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Michal Johanis

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LOCALLY FLAT BANACH SPACES

Michal Johanis, Praha

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Abstract. The notion of functions dependent locally on finitely many coordinates plays an important role in the theory of smoothness and renormings on Banach spaces, especially when higher order smoothness is involved. In this note we investigate the structural properties of Banach spaces admitting (arbitrary) bump functions depending locally on finitely many coordinates.

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In the present paper we investigate the structural properties of Banach spaces admitting (arbitrary) bump functions depending locally on finitely many coordinates (LFC).

The first use of the LFC notion was the construction of C^∞-smooth and LFC renorming of c_0, due to Kuiper, which appeared in [1]. The LFC notion was explicitly introduced and investigated in the paper [21] of Pechanec, Whitfield and Zizler. In their work the authors have proved that every Banach space admitting a LFC bump is saturated with copies of c_0, providing in some sense a converse to Kuiper's result. Not surprisingly, it turns out that the LFC notion is closely related to the class of polyhedral spaces, introduced by Klee [18] and thoroughly investigated by many authors (see [17, Chapter 15] for results and references). Indeed, prior to [21], Fonf [4] has proved that every polyhedral Banach space is saturated with copies of c_0. Later, it was independently proved in [5] and [9] that every separable polyhedral Banach space admits an equivalent LFC norm. Using the last result Fonf's result is a corollary of [21].

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The notion of LFC has been exploited (at least implicitly) in a number of papers, in order to obtain very smooth bump functions, norms and partitions of unity on non-separable Banach spaces, see e.g. [23], [22], [3], [7], [8], [6], [14], [15], [16], [9], [10], [11], and the book [2]. In fact, it seems to be the only general approach to these problems. The reason is simple; it is relatively easy to check the (higher) differentiability properties of functions of several variables, while for functions defined on a Banach space this is very hard.

For separable spaces, one of the main known results is that a separable Banach space is polyhedral if and only if it admits a LFC renorming (resp. C^∞-smooth and LFC renorming) ([9]). This smoothing up result is however obtained by using the boundary of a Banach space, rather than through some direct smoothing procedure. There is a variety of open questions, well known among the workers in the area, concerning the existence and possible smoothing of general non-convex LFC functions.

In our note we try to capture the essence of the structure of Banach spaces admitting (continuous) LFC bumps that is responsible for some of the structural results (e.g. [4], [21], [6]). In fact, we introduce a formally more general notion of a locally flat space, and generalize the known structural results in this context. It is not clear to us whether the generalization is genuine. However, locally flat spaces include for example all spaces admitting a (not necessarily continuous) bump locally depending on finitely many linear (i.e. not necessarily continuous) functionals. This notion offers itself for a possible purely combinatorial characterization of locally flat spaces. Apart from the usual effort to find the essential ingredients in the theory, we feel that the more discrete and combinatorial notions have a better chance for finding characterization, e.g. among the Orlicz sequence spaces. This is crucial for finding new examples.

We use the standard Banach space notation. By $U(x,\delta)$ we denote an open ball centred at x with radius δ. By $X^\#$ we denote an algebraic dual to a vector space X.

The notion of a function, defined on a Banach space with a Schauder basis, which is locally dependent on finitely many coordinates was introduced in [21]. The following definition is a slight generalization which was used by many authors.

Definition 1. Let X be a topological vector space, $\Omega \subset X$ an open subset, E be an arbitrary set, $M \subset X^\#$ and $g\colon \Omega \to E$. We say that g *depends only on* M on a set $U \subset \Omega$ if $g(x) = g(y)$ whenever $x, y \in U$ are such that $f(x) = f(y)$ for all $f \in M$. We say that g *depends locally on finitely many coordinates from* M (LFC-M for short) if for each $x \in \Omega$ there are a neighbourhood $U \subset \Omega$ of x and a finite subset $F \subset M$ such that g depends only on F on U. We say that g *depends locally on finitely many coordinates* (LFC for short) if it is LFC-X^*.

We may equivalently say that g depends only on $\{f_1, \dots, f_n\} \subset X^\#$ on $U \subset \Omega$ if there exist a mapping $G\colon \mathbb{R}^n \to E$ such that $g(x) = G(f_1(x), \dots, f_n(x))$ for all

$x \in U$. Notice that if $g\colon \Omega \to E$ is LFC and $h\colon E \to F$ is any mapping, then also $h \circ g$ is LFC.

The canonical example of a non-trivial LFC function is the sup norm on c_0, which is LFC-$\{e_i^*\}$ away from the origin. Indeed, take any $x = (x_i) \in c_0$, $x \neq 0$. Let $n \in \mathbb{N}$ be such that $|x_i| < \|x\|_\infty/2$ for $i > n$. Then $\|\cdot\|_\infty$ depends only on $\{e_1^*, \dots, e_n^*\}$ on $U(x, \|x\|_\infty/4)$.

The following lemma shows that sometimes it is possible to join together some of the neighbourhoods in the definition of LFC:

Lemma 2. *Let X be a topological vector space, E be an arbitrary set, $g\colon X \to E$ and $M \subset X^{\#}$. Let $U_\alpha \subset X$, $\alpha \in I$ be open sets such that $U = \bigcup_{\alpha \in I} U_\alpha$ is convex and g depends only on M on each U_α, $\alpha \in I$. Then g depends only on M on the whole of U.*

Proof. Pick any $x, y \in U$ such that $f(x) = f(y)$ for all $f \in M$. Since U is convex, the line segment $[x, y] \subset U$. Since $[x, y]$ is compact, there is a finite covering $U_1, \dots, U_n \in \{U_\alpha\}_{\alpha \in I}$ of $[x, y]$. Since $[x, y]$ is connected, without loss of generality we may assume that $x \in U_1$, $y \in U_n$ and there are $x_i \in U_i \cap U_{i+1} \cap [x, y]$ for $i = 1, \dots, n-1$. As $x_i \in [x, y]$, we have $f(x) = f(y) = f(x_i)$ for all $f \in M$ and $i = 1, \dots, n-1$. Therefore $g(x) = g(x_1) = \ldots = g(x_{n-1}) = g(y)$. $\square$

A norm on a normed space is said to be LFC, if it is LFC away from the origin. Recall that a bump function (or bump) on a topological vector space X is a function $b\colon X \to \mathbb{R}$ with a bounded non-empty support.

The existence of a LFC norm (or even a continuous LFC bump) on a Banach space is known to have strong implications on the structure of the space (see e.g. [4], [21], [6]). The role of continuity in these results seems rather interesting. It turns out that the essence lies in the discrete (or combinatorial) structure of the space itself. This leads us to the following general concept:

Definition 3. Let X be a vector space, $A \subset X$, $U \subset X$ be arbitrary subsets of X. We say that A is *determined* on U by a subspace $Z \subset X$ if $U \cap (y + Z) \subset A$ for all $y \in U \cap A$.

Clearly, if A is determined on U by Z then A is determined on U by any subspace of Z.

Let us denote the set of all finite-codimensional subspaces of a vector space X by $\mathscr{FC}(X)$. If X is moreover a topological vector space, we denote by $\mathscr{FC}_c(X)$ the set of all closed finite-codimensional subspaces.

Definition 4. Let X be a topological vector space, $A \subset X$ be an arbitrary subset of X and $\mathscr{Z} \subset \mathscr{FC}(X)$. We say that A is *locally finite-dimensionally determined* by $\mathscr{Z}$ (or LFD-$\mathscr{Z}$ for short) if for any $x \in X$ there is a neighbourhood $U \subset X$ of x and $Z \in \mathscr{Z}$ such that A is determined by Z on U. We say that A is *locally finite-dimensionally determined* (or LFD) if A is LFD-$\mathscr{F}\mathscr{C}(X)$.

Fact 5. *Let X be a topological vector space, let $A \subset X$ and $M \subset X^{\#}$. The function χ_A is LFC-M if and only if A is LFD-$\mathscr{Z}$ for $\mathscr{Z} = \Big\{\bigcap_{i=1}^{n} \ker f_i\,;\ \{f_1, \ldots, f_n\} \subset M,\ n \in \mathbb{N}\Big\}$.*

Proof. A is determined on U by $\bigcap_{i=1}^{n} \ker f_i$ if and only if χ_A on U depends only on $\{f_1, \ldots, f_n\} \subset X^{\#}$. $\square$

Fact 6. *Let X be a topological vector space and $A, B \subset X$.*

(a) *X and $\emptyset$ are LFD. If A and B are LFD, then so are the sets $A \cap B$, $A \cup B$ and $X \setminus A$. In other words, all LFD subsets of X form an algebra.*

(b) *If $T\colon X \to X$ is an automorphism or a translation and A is LFD, then $T(A)$ is also LFD.*

(c) *If A and B are separated (i.e. $A \cap \overline{B} = \bar{A} \cap B = \emptyset$) and $A \cup B$ is LFD, then both A and B are LFD.*

Proof. (a): Fix $x \in X$. If U, V are neighbourhoods of x such that A is determined by Z on U and B is determined by W on V, then both $A \cap B$ and $A \cup B$ are determined by $Z \cap W$ on $U \cap V$. The rest is obvious.

(b): It is obvious, since an automorphism preserves the finite codimension of subspaces.

(c): For a fixed $x \in X$ there is a neighbourhood of x such that $A \cup B$ is determined by $Z \in \mathscr{F}\mathscr{C}(X)$ on U and $U - x$ is balanced, hence U is connected. For any $y \in U \cap A$, we have $U \cap (y + Z) \subset A \cup B$ and $U \cap (y + Z)$ is connected, which means that $U \cap (y + Z) \subset A$. $\square$

Theorem 7. *Let X be a topological vector space, $\mathscr{Z} \subset \mathscr{F}\mathscr{C}(X)$. If $A \subset X$ is LFD-$\mathscr{Z}$, then $\bar{A}$ is LFD-$\tilde{\mathscr{Z}}$, where $\tilde{\mathscr{Z}} = \{\overline{Z}, Z \in \mathscr{Z}\} \subset \mathscr{F}\mathscr{C}_c(X)$.*

Proof. Fix $x \in X$. There is an open neighbourhood of zero U and $Z \in \mathscr{Z}$ such that A is determined on $x + U$ by Z. Let V be an open neighbourhood of zero such that $V + V + V \subset U$. Choose any $y \in (x + V) \cap \bar{A}$ and $z \in \overline{Z}$ such that $y + z \in x + V$. There is a net $\{y_\gamma\} \subset A$ such that $y_\gamma \to y$ and a net $\{z_\gamma\} \subset Z$ such that $z_\gamma \to z$. We can moreover assume that $\{y_\gamma\} \subset x + U$, $\{y_\gamma\} \subset y + V$ and $\{z_\gamma\} \subset z + V$. Then $y_\gamma + z_\gamma - x = (y + z - x) + (y_\gamma - y) + (z_\gamma - z) \in V + V + V \subset U$. Thus $y_\gamma + z_\gamma \in x + U$ which together with $y_\gamma \in (x + U) \cap A$ gives $y_\gamma + z_\gamma \in A$. It follows that $y + z \in \bar{A}$, which means that A is determined on $x + V$ by $\overline{Z}$. $\square$

Similarly, we have

Theorem 8. *Let X be a topological vector space, $\Omega \subset X$ an open subset, E a Hausdorff topological space and $g\colon \Omega \to E$. If g is LFC-$X^{\#}$ and continuous, then g is LFC-X^*.*

P r o o f. Fix $x \in \Omega$. There is a neighbourhood U of x such that g depends only on $\{f_1, \dots, f_n\} \subset X^{\#}$ on U. Let $\{\tilde{f}_1, \dots, \tilde{f}_n\} \subset X^*$ be such that $\bigcap \ker \tilde{f}_i = \overline{\bigcap \ker f_i}$. Choose $y \in U$. Since $g(z) = g(y)$ for any $z \in U$ such that $z \in y + \bigcap \ker f_i$, the continuity of g implies that $g(z) = g(y)$ also for any $z \in U$ such that $z \in \overline{y + \bigcap \ker f_i}$, i.e. whenever $\tilde{f}_i(y) = \tilde{f}_i(z)$ for all $1 \leqslant i \leqslant n$. $\square$

If X is a topological vector space, let us recall that a set-valued mapping $\psi\colon X \to 2^X$ is called a cusco mapping if for each $x \in X$, $\psi(x)$ is a non-empty compact convex subset of X and for each open set U in X, $\{x \in X;\ \psi(x) \subset U\}$ is open.

Lemma 9. *Let X be a locally convex space, E be an arbitrary set and $g\colon X \to E$ be an LFC-M mapping for some $M \subset X^{\#}$. Further, let $\psi\colon X \to 2^X$ be a cusco mapping with the following property: For any finite $F \subset M$, if $x, y \in X$ are such that $f(x) = f(y)$ for all $f \in F$, then for each $w \in \psi(x)$ there is $z \in \psi(y)$ such that $f(w) = f(z)$ for all $f \in F$. Then the mapping $G\colon X \to 2^E$, $G(x) = g(\psi(x))$, is LFC-M.*

P r o o f. Let $x_0 \in X$. We can find a finite covering of the compact $\psi(x_0)$ by open sets U_i, $i = 1, \dots, n$, so that g depends only on a finite set $F_i \subset M$ on U_i. Let W be a convex neighbourhood of zero such that $\psi(x_0) + W \subset \bigcup U_i$ and put $U = \psi(x_0) + W$ and $F = \bigcup F_i$. As U is convex and $U \subset \bigcup U_i$, by Lemma 2, g depends only on F on U.

Suppose $V \subset X$ is a neighbourhood of x_0 such that $\psi(V) \subset U$. Let $x, y \in V$ be such that $f(x) = f(y)$ for all $f \in F$. Choose $w' \in G(x)$ and find $w \in \psi(x)$ for which $g(w) = w'$. Then, by the assumption on ψ, there is $z \in \psi(y)$ such that $f(w) = f(z)$ for all $f \in F$. But we have also $w \in \psi(x) \subset U$ and $z \in \psi(y) \subset U$ and hence $g(w) = g(z)$. Therefore $w' \in G(y)$ and by the symmetry we can conclude that $G(x) = G(y)$. $\square$

As we shall see, the existence of a non-empty bounded LFD set in an infinite-dimensional space has a strong impact on the structure of the space.

Definition 10. We say that a topological vector space X is *locally flat* if there exists a non-empty *bounded* LFD subset $A \subset X$.

Let X be a topological vector space, $Y \subset X$ and $Z \subset X$ be linear subspaces. As follows from the remark after Definition 3 and the fact that $\dim Y/(Y \cap Z) \leqslant \dim X/Z$, any linear subspace of a locally flat space is also locally flat.

By Theorem 7 and Fact 5, X is locally flat if and only if it admits a LFC bump function b (in general arbitrary, i.e. non-continuous). Indeed, then $(1 - \chi_{\{0\}}) \circ b$ is a characteristic function of a bounded set which is LFC.

Theorem 11. *Let X be a locally flat topological vector space. Then X has a basis of neighbourhoods of zero formed by bounded LFD sets.*

P r o o f. It suffices to show that there is a set $C \subset X$ that is a bounded LFD neighbourhood of zero in X, since then by the boundedness $\{n^{-1}C\}_{n=1}^{\infty}$ is a basis of neighbourhoods of zero.

By Fact 6 and Theorem 7 we may assume that there is a closed bounded LFD-$\mathscr{F}\mathscr{C}_c(X)$ subset A of X such that $0 \in A$. There is a neighbourhood U of zero and $Y \in \mathscr{F}\mathscr{C}_c(X)$ such that A is determined by Y on U. Put $A_0 = A \cap Y$. By Fact 6, A_0 is still a closed bounded LFD-$\mathscr{F}\mathscr{C}_c(X)$ subset of X for which $0 \in U \cap Y \subset A_0 \subset Z$.

We assume that $\operatorname{codim} Y = 1$, otherwise we repeat inductively the following construction.

Choose $e \in X \setminus Y$ and denote $B = \{se;\ |s| \leqslant 1\}$. Put $A_1 = A_0 + B$. The set A_1 is bounded and LFD-$\mathscr{F}\mathscr{C}_c(X)$: Fix any $x \in X$, $x = y + te$ for $y \in Y$ and t scalar. There is a neighbourhood V of y such that A_0 is determined on V by some $Z \in \mathscr{F}\mathscr{C}_c(X)$, $Z \subset Y$. We denote $V_Y = V \cap Y$ and put $W = V_Y + te + B$. Since Y is closed and $\operatorname{codim} Y = 1$, the product topology on $Y \oplus \operatorname{span}\{e\}$ coincides with the topology of X and thus W is a neighbourhood of x. Then for any $z \in W \cap A_1$ we have $z = z_1 + se$, where $z_1 \in V_Y \cap A_0 = V \cap A_0$ and $|s| \leqslant 1$. As A_0 is determined by Z on V, we have $V \cap (z_1 + Z) \subset A_0$ and therefore $W \cap (z + Z) = V_Y \cap (z_1 + Z) + se \subset A_0 + se \subset A_1$.

A_1 is a neighbourhood of zero in X, because $A_1 \supset (U \cap Z) + B$ and $U \cap Z$ is a neighbourhood of zero in Z and we use the same argument on product topology as above. □

Using Kolmogorov's theorem we immediately obtain

Corollary 12. *Any Hausdorff locally convex space that is locally flat is normable.*

Another consequence follows from Lemma 9.

Corollary 13. *Let X be a locally flat normed linear space. Then X has a balanced bounded LFD neighbourhood of zero.*

P r o o f. By Theorem 11 there is $A \subset X$ which is a bounded LFD neighbourhood of zero. Define a mapping $\psi\colon X \to 2^X$ by $\psi(x) = \{tx;\ |t| \leqslant 1\}$. It is easy to check that ψ is a cusco mapping. Furthermore, let $F \subset X^{\#}$, and suppose $x, y \in X$ are such that $f(x) = f(y)$ for all $f \in F$. Choose any $w \in \psi(x)$. Then $w = tx$ for some suitable t, $|t| \leqslant 1$, and we have $ty \in \psi(y)$ and $f(w) = f(tx) = f(ty)$ for all $f \in F$. The function $g = \chi_A$ is LFC by Fact 5. Thus Lemma 9 implies that the function $h(x) = \inf\limits_{|t| \leqslant 1} g(tx) = \inf g(\psi(x))$ is LFC.

Let $D = h^{-1}(\{1\})$. This set is LFD by Fact 5. We have $h(x) \leqslant g(x)$ for all $x \in X$ and hence $D \subset A$ and D is bounded. Since A is a neighbourhood of zero, there is some ball B, $B \subset A$, and we have $h(x) = 1$ for any $x \in B$. Thus $B \subset D$ and D is a neighbourhood of zero. Next, $h(tx) = \inf\limits_{|s| \leqslant 1} g(tsx) \geqslant \inf\limits_{|s| \leqslant 1} g(sx) = h(x)$ for any t, $|t| \leqslant 1$. Therefore $x \in D$ implies $tx \in D$ for all t, $|t| \leqslant 1$ and D is balanced. □

Theorem 14. *Let X be a normed linear space, $A \subset X$ be a balanced bounded LFD neighbourhood of zero. If the Minkowski functional p of A is continuous, then it is LFC away from the origin. In particular, if A is moreover convex, then p is an equivalent LFC norm.*

Proof. Without loss of generality we may assume that A is closed and LFD-$\mathscr{FC}_c(X)$.

Fix any $x \in X \setminus \{0\}$ and put $\beta = p(x)$. There is $0 < \delta < \|x\|$ such that βA is determined by a subspace $Z \in \mathscr{FC}_c(X)$ on $U(x, \delta)$. Put $t_1 = (1 + \|x\|/(\|x\| + \delta))/2$ and $t_2 = (1 + \|x\|/(\|x\| - \delta))/2$. Let V be a neighbourhood of x such that $|p(y) - p(x)| < \beta \min\{1 - t_1, t_2 - 1\}$ for $y \in V$. Put

$$U = V \cap \bigcap_{t_1 < t < t_2} U(tx, t\delta) = V \cap U(t_1 x, t_1 \delta) \cap U(t_2 x, t_2 \delta),$$

which is a neighbourhood of x, as by the definition of t_1 and t_2 both $U(t_1 x, t_1 \delta)$ and $U(t_2 x, t_2 \delta)$ are neighbourhoods of x. (The second equality follows by an easy convexity argument.)

It is easy to see that each of the sets $t\beta A$, $t_1 < t < t_2$, is determined on U by Z. Furthermore, $t_1 \beta < p(y) < t_2 \beta$ for any $y \in U$. Since A is closed, we have $y \in p(y)A$ and $y \notin tA$ for $0 < t < p(y)$. Therefore $y + z \in p(y)A$ and $y + z \notin tA$ for $t_1 \beta < t < p(y)$ whenever $z \in Z$ is such that $y + z \in U$. As A is balanced, it follows that $y + z \notin tA$ for all $0 < t < p(y)$ and hence $p(y + z) = p(y)$ whenever $z \in Z$ is such that $y + z \in U$. This means that p depends on U only on $f_1, \ldots, f_n \in X^*$ such that $Z = \bigcap \ker f_i$. □

Theorem 15 ([21]). *An infinite-dimensional locally flat Banach space X is saturated by c_0.*

P r o o f. As any subspace of X is also locally flat, it suffices to prove that $c_0 \subset X$.

Let $A \subset X$ be a non-empty bounded LFD set. Without loss of generality we may assume that $0 \in A \subset B_X$. We will inductively construct a sequence $\{x_i\} \subset X$ as follows: Set $x_0 = 0$. If $x_0, x_1, \ldots, x_{n-1}$ have already been defined, we put

$$A_n = \Bigl\{ y \in X \setminus \{0\};\ \sum_{i=0}^{n-1} \varepsilon_i x_i + \varepsilon_n y \in A \text{ for all choices of signs } \varepsilon_i = \pm 1, \qquad i = 0, \ldots, n \Bigr\}.$$

Using the fact that A is LFD we have $A_n \neq \emptyset$. Indeed, $\sum\limits_{i=0}^{n-1} \varepsilon_i x_i \in A$ for any $\varepsilon_i = \pm 1$ by the construction and in the neighbourhood of each of these points the set A is determined by some finite-codimensional subspace. Since there are finitely many of these points, the intersection of all the respective finite-codimensional subspaces is non-empty and sufficiently small vectors from this intersection belong to A. We put $M_n = \sup\limits_{y \in A_n} \|y\|$, and choose $x_n \in A_n$ such that $\|x_n\| > M_n/2$.

We claim that the series $\sum\limits_{i=1}^{\infty} x_i$ does not converge unconditionally. Indeed, assume the contrary. Then the set $S = \Bigl\{ \sum\limits_{i=1}^{n} \varepsilon_i x_i;\ \varepsilon_i = \pm 1,\ n \in \mathbb{N} \Bigr\} \subset A$ is relatively compact and we can find a finite covering of the compact $\overline{S}$ by open balls $U_1, \ldots, U_n$ with radii $\delta_1, \ldots, \delta_n$ and $Z_1, \ldots, Z_n \subset \mathscr{F}\mathscr{C}(X)$ such that A is determined on $2U_i$ by Z_i, $i = 1, \ldots, n$. We put $Z = \bigcap\limits_{i=1}^{n} Z_i$ and $\delta = \min\limits_{1 \leqslant i \leqslant n} \delta_i$. As $\dim Z = \infty$, we can choose $z \in Z$ for which $\|z\| = \delta$. Since $z \in A_n$ for any $n \in \mathbb{N}$, it follows that $\|x_n\| > M_n/2 \geqslant \delta/2$ for all $n \in \mathbb{N}$, which contradicts the convergence of $\sum\limits_{i=1}^{\infty} x_i$.

Without loss of generality we may assume that $\sum\limits_{i=1}^{\infty} x_i$ is not convergent (otherwise we change appropriately the signs of x_i). As $x_n \in A_n$, we have $\Bigl\| \sum\limits_{i=0}^{n} \varepsilon_i x_i \Bigr\| \leqslant 1$ for any choice of $\varepsilon_i = \pm 1$ and all $n \in \mathbb{N}$. Thus $\sum\limits_{i=0}^{\infty} x_i$ is weakly unconditionally Cauchy and by the Bessaga-Pełczyński theorem ([20, 2.e.4]), X contains an isomorphic copy of c_0. (The canonical basis of c_0 is equivalent to some sequence of blocks of $\{x_i\}$.) $\square$

Theorem 16. *Let $A \subset X$ be a non-empty bounded LFD-$\mathscr{Z}$ subset of a Banach space X. Denote $\mathscr{Z}^{\perp} = \bigcup\{Z^{\perp};\ Z \in \mathscr{Z}\}$. Then $\overline{\mathscr{Z}^{\perp}} = X^*$.*

Proof. Without loss of generality we may assume $A \subset B_X$. Pick any $f \in X^*$ and $\varepsilon > 0$ and notice that f is bounded on A.

Choose any $y_1 \in A$ and define inductively a sequence $\{y_n\} \subset A$. If $y_1, \ldots, y_n$ have already been defined, choose $y_{n+1} \in A$ so that

$$(1)\qquad \varepsilon\|y_{n+1} - y_n\| - f(y_{n+1}) \leqslant \min\Big\{\inf_{x\in A}(\varepsilon\|x - y_n\| - f(x)) + \frac{1}{2^n}, -f(y_n)\Big\}.$$

This is always possible. We put $y_{n+1} = y_n$ if the infimum above is attained in y_n, otherwise the infimum is strictly smaller than $-f(y_n)$ and we can choose a suitable y_{n+1} by the definition of infimum.

The sequence $\{f(y_n)\}$ is non-decreasing and bounded, thus it is convergent in $\mathbb{R}$. Moreover, $\{y_n\}$ is convergent. Indeed, $\|y_m - y_n\| \leqslant \sum\limits_{k=n}^{m-1} \|y_{k+1} - y_k\| \leqslant \sum\limits_{k=n}^{m-1} (f(y_{k+1}) - f(y_k))/\varepsilon = (f(y_m) - f(y_n))/\varepsilon$ for $m > n$, and since $\{f(y_n)\}$ is Cauchy, so is $\{y_n\}$, and there is $x_0 \in X$ such that $y_n \to x_0$.

Finally, (1) implies

$$(2)\quad f(x) - f(y_{n+1}) \leqslant \varepsilon\|x - y_n\| + \frac{1}{2^n} - \varepsilon\|y_{n+1} - y_n\| \text{ for every } x \in A \text{ and } n \in \mathbb{N}.$$

Let $\delta > 0$ and $Z \in \mathscr{Z}$ be such that A is determined by Z on $U(x_0, \delta)$. Let n_0 be such that $y_n \in U(x_0, \delta/2)$ for every $n > n_0$. Pick any $z \in Z$. Choose $t > 0$ sufficiently small so that $\|tz\| < \delta/2$. Then $y_{n+1} + tz \in U(x_0, \delta)$ for every $n > n_0$ and consequently $y_{n+1} + tz \in A$ and by (2),

$$\begin{aligned} f(z) &= \frac{1}{t}(f(y_{n+1} + tz) - f(y_{n+1})) \\ &\leqslant \frac{1}{t}\Big(\varepsilon\|y_{n+1} - y_n + tz\| + \frac{1}{2^n} - \varepsilon\|y_{n+1} - y_n\|\Big) \leqslant \varepsilon\|z\| + \frac{1}{t2^n} \end{aligned}$$

for any $n > n_0$. It follows that $f(z) \leqslant \varepsilon\|z\|$ for any $z \in Z$. By the Hahn-Banach theorem we can find $g \in X^*$ such that $g = f$ on Z and $\|g\| \leqslant \varepsilon$. Clearly, $f - g \in \mathscr{Z}^{\perp}$ and $\|f - (f - g)\| \leqslant \varepsilon$. $\square$

The next corollary removes the assumption of continuity in a theorem from [6].

Corollary 17. *Let X be a Banach space, $M \subset X^*$ and assume X admits an arbitrary LFC-M bump function. Then $\overline{\operatorname{span}}\, M = X^*$.*

P r o o f. Let b be the LFC-M bump function. Put $A = \{x \in X;\ b(x) \neq 0\}$ and $\mathscr{Z} = \Big\{\bigcap\limits_{i=1}^{n} \ker f_i;\ \{f_1, \ldots, f_n\} \subset M,\ n \in \mathbb{N}\Big\}$. Then A is a non-empty bounded LFD-$\mathscr{Z}$ set (Fact 5), $\mathscr{Z}^\perp \subset \operatorname{span} M$ and by Theorem 16, $\overline{\operatorname{span}}\, M = X^*$. $\square$

Corollary 18. *Any infinite-dimensional locally flat Banach space is a c_0-saturated Asplund space.*

P r o o f. Let X be an infinite-dimensional locally flat Banach space. Then X is c_0-saturated by Theorem 15. Since local flatness passes to subspaces, it is enough to show that X^* is separable provided that X is separable.

By the Lindelöf property of X, there exists a countable collection $\mathscr{Z} = \{Z_i\} \subset \mathcal{FC}(X)$ such that A is LFD-$\mathscr{Z}$. If $Z \in \mathscr{FC}(X)$, then $Z^\perp \subset X^*$ is a subspace with $\dim Z^\perp \leqslant \operatorname{codim} Z$. As Z_i is finite-codimensional, we can find $\{f_{i,j}\}_{j=1}^{n_i} \subset Z_i^\perp$, such that $Z_i^\perp = \operatorname{span}\{f_{i,j}\}_{j=1}^{n_i}$, where $n_i \leqslant \operatorname{codim} Z_i$. Notice that we have $\mathscr{Z}^\perp = \bigcup\limits_i Z_i^\perp \subset \operatorname{span} \bigcup\limits_i Z_i^\perp = \operatorname{span} \bigcup\limits_i \{f_{i,1}, \ldots, f_{i,n_i}\}$ and so by Theorem 16, $X^* = \overline{\operatorname{span}} \bigcup\limits_i \{f_{i,1}, \ldots, f_{i,n_i}\}$, hence it is separable. $\square$

However, not all c_0-saturated Asplund spaces are locally flat: Theorem 19 below is a strengthening of a theorem from [19]. (Leung's statement is that the Orlicz sequence space h_M does not admit a LFC norm if M satisfies the condition in Theorem 19.) Leung in [19] constructed a c_0-saturated Orlicz sequence space satisfying the condition in Theorem 19. His space is therefore a separable c_0-saturated Asplund space that is not locally flat. For further results concerning LFC bump functions on Orlicz sequence spaces see [13].

Theorem 19. *Let M be a non-degenerate Orlicz function for which there exists a sequence $\{t_n\}$ decreasing to 0 such that*

$$\sup_n \frac{M(Kt_n)}{M(t_n)} < \infty \quad \text{for all } 0 < K < \infty.$$

Then the Orlicz sequence space h_M is not locally flat.

Before proving Theorem 19, we make the following observation: If X has a shrinking Schauder basis, we may study only the sets that are locally determined by the subspaces generated by natural projections associated with the basis. This follows from the following reformulation of [12, Corollary 10] using Fact 5. (Notice also, that taking into account Corollary 17, Schauder bases that are shrinking emerge quite naturally in this context.)

Theorem 20. *Let X be a Banach space with a shrinking Schauder basis $\{e_i\}$, $A \subset X$ be LFD-$\mathscr{F}\mathscr{C}_c(X)$ and $\varepsilon > 0$. Then there is a (shrinking) Schauder basis $\{x_i\}$ of X, $(1+\varepsilon)$-equivalent to $\{e_i\}$, such that A is LFD-$\mathscr{Z}$ for $\mathscr{Z} = \{\overline{\operatorname{span}}\{x_i\}_{i=n}^{\infty};\ n \in \mathbb{N}\}$.*

P r o o f of Theorem 19. Suppose that there is a non-empty bounded $A \subset h_M$ which is LFD. Without loss of generality we may assume that $0 \in A \subset B_X$ and A is LFD-$\mathscr{Z}$, where $\mathscr{Z} = \{\overline{\operatorname{span}}\{e_i\}_{i=n}^{\infty};\ n \in \mathbb{N}\}$. (Since h_M is c_0-saturated by Theorem 15, it does not contain ℓ_1. As $\{e_i\}$ is unconditional, it is shrinking by James's theorem. Now consider $T(\bar{A})$, where $T\colon X \to X$ is an equivalence isomorphism of the bases $\{x_i\}$ and $\{e_i\}$ from Theorem 20.)

Notice that the vectors with coordinates in the set $\{t_n\} \cup \{0\}$ have the property of "bounded completeness": If $\left\|\sum\limits_{i=1}^{k} t_{m_i} e_i\right\| \leqslant 1$ for all $k \in \mathbb{N}$, where $m_i \in \mathbb{N} \cup \{0\}$ are not necessarily distinct (we put $t_0 = 0$), then $\sum\limits_{i=1}^{\infty} t_{m_i} e_i$ converges in h_M. Indeed, it follows that $\sum\limits_{i=1}^{k} M(t_{m_i}) \leqslant 1$ for all $k \in \mathbb{N}$. For all $0 < K < \infty$ and all $k \in \mathbb{N}$,

$$\sum_{i=1}^{k} M(Kt_{m_i}) \leqslant \sup_n \frac{M(Kt_n)}{M(t_n)} \sum_{i=1}^{k} M(t_{m_i}) \leqslant \sup_n \frac{M(Kt_n)}{M(t_n)}.$$

Consequently, $\sum\limits_{i=1}^{\infty} M(Kt_{m_i}) < \infty$ for all $0 < K < \infty$, and the sum $\sum\limits_{i=1}^{\infty} t_{m_i} e_i$ converges in h_M.

We construct a sequence $\{x_k\} \subset A$ by induction. Put $x_0 = 0 \in A$ and define natural numbers $m_0 = n_0 = 1$. If $m_{k-1} \in \mathbb{N}$, $n_{k-1} \in \mathbb{N}$ and $x_{k-1} \in A$ are already defined, we put

$$M_k = \{(m,n) \in \mathbb{N}^2;\ m \geqslant m_{k-1},\ n > n_{k-1} \text{ and } x_{k-1} + t_m e_n \in A\}.$$

As A is determined by some $W \in \mathscr{Z}$ on a neighbourhood of x_{k-1}, where W contains all e_n for n big enough, and $t_m \to 0$, we can see that $M_k \neq \emptyset$. Let $(m_k, n_k) = \min M_k$ in the lexicographic ordering of $\mathbb{N}^2$ and put $x_k = x_{k-1} + t_{m_k} e_{n_k}$.

Since $\{x_k\} \subset A \subset B_X$ and $x_k = \sum\limits_{i=1}^{k} t_{m_i} e_{n_i}$, by the above argument $x_k \to x \in h_M$. We can find $\delta > 0$ so that A is determined by some $Z \in \mathscr{Z}$ on $U(x, \delta)$. There is $N \in \mathbb{N}$ such that $\{e_i\}_{i>N} \subset Z$. Because x_k converges, we have $m_k \to \infty$ and so there is $j \in \mathbb{N}$ such that $x_j \in U(x, \delta/2)$, $\|t_{m_j} e_1\| < \delta/2$, $m_j < m_{j+1}$ and $n_j > N$. Then $x_j + t_{m_j} e_{n_j+1} \in A$ and therefore $(m_j, n_j+1) \in M_{j+1}$. But $(m_j, n_j+1) < (m_{j+1}, n_{j+1})$, which is a contradiction. $\square$

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Author's address: Michal Johanis, Department of Mathematical Analysis, Charles University, Sokolovská 83, 186 75 Praha 8, Czech Republic, e-mail: `johanis@karlin.mff.cuni.cz`.