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BOUNDARY FUNCTIONS ON A BOUNDED BALANCED DOMAIN

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Abstract. We solve the following Dirichlet problem on the bounded balanced domain Ω with some additional properties: For $p > 0$ and a positive lower semi-continuous function u on $\partial\Omega$ with $u(z) = u(\lambda z)$ for $|\lambda| = 1$, $z \in \partial\Omega$ we construct a holomorphic function $f \in \mathbb{O}(\Omega)$ such that $u(z) = \int_{\mathbb{D}z} |f|^p d\mathfrak{L}^2_{\mathbb{D}z}$ for $z \in \partial\Omega$, where $\mathbb{D} = \{\lambda \in \mathbb{C}\colon |\lambda| < 1\}$.

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1. Preface

Let us denote $\mathbb{D} = \{\lambda \in \mathbb{C}\colon |\lambda| < 1\}$. Assume that $\Omega \subset \mathbb{C}^n$ is a bounded balanced domain (i.e. $\mathbb{D}\Omega = \Omega$). We solve the following Dirichlet problem: For $p > 0$ and a positive lower semi-continuous function u on $\partial\Omega$ with $u(z) = u(\lambda z)$ for $|\lambda| = 1$, $z \in \partial\Omega$ we construct a holomorphic function[1] $f \in \mathbb{O}(\Omega)$ such that[2] $u(z) = \int_{\mathbb{D}z} |f|^p \, d\mathfrak{L}^2_{\mathbb{D}z}$ for $z \in \partial\Omega$. The case when $p = 2$ and Ω is a unit ball $\mathbb{B}^n$ was solved in the paper [6, Theorem 2.9]. Now we generalize this result. In fact, our methods can be used for a bounded balanced domain Ω wich fulfils the following

Condition 1. There exists a positive constant θ and a natural number K such that if a function g is continuous on $\partial\Omega$ and $g(z) = g(\lambda z) > 0$ when $|\lambda| = 1$, $z \in \partial\Omega$, then there exists a natural number N_0 and a sequence of homogeneous polynomials p_m of degree m such that

(1) $|p_m(z)| < g(z)$ for $m > N_0$ and $z \in \partial\Omega$,
(2) $\theta g(z) < \max\limits_{j=0,1,\dots,K-1} |p_{mK+j}(z)|$ for $m > N_0$ and $z \in \partial\Omega$.

[1] By $\mathbb{O}(\Omega)$ we denote the space of all holomorphic functions on Ω.
[2] $\mathbb{D}z = \{\lambda z\colon |\lambda| < 1\}$, $\mathfrak{L}^2_{\mathbb{D}z}$ denotes Lebesgue measure on $\mathbb{D}z$.

The above condition is true when Ω is the unit ball $\mathbb{B}^n$ (see [7, Theorem 2.7]). However, the last construction of homogeneous polynomials [7, Lemma 2.5] suggests that Condition 1 will be satisfied in more complicated domains. In fact, it will be fulfilled (see [8, Theorem 2.5]) at least for the class of bounded circular strictly convex domains with C^2 boundary. The result [6, Theorem 2.7] was proved by using some properties of homogeneous polynomials on the unit ball while in [7] we constructed similar polynomials in the case when Ω is a bounded circular strictly convex domain with C^2 boundary. For this reason Condition 1 is the main assumption for the present paper.

Our construction is to enable us to give a simple description of exceptional sets of the form

$$E^p(f) = \left\{ z \in \partial\Omega \colon \int_{\mathbb{D}z} |f|^p \, \mathrm{d}\mathfrak{L}^2_{\mathbb{D}z} = \infty \right\}.$$

The exceptional sets were presented in the papers: [1], [2], [3], [4], [5], [6], [7].

2. Solution

The following fact will simplify the integration of holomorphic functions.

Lemma 1. *Assume that $p > 0$, $f \in C(\overline{\Omega})$, $\varepsilon, \delta \in (0,1)$. If $g_m \in C(\overline{\Omega})$ and $g_m \to 0$ uniformly on any compact subset of Ω, then there exists m_0 such that*

$$\int_{\mathbb{D}z} |f + g_m|^p \, \mathrm{d}\mathfrak{L}^2_{\mathbb{D}z} \geqslant -\varepsilon + \int_{\mathbb{D}z} |f|^p \, \mathrm{d}\mathfrak{L}^2_{\mathbb{D}z} + \delta^p \int_{\mathbb{D}z} |g_m|^p \, \mathrm{d}\mathfrak{L}^2_{\mathbb{D}z},$$
$$\int_{\mathbb{D}z} |f + g_m|^p \, \mathrm{d}\mathfrak{L}^2_{\mathbb{D}z} \leqslant \varepsilon + \int_{\mathbb{D}z} |f|^p \, \mathrm{d}\mathfrak{L}^2_{\mathbb{D}z} + \delta^{-p} \int_{\mathbb{D}z} |g_m|^p \, \mathrm{d}\mathfrak{L}^2_{\mathbb{D}z}$$

for $m > m_0$, $z \in \partial\Omega$.

Proof. Let $M := \sup_{z\in\Omega} |f(z)|$. There exists a number $r \in (\frac{1}{2}, 1)$ such that $(\pi(1-r^2)M^p)/(1-\delta)^p \leqslant \varepsilon/8$. Let $D(z) = \{w \in \mathbb{D}z \colon r \leqslant \|w\|\}$. We consider the following function:

$$\Psi \colon \partial\Omega \times \overline{\mathbb{D}} \ni (z,\xi) \to \int_{|\lambda|\leqslant r} |f(\lambda z) + \xi|^p \, \mathrm{d}\mathfrak{L}^2(\lambda).$$

Since Ω is bounded and Ψ continuous there exists $0 < \alpha < \delta \sqrt[p]{\varepsilon/4\pi}$ with

$$|\Psi(z,0) - \Psi(z,\xi)| \leqslant \frac{\varepsilon}{4}$$

for $z \in \partial\Omega$ and $|\xi| \leqslant \alpha$. Moreover, there exists m_0 such that $|g_m(z)| \leqslant \alpha$ for $m > m_0$ and $z \in r\Omega$. Let us observe that

$$\int_{r\mathbb{D}z} |g_m|^p \,\mathrm{d}\mathfrak{L}^2_{\mathbb{D}z} \leqslant \int_{\mathbb{D}z} \delta^p \left|\frac{\varepsilon}{4\pi}\right| \mathrm{d}\mathfrak{L}^2_{\mathbb{D}z} \leqslant \frac{1}{4}\delta^p\varepsilon \tag{1}$$

and

$$\int_{D(z)} |f|^p \,\mathrm{d}\mathfrak{L}^2_{\mathbb{D}z} \leqslant \int_{D(z)} M^p \,\mathrm{d}\mathfrak{L}^2_{\mathbb{D}z} \leqslant \frac{1}{4}\varepsilon. \tag{2}$$

Since $|g_m| \leqslant \alpha$ on $r\mathbb{D}z$ we have $|\Psi(w,0) - \Psi(w, g_m(w))| \leqslant \frac{\varepsilon}{4}$ for $w \in r\mathbb{D}z$. In particular,[3]

$$\begin{aligned}\int_{r\mathbb{D}z} |f+g_m|^p \,\mathrm{d}\mathfrak{L}^2_{\mathbb{D}z} &\geqslant -\frac{1}{4}\varepsilon + \int_{r\mathbb{D}z} |f|^p \,\mathrm{d}\mathfrak{L}^2_{\mathbb{D}z} \\ &\overset{(2)}{\geqslant} -\frac{1}{2}\varepsilon + \int_{r\mathbb{D}z} |f|^p \,\mathrm{d}\mathfrak{L}^2_{\mathbb{D}z} + \delta^p \int_{r\mathbb{D}z} |g_m|^p \,\mathrm{d}\mathfrak{L}^2_{\mathbb{D}z}\end{aligned}$$

and

$$\begin{aligned}\int_{r\mathbb{D}z} |f+g_m|^p \,\mathrm{d}\mathfrak{L}^2_{\mathbb{D}z} &\leqslant \frac{1}{4}\varepsilon + \int_{r\mathbb{D}z} |f|^p \,\mathrm{d}\mathfrak{L}^2_{\mathbb{D}z} \\ &\overset{(2)}{\leqslant} \frac{1}{2}\varepsilon + \int_{r\mathbb{D}z} |f|^p \,\mathrm{d}\mathfrak{L}^2_{\mathbb{D}z} + \delta^{-p} \int_{r\mathbb{D}z} |g_m|^p \,\mathrm{d}\mathfrak{L}^2_{\mathbb{D}z}.\end{aligned}$$

Now we define the following sets:

$$\begin{aligned}B_{m,1}(z) &:= \{w \in \mathbb{D}z \colon\ r \leqslant \|w\|\,,\ |(f+g_m)(w)| \geqslant \delta |g_m(w)|\},\\ B_{m,2}(z) &:= \{w \in \mathbb{D}z \colon\ r \leqslant \|w\|,\ |f(w)| + |g_m(w)| \leqslant \delta^{-1}|g_m(w)|\},\\ C_{m,i}(z) &:= \{w \in \mathbb{D}z \colon\ r \leqslant \|w\|,\ w \notin B_{m,i}(z)\}.\end{aligned}$$

Let $w \in C_{m,1}(z)$. Since $|(f+g_m)(w)| < \delta|g_m(w)|$ we have $(1-\delta)|g_m(w)| \leqslant |f(w)| \leqslant M$ and

$$\int_{C_{m,1}(z)} |g_m|^p \,\mathrm{d}\mathfrak{L}^2_{\mathbb{D}z} \leqslant \int_{D(z)} \frac{M^p}{(1-\delta)^p} \,\mathrm{d}\mathfrak{L}^2_{\mathbb{D}z} \leqslant \frac{1}{8}\varepsilon.$$

We can estimate

$$\begin{aligned}\int_{D(z)} |f+g_m|^p \,\mathrm{d}\mathfrak{L}^2_{\mathbb{D}z} &\geqslant \int_{B_{m,1}(z)} |f+g_m|^p \,\mathrm{d}\mathfrak{L}^2_{\mathbb{D}z} \\ &\geqslant \delta^p \int_{B_{m,1}(z)} |g_m|^p \,\mathrm{d}\mathfrak{L}^2_{\mathbb{D}z} \geqslant -\frac{1}{4}\varepsilon + \delta^p \int_{D(z)} |g_m|^p \,\mathrm{d}\mathfrak{L}^2_{\mathbb{D}z} \\ &\geqslant -\frac{1}{2}\varepsilon + \int_{D(z)} |f|^p \,\mathrm{d}\mathfrak{L}^2_{\mathbb{D}z} + \delta^p \int_{D(z)} |g_m|^p \,\mathrm{d}\mathfrak{L}^2_{\mathbb{D}z}.\end{aligned}$$

[3] In fact, since $g_m \to 0$ uniformly on $r\Omega$, these two inequalities are easy consequences of the Lebesgue lemma.

Let $w \in C_{m,2}(z)$. Since $|f(w)| + |g_m(w)| > \delta^{-1}|g_m(w)|$ we have $(\delta^{-1} - 1)|g_m(w)| \leqslant |f(w)| \leqslant M$ and $|f(w)| + |g_m(w)| < M + \delta M/(1-\delta) = M/(1-\delta)$. So we may conclude

$$\int_{C_{m,2}(z)} |f + g_m|^p \, \mathrm{d}\mathfrak{L}^2_{\mathbb{D}z} \leqslant \int_{D(z)} \frac{M^p}{(1-\delta)^p} \, \mathrm{d}\mathfrak{L}^2_{\mathbb{D}z} \leqslant \frac{1}{8}\varepsilon.$$

This implies

$$\begin{aligned}\int_{D(z)} |f + g_m|^p \, \mathrm{d}\mathfrak{L}^2_{\mathbb{D}z} &\leqslant \frac{1}{8}\varepsilon + \int_{B_{m,2}(z)} |f + g_m|^p \, \mathrm{d}\mathfrak{L}^2_{\mathbb{D}z} \\ &\leqslant \frac{1}{8}\varepsilon + \delta^{-p} \int_{B_{m,2}(z)} |g_m|^p \, \mathrm{d}\mathfrak{L}^2_{\mathbb{D}z} \\ &\leqslant \frac{1}{2}\varepsilon + \int_{D(z)} |f|^p \, \mathrm{d}\mathfrak{L}^2_{\mathbb{D}z} + \delta^{-p} \int_{D(z)} |g_m|^p \, \mathrm{d}\mathfrak{L}^2_{\mathbb{D}z},\end{aligned}$$

which completes the proof. □

The next result will be the first approximation of our solution.

Lemma 2. *There exists a constant $a \in (0,1)$ and a natural number K such that if a function h is continuous on $\partial\Omega$ and $h(z) = h(\lambda z) > 0$ when $|\lambda| = 1$, $z \in \partial\Omega$, then there exists a natural number m_0 and a sequence of homogeneous polynomials q_m of degree m such that*

$$h(z) > \int_{\mathbb{D}z} \Big| \sum_{j=0}^{K-1} q_{mK+j} \Big|^p \, \mathrm{d}\mathfrak{L}^2_{\mathbb{D}z} > a h(z), \tag{3}$$

$$m h(z) t^{mp} > \Big| \sum_{j=0}^{K-1} q_{mK+j}(tz) \Big|^p \tag{4}$$

for $z \in \partial\Omega, t > 0$ and $m \geqslant m_0$.

P r o o f. Let θ and K be from Condition 1. Let $\delta = \min\{p/4\pi K^p, 1/2^{p+1}K^p\}$. There exists a natural number $m_0 > K$ and a sequence of homogeneous polynomials p_m of degree m such that $|p_m(z)|^p < \delta h(z)$ and $\theta^p \delta h(z) < \max_{j=0,\dots,K-1} |p_{mK+j}(z)|^p$ for $z \in \partial\Omega$ and $m \geqslant m_0$. Let $q_m := m^{1/p} p_m$, $w_m := \sum_{j=0}^{K-1} q_{m+j}$ and $I_{m,s,z} := \int_{\mathbb{D}z} |w_m|^s \, \mathrm{d}\mathfrak{L}^2_{\mathbb{D}z}$.

Assume that m_0 is so large that $(m+j)^{1/p} \leqslant 2m^{1/p}$ for $m \geqslant m_0$, $j = 0, \dots, K-1$. We then obtain the inequality (4):

$$|w_m(tz)| \leqslant \sum_{j=0}^{K-1} (m+j)^{1/p} t^{m+j} \delta^{1/p} h(z)^{1/p} \leqslant 2(m\delta h(z))^{1/p} K t^m < (mh(z))^{1/p} t^m \tag{5}$$

and conclude for the left-hand side of relation (3)

$$\begin{aligned} I_{m,p,z} &= \int_0^1\int_0^{2\pi} t|w_m(tz\mathrm{e}^{\mathrm{i}\varphi})|^p \,\mathrm{d}t\,\mathrm{d}\varphi \\ &\leqslant 4\pi\delta K^p h(z)\int_0^1 mt^{pm+1}\,\mathrm{d}t < \frac{4\pi\delta K^p h(z)}{p} \leqslant h(z) \end{aligned}$$

for $z \in \partial\Omega$.

Since $q_m, \ldots, q_{m+K-1}$ are homogeneous polynomials with degrees $m, m+1, \ldots, m+K-1$ we conclude that $q_m, \ldots, q_{m+K-1}$ are orthogonal polynomials, which implies that

$$I_{m,2,z} = \int_{\mathbb{D}z}\Big|\sum_{j=0}^{K-1} q_{m+j}\Big|^2 \,\mathrm{d}\mathfrak{L}^2_{\mathbb{D}z} = \sum_{j=0}^{K-1}\int_{\mathbb{D}z}|q_{m+j}|^2\,\mathrm{d}\mathfrak{L}^2_{\mathbb{D}z}.$$

Let us observe that

$$\begin{aligned} (6)\qquad I_{m,2,z} &= \sum_{j=0}^{K-1}\int_{\mathbb{D}z}|q_{m+j}|^2\,\mathrm{d}\mathfrak{L}^2_{\mathbb{D}z} \geqslant 2\pi\int_0^1 (m\theta^p\delta h(z))^{2/p} t^{2(m+K)-1}\,\mathrm{d}t \\ &\geqslant \frac{\pi\theta^2(m\delta h(z))^{2/p}}{2m}. \end{aligned}$$

Now we define

$$A(z) := \big\{\varphi \in [0,2\pi]\colon\ |w_m(z\mathrm{e}^{\mathrm{i}\varphi})| > \tfrac{1}{3}\theta(m\delta h(z))^{1/p}\big\}.$$

Since

$$\begin{aligned} |t^m w_m(z) - w_m(tz)| &\leqslant \sum_{k=m}^{m+K-1} k^{1/p}|t^m p_k(z) - p_k(tz)| \\ &\leqslant \sum_{k=m}^{m+K-1} 2m^{1/p}t^m(1-t^{k-m})|p_k(z)| \\ &\leqslant 2m^{1/p}t^m(1-t^K)K\max_{j=0,\ldots,K-1}|p_{m+j}(z)| \\ &\leqslant 2m^{1/p}t^m(1-t^K)K(\delta h(z))^{1/p}, \end{aligned}$$

there exists $r \in (0,1)$ such that

$$|t^m w_m(z\mathrm{e}^{\mathrm{i}\varphi}) - w_m(tz\mathrm{e}^{\mathrm{i}\varphi})| \leqslant \tfrac{1}{6}\theta(m\delta h(z))^{1/p}t^m$$

for $t \in (r,1)$, $z \in \partial\Omega$ and $m \geqslant m_0$. In particular, if $\varphi \in [0,2\pi] \setminus A(z)$ then

$$|w_m(tz\mathrm{e}^{\mathrm{i}\varphi})| \leqslant |t^m w_m(z\mathrm{e}^{\mathrm{i}\varphi})| + |t^m w_m(z\mathrm{e}^{\mathrm{i}\varphi}) - w_m(tz\mathrm{e}^{\mathrm{i}\varphi})| \leqslant \tfrac{1}{2}\theta(m\delta h(z))^{1/p} t^m$$

for $t \in (r,1)$. Let $c_z := \mathfrak{L}(A(z))$. Now due to (5) we have

$$\begin{aligned} I_{m,2,z} \leqslant\ & \int_r^1\!\!\int_{A(z)} 4(m\delta h(z))^{2/p} K^2 t^{2m+1}\,\mathrm{d}t\,\mathrm{d}\varphi + \int_r^1\!\!\int_{[0,2\pi]\setminus A(z)} t|w(tz\mathrm{e}^{\mathrm{i}\varphi})|^2\,\mathrm{d}t\,\mathrm{d}\varphi \\ & + 2\pi\int_0^r 4(m\delta h(z))^{2/p} K^2 t^{2m+1}\,\mathrm{d}t \\ \leqslant\ & \frac{c_z 4(m\delta h(z))^{2/p}K^2}{2m+2} + \frac{(1-c_z)\theta^2(m\delta h(z))^{2/p}}{4(2m+2)} + \frac{8\pi(m\delta h(z))^{2/p}K^2 r^{2m+2}}{2m+2}, \end{aligned}$$

which together with (6) gives the inequality

$$\pi\theta^2 \leqslant c_z 4K^2 + (1-c_z)\frac{\theta^2}{4} + 8\pi K^2 r^{2m+2}.$$

In particular, if m_0 is so large that $8\pi K^2 r^{2m_0+2} < \pi\theta^2/2 - \theta^2/4$ then we can estimate $\pi\theta^2/2 \leqslant c_z(4K^2 - \theta^2/4) < c_z 4K^2$ and conclude that $c_z > \pi\theta^2/8K^2$.

Let us observe that if $\varphi \in A(z)$ then

$$|w_m(tz\mathrm{e}^{\mathrm{i}\varphi})| \geqslant |t^m w_m(z\mathrm{e}^{\mathrm{i}\varphi})| - |t^m w_m(z\mathrm{e}^{\mathrm{i}\varphi}) - w_m(tz\mathrm{e}^{\mathrm{i}\varphi})| \geqslant \tfrac{1}{6}\theta(m\delta h(z))^{1/p} t^m,$$

so we can set $a = \frac{1}{16}(\pi\theta^{2+p}\delta)/(K^2 6^p p)$ and conclude for the right-hand side of relation (3):

$$\begin{aligned} I_{m,p,z} \geqslant\ & \int_r^1\!\!\int_{A(z)} t|w(tz\mathrm{e}^{\mathrm{i}\varphi})|^p\,\mathrm{d}t\,\mathrm{d}\varphi \\ >\ & \frac{\pi\theta^2}{8K^2}\frac{\theta^p\delta h(z)}{6^p}\int_r^1 m t^{pm+1}\,\mathrm{d}t > \frac{\pi\theta^{2+p}\delta}{8K^2 6^p 2p} h(z) \geqslant a h(z) \end{aligned}$$

for $m \geqslant m_0$ and m_0 large enough. □

We need also well Lemmas 3–4 to simplify our calculations.

Lemma 3. *There exists a constant $\theta \in (0,1)$ and $K \in \mathbb{N}$ such that if g is a complex continuous function on $\overline{\Omega}$ and h is a positive continuous function on $\partial\Omega$ with $h(z) = h(\lambda z) > 0$ when $|\lambda| = 1$, $z \in \partial\Omega$, then there exists a natural number m_0 and a sequence of holomorphic polynomials w_m such that*

$$h(z) > \int_{\mathbb{D}z} |g + w_m|^p - |g|^p \,\mathrm{d}\mathfrak{L}^2_{\mathbb{D}z} > \theta h(z), \tag{7}$$

$$mt^{mp} h(z) > |w_m(tz)|^p \tag{8}$$

for $z \in \partial\Omega$, $t \in (0,1]$, $m \in K\mathbb{N} \setminus [0, m_0]$.[4]

P r o o f. Due to Lemma 2 there exist a constant $a \in (0,1)$, a natural number m_0 and a sequence of holomorphic polynomials w_m such that

$$\frac{1}{2}h(z) > \int_{\mathbb{D}z} |w_m|^p \,\mathrm{d}\mathfrak{L}^2_{\mathbb{D}z} > \frac{a}{2}h(z),$$
$$mt^{mp} h(z) > |w_m(tz)|^p$$

for $z \in \partial\Omega$, $t \in (0,1]$, $m \in K\mathbb{N} \setminus [0, m_0]$. Let $\varepsilon, \delta \in (0,1)$ be such that $\max\{1 - \delta^p, \delta^{-p} - 1\} < \frac{1}{4}a$ and $\varepsilon < \frac{1}{8}ah(z)\max\{1 - \delta^p, \delta^{-p} - 1\} < \frac{1}{4}a$ for $z \in \partial\Omega$. Since $w_m \to 0$ uniformly on any compact subset of Ω due to Lemma 1 we can increase m_0 in such a way that

$$\int_{\mathbb{D}z} |g + w_m|^p \,\mathrm{d}\mathfrak{L}^2_{\mathbb{D}z} \geqslant -\varepsilon + \int_{\mathbb{D}z} |g|^p \,\mathrm{d}\mathfrak{L}^2_{\mathbb{D}z} + \delta^p \int_{\mathbb{D}z} |w_m|^p \,\mathrm{d}\mathfrak{L}^2_{\mathbb{D}z}, \tag{9}$$

$$\int_{\mathbb{D}z} |g + w_m|^p \,\mathrm{d}\mathfrak{L}^2_{\mathbb{D}z} \leqslant \varepsilon + \int_{\mathbb{D}z} |g|^p \,\mathrm{d}\mathfrak{L}^2_{\mathbb{D}z} + \delta^{-p} \int_{\mathbb{D}z} |w_m|^p \,\mathrm{d}\mathfrak{L}^2_{\mathbb{D}z} \tag{10}$$

for $m \in K\mathbb{N} \setminus [0, m_0]$.

Let us denote $I_{m,z} := \int_{\mathbb{D}z} |g + w_m|^p - |g|^p \,\mathrm{d}\mathfrak{L}^2_{\mathbb{D}z}$. Using (10) we may conclude for the left-hand side of inequality (7):

$$I_{m,z} \leqslant \varepsilon + \int_{\mathbb{D}z} |w_m|^p \,\mathrm{d}\mathfrak{L}^2_{\mathbb{D}z} + (\delta^{-p} - 1) \int_{\mathbb{D}z} |w_m|^p \,\mathrm{d}\mathfrak{L}^2_{\mathbb{D}z}$$
$$< \frac{ah(z)}{8} + \frac{h(z)}{2} + \frac{ah(z)}{8} < h(z).$$

Due to (9) we have for the right-hand side of inequality (7):

$$I_{m,z} \geqslant -\varepsilon + \int_{\mathbb{D}z} |w_m|^p \,\mathrm{d}\mathfrak{L}^2_{\mathbb{D}z} - (1 - \delta^p) \int_{\mathbb{D}z} |w_m|^p \,\mathrm{d}\mathfrak{L}^2_{\mathbb{D}z}$$
$$> -\frac{ah(z)}{8} + \frac{ah(z)}{2} - \frac{ah(z)}{8} = \frac{ah(z)}{4}.$$

We have just proved that it is enough to choose $\theta = \frac{1}{4}a$. $\square$

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[4] $K\mathbb{N} \setminus [0, m_0] = \{Kj\colon\ j \in \mathbb{N} \wedge j > m_0\}$.

Lemma 4. *Let $\varepsilon > 0$, let h be a positive continuous function on $\partial\Omega$ with $h(z) = h(\lambda z) > 0$ when $|\lambda| = 1$, $z \in \partial\Omega$. Moreover, let g be a complex continuous function on $\overline{\Omega}$ and T a compact subset of Ω. Then there exists a homolomorphic polynomial w on Ω such that $h(z) - \varepsilon < \int_{\mathbb{D}z} |w + g|^p - |g|^p \,\mathrm{d}\mathfrak{L}^2_{\mathbb{D}z} < h(z)$ for $z \in \partial\Omega$ and $\|w\|_T < \varepsilon$.*

Proof. Due to Lemma 3 there exist a constant $\theta \in (0,1)$ and a sequence of holomorphic polynomials w_m such that

(1) $\|w_m\|_T < \varepsilon/2^{m+1}$.

(2) $\theta h_m(z) < \int_{\mathbb{D}z} |w_m+g_m|^p - |g_m|^p \,\mathrm{d}\mathfrak{L}^2_{\mathbb{D}z} < h_m(z)$ for $z \in \partial\Omega$, where $h_1 = h$, $g_1 = g$, $h_{m+1}(z) = h_m(z) - (\int_{\mathbb{D}z} |w_m + g_m|^p - |g_m|^p \,\mathrm{d}\mathfrak{L}^2_{\mathbb{D}z})$ and $g_{m+1} = \sum\limits_{j=1}^{m} w_m + g$.

Let us observe that $0 < h_{m+1}(z) = h(z) - (\int_{\mathbb{D}z} |g_{m+1}|^p - |g|^p \,\mathrm{d}\mathfrak{L}^2_{\mathbb{D}z})$. Now due to (2) we can estimate

$$0 < h_{m+1}(z) = h_m(z) + \int_{\mathbb{D}z} |g_m|^p - |g_{m+1}|^p \,\mathrm{d}\mathfrak{L}^2_{\mathbb{D}z} < h_m(z) - \theta h_m(z) = (1-\theta) h_m.$$

Since $h_{m+1}(z) < (1-\theta)^m h_1(z)$ there exists m_0 so large that

$$0 < h_{m_0+1}(z) = h(z) - \left(\int_{\mathbb{D}z} |g_{m_0+1}|^p - |g|^p \,\mathrm{d}\mathfrak{L}^2_{\mathbb{D}z} \right) < \varepsilon$$

for $z \in \partial\Omega$. So it is enough to choose $w = \sum\limits_{m=1}^{m_0} w_m$. □

Now it is possible to present the main result of our paper:

Theorem 1. *Let u be a positive lower semi-continuous function on $\partial\Omega$ with $u(z) = u(\lambda z) > 0$ when $|\lambda| = 1$, $z \in \partial\Omega$. Then there exists a holomorphic function f on Ω such that $u(z) = \int_{\mathbb{D}z} |f|^p \,\mathrm{d}\mathfrak{L}^2_{\mathbb{D}z}$ for $z \in \partial\Omega$.*

Proof. Let T_m be an increasing sequence of compact subsets of $\Omega = \bigcup\limits_{m\in\mathbb{N}} T_m$. There exists a sequence u_m of continuous functions on $\partial\Omega$ with $u_m(z) = u_m(\lambda z) > 0$ when $|\lambda| = 1$, $z \in \partial\Omega$ and $u_m \nearrow u$. We construct a sequence of polynomials w_m such that

(1) $\|w_m\|_{T_m} < 1/2^{m+1}$,

(2) $u_m(z) - 1/2^m < \int_{\mathbb{D}z} \Big| \sum\limits_{k=1}^{m} w_k \Big|^p \,\mathrm{d}\mathfrak{L}^2_{\mathbb{D}z} < u_m(z)$ for $z \in \partial\Omega$.

To construct w_1 it is enough to use Lemma 4 for the data $(\varepsilon, h, g, T) = (\frac{1}{2}, u_1, 0, T_1)$. Assume that we have constructed $w_1, w_2, \ldots, w_m$. Now it is enough to choose a holomorphic polynomial w_{m+1} from Lemma 4 used for the data

$$(\varepsilon, h, g, T) = \left(\frac{1}{2^{m+1}}, h_{m+1}, \sum_{k=1}^{m} w_k, T_{m+1} \right),$$

where $h_{m+1}(z) = u_{m+1}(z) - \int_{\mathbb{D}z} \left| \sum_{k=1}^{m} w_k \right|^p \mathrm{d}\mathfrak{L}^2_{\mathbb{D}z}$. We can observe that

$$u_{m+1}(z) - \int_{\mathbb{D}z} \left| \sum_{k=1}^{m} w_k \right|^p \mathrm{d}\mathfrak{L}^2_{\mathbb{D}z} - \frac{1}{2^{m+1}}$$
$$< \int_{\mathbb{D}z} \left| \sum_{k=1}^{m+1} w_k \right|^p \mathrm{d}\mathfrak{L}^2_{\mathbb{D}z} - \int_{\mathbb{D}z} \left| \sum_{k=1}^{m} w_k \right|^p \mathrm{d}\mathfrak{L}^2_{\mathbb{D}z} < u_{m+1}(z) - \int_{\mathbb{D}z} \left| \sum_{k=1}^{m} w_k \right|^p \mathrm{d}\mathfrak{L}^2_{\mathbb{D}z}.$$

To complete the proof it is enough to define $f = \sum_{k=1}^{\infty} w_k$. $\square$

Theorem 2. *Let E be a subset of type G_δ in $\partial\Omega$. There exists a holomorphic function f such that $E = E^p(f)$ and $\int_{\Omega\setminus\mathbb{D}E} |f|^p \,\mathrm{d}\mathfrak{L}^{2n} < \infty$.*

P r o o f. To prove this fact it is enough to combine Theorem 1 with the methods from [6, Theorem 3.1]. $\square$

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