2R + \dots + x_nR}.$$

Since $(-E)(-E) = E$, we see that

$$\begin{aligned} & \begin{pmatrix} -e_{11} & -e_{12} & \dots & -e_{1n} \\ 0 & -e_{22} & \dots & -e_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & -e_{n2} & \dots & -e_{nn} \end{pmatrix} \begin{pmatrix} -e_{11} & -e_{12} & \dots & -e_{1n} \\ 0 & -e_{22} & \dots & -e_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & -e_{n2} & \dots & -e_{nn} \end{pmatrix} \\ & \equiv \begin{pmatrix} e_{11} & e_{12} & \dots & e_{1n} \\ 0 & e_{22} & \dots & e_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & e_{n2} & \dots & e_{nn} \end{pmatrix} \pmod{x_2R + \dots + x_nR}. \end{aligned}$$

This implies that

$$\begin{aligned} & \begin{pmatrix} -e_{22} & \dots & -e_{2n} \\ \vdots & \ddots & \vdots \\ -e_{n2} & \dots & -e_{nn} \end{pmatrix} \begin{pmatrix} -e_{22} & \dots & -e_{2n} \\ \vdots & \ddots & \vdots \\ -e_{n2} & \dots & -e_{nn} \end{pmatrix} \\ & \equiv \begin{pmatrix} e_{22} & \dots & e_{2n} \\ \vdots & \ddots & \vdots \\ e_{n2} & \dots & e_{nn} \end{pmatrix} \pmod{x_2R + \dots + x_nR}. \end{aligned}$$

As a result we have

$$A_{11}^2 = (-1)^{n+1} A_{11} (\text{mod } x_2 R + \dots + x_n R).$$

Hence,

$$u_{11} A_{11}^2 \equiv (-1)^{n+1} u_{11} A_{11} (\text{mod } x_2 R + \dots + x_n R).$$

Therefore we get

$$A_{11} \equiv (-1)^{n+1} (\text{mod } x_2 R + \dots + x_n R),$$

that is,

$$x_1 \equiv 1 (\text{mod } x_2 R + \dots + x_n R).$$

Conversely, assume that (1), (2) and (3) hold. By (1), we can find $x_1, \dots, x_n \in R$ such that $a_1 x_1 + \dots + a_i x_i + \dots + a_n x_n$ is e -clean. Let $c_1 = x_1$ and $c_i = -x_i$ ($2 \leq i \leq n$). Then $a_1 c_1 - \dots - a_i c_i - \dots - a_n c_n$ is e -clean. By (3), we can find $k_2, \dots, k_n \in R$ such that $c_1 = 1 + k_2 c_2 + \dots + k_n c_n$. Let

$$E = (e_{ij}) = \begin{pmatrix} 1 & -k_2 & \dots & -k_n \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \begin{pmatrix} e & & & \\ c_2 & 1 & & \\ \vdots & & \ddots & \\ c_n & & & 1 \end{pmatrix} \begin{pmatrix} 1 & k_2 & \dots & k_n \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}.$$

By (2), it is easy to verify that $E = E^2 \in M_n(R)$ and $\det E = e$. Let

$$U = (u_{ij}) = \begin{pmatrix} a_1 & \dots & a_n \\ 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{pmatrix} - E.$$

Then

$$\begin{aligned} \det U &= \begin{vmatrix} a_1 - e_{11} & a_2 - e_{12} & \dots & a_n - e_{1n} \\ -e_{21} & -e_{22} & \dots & -e_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -e_{n1} & -e_{n2} & \dots & -e_{nn} \end{vmatrix} \\ &= \begin{vmatrix} a_1 & a_2 & \dots & a_n \\ -e_{21} & -e_{22} & \dots & -e_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -e_{n1} & -e_{n2} & \dots & -e_{nn} \end{vmatrix} + (-1)^n \det E \\ &= (-1)^{n-1} (a_1 A_{11} + \dots + a_n A_{1n}) + (-1)^n e, \end{aligned}$$

where A_{11}, \dots, A_{1n} are algebraic complements of E corresponding to e_{11}, \dots, e_{1n} respectively. Obviously,

$$E = \begin{pmatrix} 1 + (e - c_1) & (e - c_1)k_2 & (e - c_1)k_3 & \dots & (e - c_1)k_n \\ c_2 & 1 + k_2c_2 & k_3c_2 & \dots & k_nc_2 \\ c_3 & k_2c_3 & 1 + k_3c_3 & \dots & k_nc_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_n & k_2c_n & k_3c_n & \dots & 1 + k_nc_n \end{pmatrix}$$

It is easy to see that $A_{11} = 1 + k_2c_2 + \dots + k_nc_n = c_1$. Furthermore, we see that each $A_{1i} = -c_i$ ($2 \leq i \leq n$). Clearly, there is a $u \in U(R)$ such that $a_1A_{11} + \dots + a_{1n}A_{1n} = a_1c_1 - \dots - a_1c_i - \dots - a_{1n}c_n = e + u$. Thus, $\det U = (-1)^{n-1}(e + u) + (-1)^ne = (-1)^{n-1}u \in U(R)$, and then $U \in \text{GL}_n(R)$. Therefore A is e -clean, as asserted. \square

Corollary 2.2. *Let $a_1, \dots, a_n \in R$ ($n \in \mathbb{N}$). If $[[a_1, a_2, \dots, a_n]]$ is 0-clean, then so is $[[a_1u_1, a_2u_2, \dots, a_nu_n]]$ for any $u_1, \dots, u_n \in U(R)$.*

Proof. Assume that $[[a_1, a_2, \dots, a_n]]$ is 0-clean. According to Theorem 2.1, there exist $x_1, x_2, \dots, x_n \in R$ such that $a_1x_1 + \dots + a_nx_n = u \in U(R)$ and $x_1 \equiv 1 \pmod{x_2R + \dots + x_nR}$. Thus, we deduce that $(a_1u_1)x_1 + a_2(u_1x_2) + \dots + a_n(u_1x_n) = u_1u \in U(R)$. In addition,

$$x_1 \equiv 1 \pmod{(u_1x_2)R + \dots + (u_1x_n)R}.$$

In view of Theorem 2.1, we have an idempotent $E \in M_n(R)$ and a $U \in \text{GL}_n(R)$ such that

$$\begin{pmatrix} a_1u_1 & a_2 & \dots & a_n \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} = E + U \quad \text{and} \quad \det E = 0.$$

Therefore we conclude that

$$\begin{aligned} [[a_1u_1, a_2u_2, \dots, a_nu_n]] &= \begin{pmatrix} 1 & & & \\ & u_2^{-1} & & \\ & & \ddots & \\ & & & u_n^{-1} \end{pmatrix} E \begin{pmatrix} 1 & & & \\ & u_2 & & \\ & & \ddots & \\ & & & u_n \end{pmatrix} \\ &+ \begin{pmatrix} 1 & & & \\ & u_2^{-1} & & \\ & & \ddots & \\ & & & u_n^{-1} \end{pmatrix} U \begin{pmatrix} 1 & & & \\ & u_2 & & \\ & & \ddots & \\ & & & u_n \end{pmatrix}, \end{aligned}$$

as desired. \square

Example 2.3. Let us show that $[[12, 5, 3]] \in M_3(\mathbb{Z})$ is clean, while $[[12, 5]] \in M_2(\mathbb{Z})$ is not. In view of [7, Example 4.5], $[[12, 5]] \in M_2(\mathbb{Z})$ is not clean. Since $12 \times (-2) + 5 \times 2 + 3 \times 5 = 1$ and $-2 \equiv 1 \pmod{2R+5R}$, it follows by Theorem 2.1 that $[[12, 5, 3]] \in M_3(\mathbb{Z})$ is 0-clean. In fact, we have the decomposition: $[[12, 5, 3]] = E + U$, where

$$E = \begin{pmatrix} 3 & 8 & -2 \\ -2 & -7 & 2 \\ -5 & -20 & 6 \end{pmatrix}, \quad U = \begin{pmatrix} 9 & -3 & 5 \\ 2 & 7 & -2 \\ 5 & 20 & -6 \end{pmatrix}$$

with $E = E^2$, $\det E = 0$ and $\det U = 1$. □

Note that Theorem 2.1 illustrates the process of computing “clean decompositions” of numerical examples. Let $a_1, \dots, a_n, a_{n+1} \in R$ ($n \in \mathbb{N}$). If $[[a_1, a_2, \dots, a_n]] \in M_n(R)$ is e -clean, then so is $[[a_1, a_2, \dots, a_{n+1}]] \in M_{n+1}(R)$. Example 2.3 shows that the converse is not true.

3. STABLE RANGES

Lemma 3.1. *Let $a_1, a_2, \dots, a_{n+1} \in R$ ($n \in \mathbb{N}$). If $(a_2, \dots, a_{n+1}) = 1$, then $[[a_1, a_2, \dots, a_{n+1}]] \in M_{n+1}(R)$ is 0-clean.*

Proof. Since $(a_2, \dots, a_{n+1}) = 1$, there are $x_2, \dots, x_{n+1} \in R$ such that $a_2x_2 + \dots + a_{n+1}x_{n+1} = 1$. Thus, $a_1 \times 0 + a_2x_2 + \dots + a_{n+1}x_{n+1} = 1$. It is easy to see that

$$0 \equiv 1 \pmod{x_2R + \dots + x_{n+1}R}.$$

Applying to Theorem 2.1, we complete the proof. □

Theorem 3.2. *Let $a_1, a_2, \dots, a_{n+1} \in R$ ($n \in \mathbb{N}$). If R satisfies the n -stable range condition, then the following conditions are equivalent:*

- (1) $[[a_1, a_2, \dots, a_{n+1}]]$ is 0-clean.
- (2) $(a_1, a_2, \dots, a_{n+1}) = 1$.

Proof. (1) \Rightarrow (2) By virtue of Theorem 2.1, there exist $x_1, \dots, x_{n+1} \in R$ such that $a_1x_1 + \dots + a_{n+1}x_{n+1} = u \in U(R)$; hence, $a_1x_1u^{-1} + \dots + a_{n+1}x_{n+1}u^{-1} = 1$. That is, $(a_1, a_2, \dots, a_{n+1}) = 1$.

(2) \Rightarrow (1) Since $(a_1, a_2, \dots, a_{n+1}) = 1$ in R , there exist $c_2, \dots, c_{n+1} \in R$ such that $(a_2 + a_1c_2, \dots, a_{n+1} + a_1c_{n+1}) = 1$. In view of Lemma 3.1, $[[a_1, a_2 + a_1c_2, \dots, a_{n+1} + a_1c_{n+1}]] \in M_{n+1}(R)$ is 0-clean. Thus, we have an idempotent $E \in M_{n+1}(R)$

and a $U \in \text{GL}_{n+1}(R)$ such that $[[a_1, a_2 + a_1c_2, \dots, a_{n+1} + a_1c_{n+1}]] = E + U$ and $\det E = 0$. Let

$$Q = \begin{pmatrix} 1 & c_2 & \dots & c_{n+1} \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix} \in \text{GL}_{n+1}(R).$$

Then, $Q^{-1}[[a_1, a_2, \dots, a_{n+1}]]Q = [[a_1, a_2 + a_1c_2, \dots, a_{n+1} + a_1c_{n+1}]] = E + U$. Therefore $[[a_1, a_2, \dots, a_{n+1}]] = QEQ^{-1} + QUQ^{-1}$. In addition, $QEQ^{-1} \in M_{n+1}(R)$ is an idempotent matrix, $\det QEQ^{-1} = 0$ and $QUQ^{-1} \in \text{GL}_{n+1}(R)$. Thus we complete the proof. \square

Recall that a domain ring R is a Dedekind domain provided every ideal of R is a projective R -module. The class of Dedekind domains is very large. It includes all principal ideal domains. The ring $\mathbb{Z}[\sqrt{-d}]$ is a Dedekind domain provided d is square-free and $d \not\equiv 3 \pmod{4}$. Also we note that $\mathbb{R}[x, y]/(x^2 + y^2 - 1)$, the ring of polynomial functions on a circle, is a Dedekind domain. It is well known that every Dedekind domain satisfies the 2-stable range condition.

Corollary 3.3. *Let R be a Dedekind domain and let $a_1, \dots, a_n \in R$ ($n \geq 3$). Then the following conditions are equivalent:*

- (1) $[[a_1, a_2, \dots, a_n]]$ is 0-clean.
- (2) $[[a_1, a_2, \dots, a_n]] \neq 0$ is unit-regular.
- (3) $(a_1, \dots, a_n) = 1$.

Proof. (1) \Leftrightarrow (3) Since R is a Dedekind domain, it satisfies the 2-stable range condition, and so this is clear by virtue of Theorem 3.2.

(2) \Rightarrow (3) Let $[[a_1, a_2, \dots, a_n]] \neq 0$ be unit-regular. Then there exist an idempotent $E = (e_{ij}) \in M_n(R)$ and a $U = (u_{ij}) \in \text{GL}_n(R)$ such that $[[a_1, a_2, \dots, a_n]] = EU$, i.e., $[[a_1, a_2, \dots, a_n]]U^{-1} = E$. This implies that $e_{ij} = 0$ for $i = 2, \dots, n$. Thus, $[[a_1, a_2, \dots, a_n]] = [[e_{11}, e_{12}, \dots, e_{1n}]]U$; hence, $(a_1, \dots, a_n) = e_{11}(u_{11}, \dots, u_{1n})$. Clearly, $e_{11} = e_{11}^2 \in R$, and then, $e_{11} = 1$. Thus we get $(a_1, \dots, a_n) = (u_{11}, \dots, u_{1n}) = 1$.

(3) \Rightarrow (2) Since $(a_1, \dots, a_{n-1}, a_n) = 1$, there are $x_1, \dots, x_n \in R$ such that $a_1x_1 + \dots + a_nx_n = 1$. As R satisfies the 2-stable range condition, we have b_i, c_i ($3 \leq i \leq n$) such that $(a_1 + a_3b_3 + \dots + a_nb_n, a_2 + a_3c_3 + \dots + a_nc_n) = 1$. Thus, $(a_1 + a_3b_3 + \dots + a_nb_n)x + (a_2 + a_3c_3 + \dots + a_nc_n)y = 1$ for some $x, y \in R$. One

easily checks that

$$\begin{pmatrix} a_1 & a_2 & a_3 & \dots & a_{n-1} & a_n \\ -y & x & 0 & \dots & 0 & 0 \\ -b_3 & -c_3 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -b_{n-1} & -c_{n-1} & 0 & \dots & 1 & 0 \\ -b_n & -c_n & 0 & \dots & 0 & 1 \end{pmatrix} \in \text{GL}_n(R).$$

Therefore

$$[[a_1, a_2, \dots, a_n]] = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} \begin{pmatrix} a_1 & a_2 & \dots & a_n \\ -y & x & \dots & 0 \\ -b_3 & -c_3 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -b_n & -c_n & \dots & 1 \end{pmatrix},$$

as desired. \square

The following result should be compared to the fact that the problem of deciding the cleanness of $[[a, b]] \in M_2(\mathbb{Z})$ is considerably harder (cf. [7]).

Corollary 3.4. *Let $a_1, \dots, a_n \in \mathbb{Z}$ ($n \geq 3$). Then $[[a_1, a_2, \dots, a_n]] \in M_n(\mathbb{Z})$ is clean iff $a_1 = 0$ or $a_1 = 2$ or $(a_1, \dots, a_n) = 1$.*

Proof. If $[[a_1, a_2, \dots, a_n]] \in M_n(\mathbb{Z})$ is 1-clean, then we can find an idempotent $E \in M_n(\mathbb{Z})$ and a $U = (u_{ij}) \in \text{GL}_n(\mathbb{Z})$ such that $[[a_1, a_2, \dots, a_n]] = E + U$ and $\det E = 1$. Thus, $E = \text{diag}(1, \dots, 1) \in M_n(\mathbb{Z})$. This implies that $u_{ij} = 0$ ($i \neq j$), $u_{ii} = -1$ ($2 \leq i \leq n$). Hence, $a_1 - 1 \in U(\mathbb{Z})$, i.e., $a_1 = 0, 2$. Thus we conclude that $[[a_1, a_2, \dots, a_n]] \in M_n(\mathbb{Z})$ is 1-clean if and only if either $a_1 = 0$ or $a_1 = 2$. Consequently, the result follows from Corollary 3.3. \square

We say that $0 \neq A \in M_n(R)$ has rank 1 provided there exist $P, Q \in \text{GL}_n(R)$ such that $PAQ = [[a_1, \dots, a_n]]$ for some $a_1, \dots, a_n \in R$.

Corollary 3.5. *Let R be a Dedekind domain and let $A \in M_n(R)$ ($n \geq 3$). If A has rank 1, then the following conditions are equivalent:*

- (1) A is 0-clean.
- (2) A is unit-regular.

Proof. (1) \Rightarrow (2) As A has rank 1, there exist $P, Q \in \text{GL}_n(R)$ such that $PAQ = [[a_1, \dots, a_n]]$ for some $a_1, \dots, a_n \in R$. Thus,

$$PAP^{-1} = [[a_1, \dots, a_n]]Q^{-1}P^{-1} = [[b_1, \dots, b_n]]$$

for some $b_1, \dots, b_n \in R$. This implies that $[[b_1, \dots, b_n]]$ is 0-clean. According to Corollary 3.3, $[[b_1, \dots, b_n]]$ is unit-regular. Therefore A is unit-regular.

(2) \Rightarrow (1) As in the preceding discussion, $PAP^{-1} = [[b_1, \dots, b_n]] \neq 0$ for some $b_1, \dots, b_n \in R$. Thus, $[[b_1, \dots, b_n]]$ is unit-regular. In view of Corollary 3.3, $[[b_1, \dots, b_n]]$ is 0-clean, and therefore so is A . \square

It is clear that no polynomial in the polynomial ring over a field is clean. Furthermore, [1, Example 3.3] shows that no polynomial in the polynomial ring over a commutative ring is semiclean. We end this section by noting that Theorem 2.1 provides an explicit program to represent such kind of a matrix as the sum of an idempotent matrix and an invertible matrix.

Example 3.6. Let $[[1 + xy, x^2, y]] \in M_3(\mathbb{Z}[x, y])$. Obviously, we have $(1 + xy) \cdot (1 - xy) + x^2 \cdot y + y \cdot x^2(-1 + y) = 1$. In addition, $1 - xy = 1 + y \cdot (-x) + x^2(-1 + y) \cdot 0$. Thus, we have

$$\begin{aligned} E &= \begin{pmatrix} 1 & -x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ -y & 1 & 0 \\ x^2(1-y) & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} xy & -x(1-xy) & 0 \\ -y & 1-xy & 0 \\ x^2(1-y) & x^3(1-y) & 1 \end{pmatrix}. \end{aligned}$$

Then $E = E^2 \in M_3(\mathbb{Z}[x, y])$ and $\det E = 0$. Let

$$U = \begin{pmatrix} 1+xy & x^2 & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - E = \begin{pmatrix} 1 & x(1+x-xy) & y \\ y & -1+xy & 0 \\ -x^2(1-y) & -x^3(1-y) & -1 \end{pmatrix}.$$

Then $U \in \text{GL}_3(\mathbb{Z}[x, y])$ and $\det U = 1$. This proves that

$$\begin{aligned} &[[1 + xy, x^2, y]] \\ &= \begin{pmatrix} xy & -x(1-xy) & 0 \\ -y & 1-xy & 0 \\ x^2(1-y) & x^3(1-y) & 1 \end{pmatrix} + \begin{pmatrix} 1 & x(1+x-xy) & y \\ y & -1+xy & 0 \\ -x^2(1-y) & -x^3(1-y) & -1 \end{pmatrix} \end{aligned}$$

is clean. \square

4. EXTENSIONS

In [2], Camillo and Yu proved that every element of a clean ring in which 2 is invertible is $(s, 2)$. In this section, we investigate some sufficient conditions under which an $n \times n$ matrix $[[a_1, a_2, \dots, a_n]]$ over a commutative ring is $(s, 2)$.

Theorem 4.1. Let $a_1, \dots, a_n \in R$. Then $[[a_1, a_2, \dots, a_n]]$ is $(s, 2)$ provided the following conditions hold:

- (1) There exist $x_1, \dots, x_n \in R$ such that $a_1x_1 + \dots + a_nx_n \in R$ is $(s, 2)$.
- (2) $x_1 \equiv 1 \pmod{x_2R + \dots + x_nR}$.

Proof. By (1), we can find $x_1, \dots, x_n \in R$ such that $a_1x_1 + \dots + a_ix_i + \dots + a_nx_n$ is $(s, 2)$. Let $c_1 = x_1$ and $c_i = -x_i$ ($2 \leq i \leq n$). Then $a_1c_1 - \dots - a_ic_i - \dots - a_nc_n = u + v$ for some $u, v \in U(R)$. Let

$$U = (e_{ij}) = \begin{pmatrix} 1 & -k_2 & \dots & -k_n \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \begin{pmatrix} u & & & \\ c_2 & 1 & & \\ \vdots & & \ddots & \\ c_n & & & 1 \end{pmatrix} \begin{pmatrix} 1 & k_2 & \dots & k_n \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}.$$

Obviously, $U \in \text{GL}_n(R)$ and $\det U = u$. Let

$$V = (v_{ij}) = \begin{pmatrix} a_1 & \dots & a_n \\ 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{pmatrix} - U.$$

Then

$$\begin{aligned} \det V &= \begin{vmatrix} a_1 - u_{11} & a_2 - u_{12} & \dots & a_n - u_{1n} \\ -u_{21} & -u_{22} & \dots & -u_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -u_{n1} & -u_{n2} & \dots & -u_{nn} \end{vmatrix} \\ &= \begin{vmatrix} a_1 & a_2 & \dots & a_n \\ -u_{21} & -u_{22} & \dots & -u_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -u_{n1} & -u_{n2} & \dots & -u_{nn} \end{vmatrix} + (-1)^n \det U \\ &= (-1)^{n-1} (a_1 A_{11} + \dots + a_n A_{1n}) + (-1)^n u, \end{aligned}$$

where A_{11}, \dots, A_{1n} are algebraic complements of U corresponding to u_{11}, \dots, u_{1n} , respectively. Clearly,

$$U = \begin{pmatrix} 1 + (u - c_1) & (u - c_1)k_2 & (u - c_1)k_3 & \dots & (u - c_1)k_n \\ c_2 & 1 + k_2c_2 & k_3c_2 & \dots & k_nc_2 \\ c_3 & k_2c_3 & 1 + k_3c_3 & \dots & k_nc_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_n & k_2c_n & k_3c_n & \dots & 1 + k_nc_n \end{pmatrix}$$

It is easy to see that $A_{11} = 1 + k_2c_2 + \dots + k_nc_n = c_1$. As in the proof of Theorem 2.1, we see that $A_{1i} = -c_i$ ($2 \leq i \leq n$). Thus, $a_1A_{11} + \dots + a_{1n}A_{1n} = a_1c_1 - \dots - a_1c_i - \dots - a_nc_n = u + v$. Hence, $\det U = (-1)^{n-1}(u + v) + (-1)^n u = (-1)^{n-1}v \in U(R)$, and so $U \in \text{GL}_n(R)$. Consequently, we conclude that A is $(s, 2)$, as desired. \square

Corollary 4.2. *Let $a_1, \dots, a_n \in R$. Then $[[a_1, a_2, \dots, a_{n+1}]] \in M_{n+1}(R)$ is $(s, 2)$ provided the following conditions hold:*

- (1) *There exist $u, v \in U(R)$ such that $1 = u + v$.*
- (2) *R satisfies the n -stable range condition.*
- (3) *$(a_1, \dots, a_{n+1}) = 1$.*

Proof. By (2) and (3), there exist $c_2, \dots, c_{n+1} \in R$ such that $(a_2 + a_1c_2, \dots, a_{n+1} + a_1c_{n+1}) = 1$. Let $b_i = a_i + a_1c_i$ ($2 \leq i \leq n$). Then there are $x_2, \dots, x_{n+1} \in R$ such that $b_2x_2 + \dots + b_{n+1}x_{n+1} = 1$. By (1), $a_1 \times 0 + b_2x_2 + \dots + b_{n+1}x_{n+1} = u + v$. Clearly,

$$0 \equiv 1 \pmod{x_2R + \dots + x_{n+1}R}.$$

By Theorem 4.1, there are $U, V \in \text{GL}_{n+1}(R)$ such that $[[a_1, b_2, \dots, b_{n+1}]] = U + V$. Let

$$Q = \begin{pmatrix} 1 & c_2 & \dots & c_{n+1} \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix} \in \text{GL}_{n+1}(R).$$

Then $[[a_1, a_2, \dots, a_{n+1}]]Q = [[a_1, a_2 + a_1c_2, \dots, a_{n+1} + a_1c_{n+1}]] = U + V$. This implies that $[[a_1, a_2, \dots, a_{n+1}]] = UQ^{-1} + VQ^{-1}$, as required. \square

Example 4.3. Let $R = \{0, e, a, b\}$ be a set. Define operations by the following tables:

+	0	e	a	b
0	0	e	a	b
e	e	0	b	a
a	a	b	0	e
b	b	a	e	0

×	0	e	a	b
0	0	0	0	0
e	0	e	a	b
a	0	a	b	e
b	0	b	e	a

Then R is a field with four elements. In this case, $2 \notin U(R)$ and the identity $e \in R$ is the sum $a + b$ of two units $a, b \in R$. Let $[[e + x, x^2, e - x]] \in M_3(R[x])$. Then $(e + x)(e - x) + x^2 \times 1 + (e - x) \times 0 = e$. Clearly, $R[x]$ satisfies the 2-stable range condition. According to Corollary 4.2, $[[e + x, x^2, e - x]] \in M_3(R[x])$ is $(s, 2)$. \square

Theorem 4.4. *Let $a_1, \dots, a_n \in R$. If R satisfies the 1-stable range condition, then $[[a_1, a_2, \dots, a_n]]$ is $(s, 2)$ iff the following conditions hold:*

- (1) *There exist $x_1, \dots, x_n \in R$ such that $a_1x_1 + \dots + a_nx_n$ is $(s, 2)$.*
- (2) *$x_1 \equiv 1 \pmod{x_2R + \dots + x_nR}$.*

Proof. “ \Leftarrow ” is clear by Theorem 4.1.

“ \Rightarrow ” Suppose that $[[a_1, a_2, \dots, a_n]]$ is $(s, 2)$. Then we have two matrices $U = (u_{ij}), V = (v_{ij}) \in \text{GL}_n(R)$ such that $[[a_1, a_2, \dots, a_n]] = U + V$. Thus,

$$\begin{pmatrix} a_1 - u_{11} & a_2 - u_{12} & \dots & a_n - u_{1n} \\ -u_{21} & -u_{22} & \dots & -u_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -u_{n1} & -u_{n2} & \dots & -u_{nn} \end{pmatrix} = V.$$

Hence, we get

$$\begin{vmatrix} a_1 & a_2 & \dots & a_n \\ -u_{21} & -u_{22} & \dots & -u_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -u_{n1} & -u_{n2} & \dots & -u_{nn} \end{vmatrix} + (-1)^n \det U = \det V.$$

It follows that $a_1A_{11} + a_2A_{12} + \dots + a_nA_{1n} = (-1)^{n+1}u + v$, where $u = \det U$, $v = \det V$ and A_{1i} ($1 \leq i \leq n$) is the algebraic complement corresponding to a_i ($1 \leq i \leq n$). Let each $x_i = A_{1i}$. Then $a_1x_1 + a_2x_2 + \dots + a_nx_n$ is $(s, 2)$. Obviously, $(-u_{21})A_{11} + (-u_{22})A_{12} + \dots + (-u_{2n})A_{1n} = 0$, and thus,

$$(-u_{21})A_{11} \equiv 0 \pmod{x_2R + \dots + x_nR}.$$

Furthermore, $u_{11}A_{11} + u_{12}A_{12} + \dots + u_{1n}A_{1n} = (-1)^{n+1}u$, and then

$$u_{11}A_{11} \equiv (-1)^{n+1}u \pmod{x_2R + \dots + x_nR}.$$

Since $u \in U(R)$, we see that

$$-u_{21} \equiv 0 \pmod{x_2R + \dots + x_nR}.$$

Likewise, we show that

$$-u_{31}, \dots, -u_{n1} \equiv 0 \pmod{x_2R + \dots + x_nR}.$$

Therefore

$$\begin{pmatrix} -u_{11} & -u_{12} & \cdots & -u_{1n} \\ 0 & -u_{22} & \cdots & -u_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & -u_{n2} & \cdots & -u_{nn} \end{pmatrix} \text{ is invertible (mod } x_2R + \cdots + x_nR).$$

This yields that $A_{11} \in R$ is invertible modulo $x_2R + \cdots + x_nR$. That is, there exists a $v \in R$ such that $r := A_{11}v - 1 \in x_2R + \cdots + x_nR$. Since R satisfies the 1-stable range condition, it follows from $A_{11}v - r = 1$ that $w := A_{11} - rz \in U(R)$ for some $z \in R$. Let $x'_i = A_{1i}w^{-1}$ ($1 \leq i \leq n$). Then $a_1x'_1 + \cdots + a_nx'_n = (-1)^{n+1}uw^{-1} + vw^{-1} \in R$ is $(s, 2)$. In addition, $x'_1 \equiv 1 \pmod{x_2R + \cdots + x_nR}$, and we are done. \square

Corollary 4.5. *Let R be a strongly π -regular ring. Then $[[a_1, a_2, \dots, a_n]]$ is $(s, 2)$ iff the following conditions hold:*

- (1) *There exist $x_1, \dots, x_n \in R$ such that $a_1x_1 + \cdots + a_nx_n$ is $(s, 2)$.*
- (2) *$x_1 \equiv 1 \pmod{x_2R + \cdots + x_nR}$.*

P r o o f. Since every strongly π -regular ring satisfies the 1-stable range condition, we complete the proof by Theorem 4.4. \square

Example 4.6. Let $R = \{a + bt : a, b \in \mathbb{Z}/2\mathbb{Z}, t^2 = 0\}$. Then neither 1 nor $1 + t$ is $(s, 2)$. For any $a + bt \in R$, $(a + bt)^2 = (a + bt)^4$; hence, R is strongly π -regular. As $1 \times 1 + (1 + t) \times 1$ is $(s, 2)$ and $1 \equiv 1 \pmod{R}$, it follows by Corollary 4.5 that $[[1, 1 + t]]$ is $(s, 2)$. In fact, we have the decomposition:

$$[[1, 1 + t]] = \begin{pmatrix} 0 & 1 + t \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

In this case, $2 \notin U(R)$. \square

References

- [1] *V. P. Camillo and D. A. Khurana:* Characterization of unit regular rings. *Comm. Algebra* 29 (2001), 2293–2295.
- [2] *V. P. Camillo and H. P. Yu:* Exchange rings, units and idempotents. *Comm. Algebra* 22 (1994), 4737–4749.
- [3] *H. Chen:* Exchange rings with Artinian primitive factors. *Algebr. Represent. Theory* 2 (1999), 201–207.
- [4] *H. Chen:* Separative ideals, clean elements, and unit-regularity. *Comm. Algebra* 34 (2006), 911–921.
- [5] *J. W. Fisher and R. L. Snider:* Rings generated by their units. *J. Algebra* 42 (1976), 363–368.
- [6] *M. Henriksen:* Two classes of rings generated by their units. *J. Algebra* 31 (1974), 182–193.

- [7] *D. Khurana and T. Y. Lam*: Clean matrices and unit-regular matrices. *J. Algebra* *280* (2004), 683–698.
- [8] *T. Y. Lam*: A crash course on stable range, cancellation, substitution and exchange. *J. Algebra Appl.* *3* (2004), 301–343.
- [9] *W. K. Nicholson and K. Varadarjan*: Countable linear transformations are clean. *Proc. Amer. Math. Soc.* *126* (1998), 61–64.
- [10] *W. K. Nicholson and Y. Zhou*: Clean rings: A survey, *Advances in Ring Theory. Proceedings of the 4th China-Japan-Korea International Conference* (2004), 181–198.
- [11] *R. Raphael*: Rings which are generated by their units. *J. Algebra* *28* (1974), 199–205.
- [12] *K. Samei*: Clean elements in commutative reduced rings. *Comm. Algebra* *32* (2004), 3479–3486.

Author's address: Huanyin Chen, Department of Mathematics, Hangzhou Normal University, Hangzhou 310036, China, e-mail: huanyinchen@yahoo.cn.