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INTEGRAL FORMULAS FOR CLOSED SPACELIKE HYPERSURFACES IN ANTI-DE SITTER SPACE $H_1^{n+1}(-1)$

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Abstract. In this paper, we study closed k-maximal spacelike hypersurfaces M^n in anti-de Sitter space $H_1^{n+1}(-1)$ with two distinct principal curvatures and give some integral formulas about these hypersurfaces.

Keywords: anti-de Sitter space, kth mean curvature, Gauss equations

MSC 2010: 53B30, 53C50, 53Z05, 53C40

1. Introduction

Let $M_1^{n+1}(c)$ be an $(n+1)$-dimensional Lorentzian manifold of constant curvature c, which we also call a *Lorentzian space form*. When $c > 0$, then $M_1^{n+1}(c) = S_1^{n+1}(c)$ (i.e. $(n+1)$-dimensional de Sitter space); when $c = 0$, then $M_1^{n+1}(c) = L^{n+1}$ (i.e. $(n+1)$-dimensional Lorentz-Minkowski space); when $c < 0$, then $M_1^{n+1}(c) = H_1^{n+1}(c)$ (i.e. $(n+1)$-dimensional anti-de Sitter space). A hypersurface M of $M_1^{n+1}(c)$ is said to be spacelike if the induced metric on M from that of the ambient space is positive definite.

E. Calabi [1] was the first to study the Bernstein problem for a maximal spacelike entire graph in L^{n+1} and proved that it has to be a hyperplane provided $n \leqslant 4$. S. Y. Cheng and S. T. Yau [3] proved that the conclusion remains true for all n. As a generalization of the Bernstein type problem, Cheng-Yau [3] and T. Ishihara [5] proved that a complete maximal spacelike submanifold M of $M_1^{n+1}(c)$ $(c \geqslant 0)$ is totally geodesic.

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In recent years, the main stream of investigation has turned towards more general classes of Lorentz ambient spaces, dealing with problems concerning the kth mean curvatures H_k of M. When $H_k = 0$, a spacelike hypersurface M is said to be k-maximal. In particular, what can we say about k-maximal spacelike hypersurfaces in $H_1^{n+1}(-1)$?

On the other hand, some interesting results for hypersurfaces with two distinct principal curvatures can be found in [2], [4], [10], [11], [12].

In this paper we study closed k-maximal spacelike hypersurfaces M^n in the anti-de Sitter space $H_1^{n+1}(-1)$ with two distinct principal curvatures. We prove the following result:

Theorem 1.1. *For $n \geqslant 3$, let M be an n-dimensional closed k-maximal not $(k+1)$-maximal connected spacelike hypersurface in the anti-de Sitter space $H_1^{n+1}(-1)$ with two distinct principal curvatures, then*

$$\int_M \Big\{S - \frac{n(k^2-2k+n)}{k(n-k)}\Big\}\,\mathrm{d}M > 0, \tag{1.1}$$

where S is the squared norm of the second fundamental form of M.

2. Preliminaries

Let M be an n-dimensional closed spacelike hypersurface of the anti-de Sitter space $H_1^{n+1}(-1)$. For any $p \in M$ we choose a local orthonormal frame $e_1, \ldots, e_n, e_{n+1}$ in $H_1^{n+1}(-1)$ around p such that $e_1, \ldots, e_n$ are tangent to M. Take the corresponding dual coframe $\omega_1, \ldots, \omega_n, \omega_{n+1}$ with the matrix of connection one forms being ω_{ij}. The metric of $H_1^{n+1}(-1)$ is given by $\overline{ds^2} = \sum\limits_i \omega_i^2 - \omega_{n+1}^2$.

In this paper we shall make use of the following convention on the ranges of indices:

$$1 \leqslant i, j, k, \ldots \leqslant n, \quad 1 \leqslant a, b, \ldots \leqslant n-1.$$

A well-known argument [3] shows that the forms ω_{in+1} may be expressed as $\omega_{in+1} = \sum\limits_j h_{ij}\omega_j$, $h_{ij} = h_{ji}$. The second fundamental form is $B = \sum\limits_{i,j} h_{ij}\omega_i \bigotimes \omega_j$. The mean curvature of M is given by $H = \frac{1}{n}\sum\limits_i h_{ii}$.

The Gauss equations are

$$R_{ijkl} = -(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) - (h_{ik}h_{jl} - h_{il}h_{jk}), \tag{2.1}$$

$$R_{ij} = -(n-1)\delta_{ij} - nHh_{ij} + \sum_k h_{ik}h_{kj}, \tag{2.2}$$

$$n(n-1)(r+1) = -n^2H^2 + S, \tag{2.3}$$

where r is the normalized scalar curvature of M and the squared norm of the second fundamental form is

$$S = \sum_{i,j} (h_{ij})^2. \tag{2.4}$$

The Codazzi equations are

$$h_{ijk} = h_{ikj}, \tag{2.5}$$

where the covariant derivative of h_{ij} is defined by

$$\sum_k h_{ijk}\omega_k = dh_{ij} + \sum_k h_{kj}\omega_{ki} + \sum_k h_{ik}\omega_{kj}. \tag{2.6}$$

The second covariant derivative of h_{ij} is defined by

$$\sum_l h_{ijkl}\omega_l = dh_{ijk} + \sum_l h_{ljk}\omega_{li} + \sum_l h_{ilk}\omega_{lj} + \sum_l h_{ijl}\omega_{lk}. \tag{2.7}$$

By exterior differentiation of (2.6), we obtain the Ricci identities

$$h_{ijkl} - h_{ijlk} = \sum_m h_{mj}R_{mikl} + \sum_m h_{im}R_{mjkl}. \tag{2.8}$$

We may choose a frame field $\{e_1, \ldots, e_{n+1}\}$ such that

$$\omega_{in+1} = \lambda_i\omega_i, \quad \text{that is} \quad h_{ij} = \lambda_i\delta_{ij}, \;\; i = 1, 2, \ldots, n, \tag{2.9}$$

where λ_i are the principal curvatures. If we assume that M is a closed spacelike hypersurface with two distinct principal curvatures, one of the principal curvatures of M is simple (i.e. of multiplicity 1). Then we may put

$$\lambda_1 = \lambda_2 = \ldots = \lambda_{n-1} = \lambda; \quad \lambda_n = \mu; \;\; \lambda \neq \mu. \tag{2.10}$$

Then the kth mean curvature H_k of the hypersurface can be given in such a way that

$$S_k = (-1)^k C_n^k H_k = \sum_{1\leqslant i_1 < i_2 < \ldots < i_k \leqslant n} \lambda_{i_1} \ldots \lambda_{i_k}, \tag{2.11}$$

where $C_n^k = n!/k!(n-k)!$.

From (2.10) and $H_k \equiv 0$ we deduce that

$$S_k = (-1)^k C_n^k H_k = C_{n-1}^k \lambda^k + C_{n-1}^{k-1} \lambda^{k-1} \mu \equiv 0, \tag{2.12}$$

and it follows that

$$\lambda^{k-1}[(n-k)\lambda + k\mu] \equiv 0. \tag{2.13}$$

On the other hand, we know that

$$S_{k+1} = \lambda^k (C_{n-1}^{k+1} \lambda + C_{n-1}^k \mu) \neq 0, \tag{2.14}$$

then it follows form (2.13) and (2.14) that

$$\lambda \neq 0, \quad (n-k)\lambda + k\mu \equiv 0. \tag{2.15}$$

By making use of the similar methods in [9], we can prove the following

Lemma 2.1 ([9]). *Let M be an n-dimensional spacelike hypersurface in $H_1^{n+1}(-1)$ such that the multiplicities of all its principal curvatures are constant. Then the distribution of the space of principal vectors corresponding to each principal curvature is completely integrable. In particular, if the multiplicity of a principal curvature is greater than 1, then this principal curvature is constant on each integral submanifold of the corresponding distribution of the space of principal vectors.*

By Lemma 2.1 and (2.10) we have

$$\lambda_{,1} = \ldots = \lambda_{,n-1} = 0, \quad \mu_{,1} = \ldots = \mu_{,n-1} = 0. \tag{2.16}$$

By means of (2.6) and (2.9) we obtain

$$h_{ijk}\omega_k = \delta_{ij} \mathrm{d}\lambda_j + (\lambda_i - \lambda_j)\omega_{ij}. \tag{2.17}$$

Summarizing the above arguments we obtain

$$h_{ijk} = 0, \quad \text{if } i \neq j, \ \lambda_i = \lambda_j, \tag{2.18}$$

$$h_{aab} = 0, \quad h_{aan} = \lambda_{,n}, \tag{2.19}$$

$$h_{nna} = 0, \quad h_{nnn} = \mu_{,n}. \tag{2.20}$$

Proposition 2.1. *For $n \geqslant 3$ let M be an n-dimensional k-maximal not $(k+1)$-maximal connected spacelike hypersurface in $H_1^{n+1}(-1)$ with two distinct principal curvatures. Then M is a locus of the moving $(n-1)$-dimensional submanifold $M_1^{n-1}(s)$ along which the principal curvature λ of multiplicity $(n-1)$ is constant and λ satisfies the following ordinary differential equation of order 2:*

$$\frac{\mathrm{d}^2\lambda}{\mathrm{d}s^2} = \frac{n+k}{n\lambda}\Big(\frac{\mathrm{d}\lambda}{\mathrm{d}s}\Big)^2 + \frac{n(n-k)\lambda^3}{k^2} - \frac{n\lambda}{k}. \tag{2.21}$$

3. Proof of Theorem 1.1

We first prove the following key lemma.

Lemma 3.1. *For $n \geqslant 3$, let M be an n-dimensional k-maximal not $(k+1)$-maximal spacelike hypersurface in $H_1^{n+1}(-1)$ with two distinct principal curvatures. Then we have*

$$\frac{1}{S}\sum_k (S_{,k})^2 = \frac{4n(k^2-2k+n)}{(3n-2)k^2-2nk+n^2}\sum_{i,j,k} h_{ijk}^2. \tag{3.1}$$

Proof. Let $\lambda_1 = \ldots = \lambda_{n-1} = \lambda$, $\lambda_n = \mu$, then we have $(n-k)\lambda + k\mu = 0$. A direct calculation then gives

$$S = (n-1)\lambda^2 + \mu^2 = \frac{n(k^2-2k+n)\lambda^2}{k^2}, \tag{3.2}$$

$$S_{,i} = \frac{2n(k^2-2k+n)}{k^2}\lambda\lambda_{,i}.$$

By (3.2) and (2.16) we have

$$\frac{1}{S}\sum_k (S_{,k})^2 = \frac{1}{S}(S_{,n})^2 = \frac{4n(k^2-2k+n)}{k^2}(\lambda_{,n})^2. \tag{3.3}$$

On the other hand, (2.18), (2.19) and (2.20) yield

$$\begin{aligned} \sum_{i,j,k} h_{ijk}^2 &= \sum_{a,b,c} h_{abc}^2 + 3\sum_{a,b} h_{abn}^2 + 3\sum_a h_{ann}^2 + h_{nnn}^2 \\ &= 3\sum_a h_{naa}^2 + h_{nnn}^2 = 3(n-1)(\lambda_{,n})^2 + (\mu_{,n})^2 \\ &= \frac{(3n-2)k^2-2nk+n^2}{k^2}(\lambda_{,n})^2. \end{aligned} \tag{3.4}$$

We have completed the proof of Lemma 3.1. □

Lemma 3.2. *Let M be an n-dimensional spacelike hypersurface in $H_1^{n+1}(-1)$. Then we have*

$$\frac{1}{2}\Delta S=\sum_{i,j,k}h_{ijk}^2+\sum_i\lambda_i(nH)_{,ii}+\frac{1}{2}\sum_{i,j}R_{ijij}(\lambda_i-\lambda_j)^2, \tag{3.5}$$

where λ_i are the principal curvatures of M and $(\cdot)_{,ij}$ is the covariant derivative relative to the induced metric.

P r o o f of Theorem 1.2. Since $\lambda\neq 0$ and $H_k=0$, we deduce from (2.13) that

$$(n-k)\lambda+k\mu=0 \tag{3.6}$$

on M. First, we compute

$$\frac{1}{2}\Delta(\ln S)=\frac{1}{2}\sum_k(\ln S)_{,kk}=\frac{1}{2}\sum_k\Big(\frac{S_{,k}}{S}\Big)_{,k}=\frac{1}{2}\frac{\Delta S}{S}-\frac{1}{2}\frac{\sum_k(S_{,k})^2}{S^2}. \tag{3.7}$$

Using Lemma 3.2 and the Gauss equation $R_{anan}=-1-\lambda\mu$, we obtain

$$\begin{aligned}\frac{1}{2}\Delta S&=\sum_{i,j,k}h_{ijk}^2+\frac{1}{2}\sum_{i,j}R_{ijij}(\lambda_i-\lambda_j)^2+\sum_i\lambda_i(nH)_{,ii}\\&=\sum_{i,j,k}h_{ijk}^2+\sum_a R_{anan}(\lambda-\mu)^2+\sum_i\lambda_i(nH)_{,ii}\\&=\sum_{i,j,k}h_{ijk}^2+(n-1)(-1-\lambda\mu)(\lambda-\mu)^2+\sum_i\lambda_i(nH)_{,ii}\\&=\sum_{i,j,k}h_{ijk}^2+\frac{(n-1)n^2}{k^2}\Big[-1+\frac{n-k}{k}\lambda^2\Big]\lambda^2+\sum_i\lambda_i(nH)_{,ii}.\end{aligned} \tag{3.8}$$

From (2.7) and (2.9) we have

$$\lambda_{,ij}\omega_j=\mathrm{d}\lambda_{,i}+\lambda_{,j}\omega_{ji}. \tag{3.9}$$

From (2.17), (2.18), (2.19) and (2.20) we obtain

$$\omega_{an}=\frac{\lambda_{,n}}{\lambda-\mu}\,\omega_a.$$

Therefore, we have $\mathrm{d}\omega_n=\sum_i\omega_{ni}\wedge\omega_i=0$, which shows that we may put

$$\omega_n=\mathrm{d}s.$$

Then we have from $(n-k)\lambda + k\mu = 0$ that

$$\omega_{an} = \frac{k\lambda_{,n}}{n\lambda}\,\omega_a = (\log\lambda^{k/n})'\omega_a,$$

where the prime denotes the derivative with respect to s.

Let $i = a$ in (3.9). We see from (2.16), (2.18), (2.19) and (2.20) that

$$\lambda_{,aj}\omega_j = \mathrm{d}\lambda_{,a} + \lambda_{,j}\omega_{ja} = \lambda_{,n}\omega_{na} = \lambda_{,n}\frac{\lambda_{,n}}{\mu-\lambda}\,\omega_a = -\frac{k}{n\lambda}(\lambda_{,n})^2\omega_a. \tag{3.10}$$

It follows that

$$\lambda_{,aa} = -\frac{k}{n\lambda}(\lambda_{,n})^2. \tag{3.11}$$

Let $i = n$ in (3.9). We know from (2.16) and (2.21) that

$$\lambda_{,nj}\omega_j = \mathrm{d}\lambda_{,n} + \lambda_{,j}\omega_{jn} = \mathrm{d}\lambda_{,n} = \left\{\frac{n+k}{n\lambda}(\lambda_{,n})^2 + \frac{n(n-k)\lambda^3}{k^2} - \frac{n\lambda}{k}\right\}\omega_n. \tag{3.12}$$

It follows that

$$\lambda_{,nn} = \frac{n+k}{n\lambda}(\lambda_{,n})^2 + \frac{n(n-k)\lambda^3}{k^2} - \frac{n\lambda}{k}. \tag{3.13}$$

Putting (3.11) and (3.13) into (3.8), we have

$$\begin{aligned}
\frac{1}{2}\Delta S &= \sum_{i,j,k} h_{ijk}^2 + \frac{(n-1)n^2}{k^2}\Big[-1 + \frac{n-k}{k}\lambda^2\Big]\lambda^2 + \sum_i \lambda_i (nH)_{,ii} \\
&= \sum_{i,j,k} h_{ijk}^2 + \frac{(n-1)n^2}{k^2}\Big[-1 + \frac{n-k}{k}\lambda^2\Big]\lambda^2 \\
&\quad + (n-1)\lambda\frac{n(k-1)}{k}\lambda_{,aa} + \mu\frac{n(k-1)}{k}\lambda_{,nn} \\
&= \sum_{i,j,k} h_{ijk}^2 + \frac{(n-1)n^2}{k^2}\Big[-1 + \frac{n-k}{k}\lambda^2\Big]\lambda^2 + (n-1)(1-k)(\lambda_{,n})^2 \\
&\quad - \frac{n(n-k)(k-1)}{k^2}\Big[\frac{n+k}{n}(\lambda_{,n})^2 + \frac{n(n-k)}{k^2}\lambda^4 - \frac{n}{k}\lambda^2\Big] \\
&= \Big\{1 - \frac{(k-1)[(n-2)k^2+n^2]}{(3n-2)k^2 - 2nk + n^2}\Big\}\sum_{i,j,k} h_{ijk}^2 \\
&\quad + \frac{n^2(k^2-2k+n)}{k^4}\lambda^2\{-k + (n-k)\lambda^2\}.
\end{aligned} \tag{3.14}$$

Putting (3.14), (3.1) and (3.2) into (3.7), we have

$$
\begin{aligned}
(3.15)\quad & \frac{1}{2}\Delta(\ln S) \\
&= \frac{1}{2}\frac{\Delta S}{S} - \frac{1}{2}\frac{\sum_k (S_{,k})^2}{S^2} \\
&= \frac{1}{2S}\Big\{\Big[1 - \frac{(k-1)((n-2)k^2+n^2)}{(3n-2)k^2-2nk+n^2}\Big]\sum_{i,j,k} h_{ijk}^2 \\
&\qquad + \frac{n^2(k^2-2k+n)}{k^4}\lambda^2[-k+(n-k)\lambda^2]\Big\} - \frac{1}{2}\frac{\sum_k (S_{,k})^2}{S^2} \\
&= \Big\{\frac{(3n-2)k^2-2nk+n^2-(k-1)[(n-2)k^2+n^2]}{4n(k^2-2k+n)} - \frac{1}{2}\Big\}\frac{\sum_k (S_{,k})^2}{S^2} \\
&\quad + \frac{n}{k}\Big\{-1+\frac{k(n-k)}{n(k^2-2k+n)}S\Big\} \\
&= -\frac{k(n-2)(k-1)^2}{4n(k^2-2k+n)}\frac{\sum_k (S_{,k})^2}{S^2} + \frac{n}{k}\Big\{-1+\frac{k(n-k)}{n(k^2-2k+n)}S\Big\} \\
&\leqslant \frac{n}{k}\Big\{-1+\frac{k(n-k)}{n(k^2-2k+n)}S\Big\}.
\end{aligned}
$$

Integrating (3.15) over M, we get

$$\int_M \Big\{S - \frac{n(k^2-2k+n)}{k(n-k)}\Big\}\,\mathrm{d}M \geqslant 0.$$

Equality holds if and only if M has constant principal curvatures and M is not compact. In our case, M is compact, so equality dose not hold. This completes the proof of our Theorem 1.1 from introduction.

References

[1] *E. Calabi*: Examples of Bernstein problems for nonlinear equations. Proc. Symp. Pure Math. *15* (1970), 223–230.

[2] *L. Cao, G. Wei*: A new characterization of hyperbolic cylinder in anti de Sitter space $H_1^{n+1}(-1)$. J. Math. Anal. Appl. *329* (2007), 408–414.

[3] *S. Y. Cheng, S. T. Yau*: Maximal space-like hypersurfaces in the Lorentz-Minkowski spaces. Ann. Math. *104* (1976), 407–419.

[4] *Z. Hu, M. Scherfner, S. Zhai*: On spacelike hypersurfaces with constant scalar curvature in the de Sitter space. Differ. Geom. Appl. *25* (2007), 594–611.

[5] *T. Ishihara*: Maximal spacelike submanifolds of a pseudo-Riemannian space of constant curvature. Mich. Math. J. *35* (1988), 345–352.

[6] *H. Li*: Hypersurfaces with constant scalar curvature in space forms. Math. Ann. *305* (1996), 665–672.

[7] *H. Li*: Global rigidity theorems of hypersurfaces. Ark. Mat. *35* (1997), 327–351.

[8] *H. Omori*: Isometric immersions of Riemannian manifolds. J. Math. Soc. Japan. *19* (1967), 205–214.

[9] *T. Otsuki*: Minimal hypersurfaces in a Riemannian manifold of constant curvature. Am. J. Math. *92* (1970), 145–173.
[10] *O. Perdomo*: Rigidity of minimal hypersurfaces of spheres with two principal curvatures. Arch. Math. *82* (2004), 180–184.
[11] *Q. L. Wang*: Rigidity of Clifford minimal hypersurfaces. Monatsh. Math. *140* (2003), 163–167.
[12] *G. X. Wei*: Rigidity theorem for hypersurfaces in a unit sphere. Monatsh. Math. *149* (2006), 343–350.

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