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DENJOY INTEGRAL AND HENSTOCK-KURZWEIL INTEGRAL IN VECTOR LATTICES, II

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Abstract. In a previous paper we defined a Denjoy integral for mappings from a vector lattice to a complete vector lattice. In this paper we define a Henstock-Kurzweil integral for mappings from a vector lattice to a complete vector lattice and consider the relation between these two integrals.

Keywords: derivative, Denjoy integral, Henstock-Kurzweil integral, fundamental theorem of calculus, vector lattice, Riesz space

MSC 2010: 46G05, 46G12

1. Introduction

The purpose of our research is to consider some derivatives and some integrals of mappings in vector spaces and to study their relations, for instance, the fundamental theorem of calculus, inclusive relations between integrals and so on. To this aim we refer to the Fréchet derivative, the Denjoy integral of mappings from an abstract space to the real line in [4], [5], [18] and the Henstock-Kurzweil integral of mappings from the division space or the real line to a complete vector lattice in [16], [17], [2]. From the above theories to consider both derivatives and integrals of mappings in vector spaces a domain of mappings may be needed an interval structure and linearity and a range of mappings may be needed a convergence structure and linearity. Hereafter we consider that both the domain and the range of mappings are vector lattices.

In the previous paper [12] we defined the Denjoy integral of mappings from a vector lattice to a complete vector lattice. In this paper we define the Henstock-Kurzweil integral of mappings from a vector lattice to a complete vector lattice. For these two integrals we show that all Denjoy integralable mappings are also Henstock-Kurzweil integrable. We use the symbols and definitions from [12] without further notice.

Definition 2.1. Let X be a vector lattice with unit and let $D \subset X$.

A mapping $\delta\colon D \times \mathscr{K}_X \longrightarrow (0, \infty)$ is said to be a gauge if it satisfies the following condition:

(G) $\alpha\delta(\xi, \alpha e) = \delta(\xi, e)$ for any $\xi \in D$, for any $e \in \mathscr{K}_X$ and for any $\alpha \in \mathscr{K}_{\mathbb{R}}$.

Definition 2.2. Let X be a vector lattice with unit, $\xi \in D \subset X$ and δ a gauge. The subset

$$O_D(\xi, \delta) = \left(\bigcup_{e \in \mathscr{K}_X} [\xi - \delta(\xi, e)e, \xi + \delta(\xi, e)e]^e \right) \cap D$$

is called a δ-neighborhood of ξ in D. When $D = X$, then it is denoted by $O(\xi, \delta)$ simply.

Remark 2.1. By [12, Lemma 3.7] we have that $\xi \in [\xi - \delta(\xi, e)e, \xi + \delta(\xi, e)e]^e$. Therefore $\xi \in O_D(\xi, \delta)$.

Definition 2.3. Let X be a vector lattice with unit, $a, b \in D \in \mathcal{CO}_X$ and δ a gauge.

If $a \neq b$, then the following set

$$\left\{ (\langle x_{k-1} | x_k \rangle, \xi_k) \;\middle|\; \begin{array}{l} x_k \in \langle a|b \rangle \ (k = 0, \dots, K), x_0 = a, x_K = b, \\ \xi_k \in D \ (k = 1, \dots, K) \end{array} \right\}$$

is said to be a δ-fine division of $\langle a|b \rangle$ if it satisfies the following (NOL)(DF) and it is said to be a δ-fine McShane division of $\langle a|b \rangle$ if it satisfies the following (NOL)(DFMS), respectively:

(NOL) There exists $x \in |\mathscr{K}_X|$ such that $x_k - x_{k-1} \in \overline{Q}(x)$ for any $k = 1, \dots, K$.

(DF) $\xi_k \in \langle x_{k-1} | x_k \rangle \subset O_D(\xi_k, \delta)$ for any $k = 1, \dots, K$.

(DFMS) $\langle x_{k-1} | x_k \rangle \subset O_D(\xi_k, \delta)$ for any $k = 1, \dots, K$.

If $a = b$, then $\{(\langle a|a \rangle, \xi)\}$ is said to be a δ-fine division of $\langle a|b \rangle$ if it satisfies (DF) and a δ-fine McShane division of $\langle a|b \rangle$ if it satisfies (DFMS).

Remark 2.2. By [12, Remark 3.4] if $\{(\langle x_{k-1}|x_k \rangle, \xi_k)\colon k = 1, \dots, K\}$ is a δ-fine division or a δ-fine McShane division of $\langle a|b \rangle$, then $\{(\langle x_k|x_{k-1} \rangle, \xi_k)\colon k = K, \dots, 1\}$ is also so of $\langle b|a \rangle$.

Remark 2.3. By [12, Lemma 3.6] if X satisfies the principal projection property, then by (NOL) we have that $b - a \in \overline{Q}(x)$. Therefore if $a < b$, then $x_{k-1} < x_k$. And if $b - a, c - b \in \overline{Q}(x)$, then the division connected a δ-fine division of $\langle a|b \rangle$ and a δ-fine division of $\langle b|c \rangle$ is also so of $\langle a|c \rangle$ and the division connected a δ-fine McShane division of $\langle a|b \rangle$ and a δ-fine McShane division of $\langle b|c \rangle$ is also so of $\langle a|c \rangle$.

Theorem 2.1. *Let X be a vector lattice with unit, $a, b \in D \in \mathcal{CO}_X$ and δ a gauge.*

Then there always exists a δ-fine division of $\langle a|b\rangle$. Therefore there exists also a δ-fine McShane division.

P r o o f. Since this is clear in the case of $a = b$, we show in the case of $a \neq b$. Let $x \in \langle a|b\rangle$. Then there exist $\varphi \in \mathbf{CSSMP(a,b)}$ and a natural number i such that $x \in \varphi^i([0,1])$. Let φ^i be **CSIP**. Then since

$$\varphi^i([0,1]) \subset \bigcup_{x\in\varphi^i([0,1])} \Big[x - \frac{1}{3}\delta(x, e^i_\varphi)e^i_\varphi, x + \frac{1}{3}\delta(x, e^i_\varphi)e^i_\varphi\Big]^{e^i_\varphi},$$

by [12, Lemma 3.9] there exist $I_{i,k} \subset [0,1]$ and $x_{i,k} \in \varphi^i(I_{i,k})$ $(k = 1, \dots, K_i)$ such that

$$\varphi^i([0,1]) \cap \Big[x_{i,k} - \frac{1}{3}\delta(x_{i,k}, e^i_\varphi)e^i_\varphi, x_{i,k} + \frac{1}{3}\delta(x_{i,k}, e^i_\varphi)e^i_\varphi\Big]^{e^i_\varphi} = \varphi^i(I_{i,k}),$$

$$[0,1] = \bigcup_{k=1}^{K_i} I_{i,k}.$$

Let $\alpha_{i,k}$ be the left end of $I_{i,k}$ and $\beta_{i,k}$ the right end of $I_{i,k}$. Order $I_{i,k}$ according to the increasing $\alpha_{i,k}$ and denote them by $I_{i,k}$'s again. Without loss of generality it may be assumed that no $I_{i,k}$ is covered by the union of other $I_{i,k}$'s because the above formulae are true even if any $I_{i,k}$ covered by the union of the other $I_{i,k}$'s is excepted. Then

$$\begin{aligned}
&0 = \alpha_{i,1} < \alpha_{i,2},\\
&\alpha_{i,k} < \beta_{i,k-1} < \alpha_{i,k+1} < \beta_{i,k} \ (k = 2, \dots, K_i - 1),\\
&\beta_{i,K-1} < \beta_{i,K} = 1.
\end{aligned}$$

Let

$$\begin{aligned}
&\gamma_{i,0} = \alpha_{i,1} = 0,\\
&\alpha_{i,k} < \gamma_{i,k-1} < \beta_{i,k-1}, \text{ where}\\
&\qquad x_{i,k-1} < \varphi^i(\gamma_{i,k-1}) < x_{i,k} \text{ if } x_{i,k-1} < x_{i,k}\\
&\qquad \text{and } x_{i,k-1} > \varphi^i(\gamma_{i,k-1}) > x_{i,k} \text{ if } x_{i,k-1} > x_{i,k}\\
&\qquad\qquad (k = 2, \dots, K_i),\\
&\gamma_{K_i} = \beta_{K_i} = 1.
\end{aligned}$$

In the case of $x_{i,k-1} < x_{i,k}$, since

$$
\begin{aligned}
0 < x_{i,k-1} - \varphi^i(\gamma_{i,k-2}) &\leqslant \frac{1}{3}\delta(x_{i,k-1}, e^i_\varphi)e^i_\varphi, \\
0 < \varphi^i(\gamma_{i,k-1}) - x_{i,k-1} &\leqslant \frac{1}{3}\delta(x_{i,k-1}, e^i_\varphi)e^i_\varphi, \\
0 < x_{i,k} - \varphi^i(\gamma_{i,k-1}) &\leqslant \frac{1}{3}\delta(x_{i,k}, e^i_\varphi)e^i_\varphi
\end{aligned}
$$

and

$$
0 < \varphi^i(\gamma_{i,k}) - x_{i,k} \leqslant \frac{1}{3}\delta(x_{i,k}, e^i_\varphi)e^i_\varphi,
$$

we have that

$$
\begin{aligned}
0 < \varphi^i(\gamma_{i,k-1}) - \varphi^i(\gamma_{i,k-2}) &\leqslant \frac{2}{3}\delta(x_{i,k-1}, e^i_\varphi)e^i_\varphi, \\
0 < \varphi^i(\gamma_{i,k}) - \varphi^i(\gamma_{i,k-1}) &\leqslant \frac{2}{3}\delta(x_{i,k}, e^i_\varphi)e^i_\varphi.
\end{aligned}
$$

In the case of $x_{i,k-1} > x_{i,k}$, since

$$
\begin{aligned}
0 < x_{i,k-1} - \varphi^i(\gamma_{i,k-2}) &\leqslant \frac{1}{3}\delta(x_{i,k-1}, e^i_\varphi)e^i_\varphi, \\
0 < x_{i,k-1} - \varphi^i(\gamma_{i,k-1}) &\leqslant \frac{1}{3}\delta(x_{i,k-1}, e^i_\varphi)e^i_\varphi, \\
0 < \varphi^i(\gamma_{i,k-1}) - x_{i,k} &\leqslant \frac{1}{3}\delta(x_{i,k}, e^i_\varphi)e^i_\varphi
\end{aligned}
$$

and

$$
0 < \varphi^i(\gamma_{i,k}) - x_{i,k} \leqslant \frac{1}{3}\delta(x_{i,k}, e^i_\varphi)e^i_\varphi,
$$

we have that

$$
\begin{aligned}
0 < \varphi^i(\gamma_{i,k-1}) - \varphi^i(\gamma_{i,k-2}) &< \frac{1}{3}\delta(x_{i,k-1}, e^i_\varphi)e^i_\varphi, \\
0 < \varphi^i(\gamma_{i,k}) - \varphi^i(\gamma_{i,k-1}) &< \frac{1}{3}\delta(x_{i,k}, e^i_\varphi)e^i_\varphi.
\end{aligned}
$$

In either case by (CS3)

$$
\begin{aligned}
\varphi^i(\gamma_{i,k-2}) &\in \langle \varphi^i(\gamma_{i,k-2}) | \varphi^i(\gamma_{i,k-1}) \rangle \\
&\subset [\varphi^i(\gamma_{i,k-2}), \varphi^i(\gamma_{i,k-1})] \\
&\subset [\varphi^i(\gamma_{i,k-2}) - \delta(x_{i,k-1}, e^i_\varphi)e^i_\varphi, \varphi^i(\gamma_{i,k-2}) + \delta(x_{i,k-1}, e^i_\varphi)e^i_\varphi]^{e^i_\varphi} \\
&\subset O_D(\varphi^i(\gamma_{i,k-2}), \delta), \\
\varphi^i(\gamma_{i,k-1}) &\in \langle \varphi^i(\gamma_{i,k-1}) | \varphi^i(\gamma_{i,k}) \rangle \\
&\subset [\varphi^i(\gamma_{i,k-1}), \varphi^i(\gamma_{i,k})] \\
&\subset [\varphi^i(\gamma_{i,k-1}) - \delta(x_{i,k}, e^i_\varphi)e^i_\varphi, \varphi^i(\gamma_{i,k-1}) + \delta(x_{i,k}, e^i_\varphi)e^i_\varphi]^{e^i_\varphi} \\
&\subset O_D(\varphi^i(\gamma_{i,k-1}), \delta).
\end{aligned}
$$

In the case where φ^i is **CSDP** the above formulae can be proved similarly. Therefore

$$\{(\langle\varphi^i(\gamma_{i,k-1})|\varphi^i(\gamma_{i,k})\rangle, \varphi^i(\gamma_{i,k-1}))\colon\ k=1,\dots,K_i,\ i=1,\dots,r_\varphi\}$$

satisfies (DF). By (CS2) there exists $x \in |\mathscr{K}_X|$ such that $\varphi^i(1)-\varphi^i(0) \in \overline{Q}(x)$ for any i. Since $0 < x_{i,k} - x_{i,k-1} \leqslant \varphi^i(1)-\varphi^i(0)$ if φ^i is **CSIP** and $0 > x_{i,k}-x_{i,k-1} \geqslant \varphi^i(1)-\varphi^i(0)$ if φ^i is **CSDP**, we obtain by [12, Lemma 3.5] that $x_{i,k}-x_{i,k-1} \in \overline{Q}(x)$. Therefore it satisfies (NOL). □

Definition 2.4. Let X be a vector lattice with unit, Y a complete vector lattice, $D \in \mathcal{CO}_X$ and $f\colon\ D \longrightarrow \mathscr{L}(X,Y)$.

For $a, b \in D$ f is said to be Henstock-Kurzweil integrable on $\langle a|b\rangle$ if there exists $I(f;a,b) \in Y$ and $\{v_e\} \in \mathscr{U}_Y^s(\mathscr{K}_X, \geqslant)$ such that for any $e \in \mathscr{K}_X$ there exists a gauge δ such that for any δ-fine division $\{(\langle x_{k-1}|x_k\rangle, \xi_k)\colon\ k=1,\dots,K\}$ of $\langle a|b\rangle$

$$\left|\sum_{k=1}^{K} f(\xi_k)(x_k-x_{k-1}) - I(f;a,b)\right| \leqslant v_e.$$

$I(f;a,b)$ is said to be a Henstock-Kurzweil integral of f on $\langle a|b\rangle$, denoted by

$$I(f;a,b) = o\text{-}(HK)\int_a^b f(x)\,\mathrm{d}x.$$

If for any $a, b \in D$ f is Henstock-Kurzweil integrable on $\langle a|b\rangle$, then it is Henstock-Kurzweil integrable on D. Let $(\mathrm{HK})(\langle\mathbf{a}|\mathbf{b}\rangle, \mathbf{Y})$ and $(\mathrm{HK})(\mathbf{D}, \mathbf{Y})$ be the class of Henstock-Kurzweil integrable mappings on $\langle a|b\rangle$ and D, respectively. The mapping

$$\begin{array}{rcccl} F_a\colon\ D & \longrightarrow & Y & & \\ \cup & & \cup & & \\ x & \longmapsto & I(f;a,x) & = & o\text{-}(HK)\int_a^x f(x)\,\mathrm{d}x \end{array}$$

is called the Henstock-Kurzweil primitive of f.

We had to show the uniqueness of the Denjoy integral but the uniqueness of the Henstock-Kurzweil integral is clear by its definition. However some properties are not clear for the Henstock-Kurzweil integral. We show in what follows

Theorem 2.2. *Let X be a vector lattice with unit, Y a complete vector lattice, $a,b\in D\in\mathcal{CO}_X$ and $\alpha,\beta\in\mathbb{R}$.*

Then if $f,g\in(\mathrm{HK})(\langle\mathbf{a}|\mathbf{b}\rangle,\mathbf{Y})$, then $\alpha f+\beta g\in(\mathrm{HK})(\langle\mathbf{a}|\mathbf{b}\rangle,\mathbf{Y})$ and

$$o\text{-}(HK)\int_a^b(\alpha f(x)+\beta g(x))\,\mathrm{d}x=\alpha o\text{-}(HK)\int_a^b f(x)\,\mathrm{d}x+\beta o\text{-}(HK)\int_a^b g(x)\,\mathrm{d}x.$$

P r o o f. Since this is clear when $\alpha=0$ and $\beta=0$, we show except this case. By assumption there exist $\{v_{f,e}\},\{v_{g,e}\}\in\mathscr{U}_Y^s(\mathscr{K}_X,\geqslant)$ such that for any $e\in\mathscr{K}_X$ there exist gauges δ_f, δ_g such that for any δ_f-fine division $\{(\langle x_{k-1}|x_k\rangle,\xi_k)\colon\ k=1,\dots,K\}$ of $\langle a|b\rangle$

$$\Bigl|\sum_{k=1}^K f(\xi_k)(x_k-x_{k-1})-I(f;a,b)\Bigr|\leqslant v_{f,e}$$

and for any δ_g-fine division $\{(\langle x_{k-1}|x_k\rangle,\xi_k)\colon\ k=1,\dots,K\}$ of $\langle a|b\rangle$

$$\Bigl|\sum_{k=1}^K g(\xi_k)(x_k-x_{k-1})-I(g;a,b)\Bigr|\leqslant v_{g,e}.$$

Let $\delta(\cdot,\cdot)=\delta_f(\cdot,\cdot)\wedge\delta_g(\cdot,\cdot)$. Then both the above formulae are true for any δ-fine division of $\langle a|b\rangle$ and

$$\begin{aligned}&\Bigl|\sum_{k=1}^K(\alpha f+\beta g)(\xi_k)(x_k-x_{k-1})-(\alpha I(f;a,b)+\beta I(g;a,b))\Bigr|\\&\quad\leqslant|\alpha|\Bigl|\sum_{k=1}^K f(\xi_k)(x_k-x_{k-1})-I(f;a,b)\Bigr|+|\beta|\Bigl|\sum_{k=1}^K g(\xi_k)(x_k-x_{k-1})-I(g;a,b)\Bigr|\\&\quad\leqslant|\alpha|v_{f,e}+|\beta|v_{g,e}.\end{aligned}$$

Therefore we conclude by [12, Remark 2.1] that $I(\alpha f+\beta g;a,b)=\alpha I(f;a,b)+\beta I(g;a,b)$. □

Lemma 2.1. *Let X be a vector lattice with unit, Y a complete vector lattice, $a,b\in D\in\mathcal{CO}_X$ and $f\colon\ D\longrightarrow\mathscr{L}(X,Y)$.*

Then $f\in(\mathrm{HK})(\langle\mathbf{a}|\mathbf{b}\rangle,\mathbf{Y})$ if and only if there exists $\{v_e\}\in\mathscr{U}_Y^s(\mathscr{K}_X,\geqslant)$ such that for any $e\in\mathscr{K}_X$ there exists a gauge δ such that for any δ-fine divisions $\{(\langle x_{1,k-1}|x_{1,k}\rangle,\xi_{1,k})\colon\ k=1,\dots,K_1\}$ and $\{(\langle x_{2,k-1}|x_{2,k}\rangle,\xi_{2,k})\colon\ k=1,\dots,K_2\}$ of $\langle a|b\rangle$

$$\Bigl|\sum_{k=1}^{K_1}f(\xi_{1,k})(x_{1,k}-x_{1,k-1})-\sum_{k=1}^{K_2}f(\xi_{2,k})(x_{2,k}-x_{2,k-1})\Bigr|\leqslant v_e.$$

Proof. The necessity is clear. We show the sufficiency. By assumption and $(U3)^s$ there exists $\{v_e\} \in \mathscr{U}_Y^s(\mathscr{K}_X, \geqslant)$ such that for any $e \in \mathscr{K}_X$ there exist gauges δ_1 and δ_2 with $\delta_2 \leqslant \delta_1$ such that for any δ_1-fine division $\{(\langle x_{1,k-1}|x_{1,k}\rangle, \xi_{1,k})\colon k = 1, \ldots, K_1\}$ and for any δ_2-fine division $\{(\langle x_{2,k-1}|x_{2,k}\rangle, \xi_{2,k})\colon k = 1, \ldots, K_2\}$ of $\langle a|b\rangle$

$$\left|\sum_{k=1}^{K_1} f(\xi_{1,k})(x_{1,k} - x_{1,k-1}) - \sum_{k=1}^{K_2} f(\xi_{2,k})(x_{2,k} - x_{2,k-1})\right| \leqslant v_{\theta(e,1)e} \leqslant \frac{1}{2}v_e,$$

where $\theta(e,p)$ is from the proof of [12, Lemma 2.1]. Next, there exists a gauge $\delta_3 \leqslant \delta_2$ such that for any δ_3-fine division $\{(\langle x_{3,k-1}|x_{3,k}\rangle, \xi_{3,k})\colon k = 1, \ldots, K_3\}$

$$\left|\sum_{k=1}^{K_2} f(\xi_{2,k})(x_{2,k} - x_{2,k-1}) - \sum_{k=1}^{K_3} f(\xi_{3,k})(x_{3,k} - x_{3,k-1})\right| \leqslant v_{\theta(e,2)e} \leqslant \frac{1}{2^2}v_e.$$

Similarly, we prove that there exists a gauge $\delta_{p+1} \leqslant \delta_p$ such that for any δ_{p+1}-fine division $\{(\langle x_{p+1,k-1}|x_{p+1,k}\rangle, \xi_{p+1,k})\colon k = 1, \ldots, K_{p+1}\}$

$$\left|\sum_{k=1}^{K_p} f(\xi_{p,k})(x_{p,k} - x_{p,k-1}) - \sum_{k=1}^{K_{p+1}} f(\xi_{p+1,k})(x_{p+1,k} - x_{p+1,k-1})\right| \leqslant v_{\theta(e,p)e} \leqslant \frac{1}{2^p}v_e.$$

Let

$$I = \bigwedge_{p=1}^{\infty} \bigvee_{q=p}^{\infty} \sum_{k=1}^{K_q} f(\xi_{q,k})(x_{q,k} - x_{q,k-1}).$$

Since Y is complete, it holds that

$$\left|\sum_{k=1}^{K_1} f(\xi_{1,k})(x_{1,k} - x_{1,k-1}) - I\right| \\ \leqslant \sum_{p=1}^{\infty}\left|\sum_{k=1}^{K_p} f(\xi_{p,k})(x_{p,k} - x_{p,k-1}) - \sum_{k=1}^{K_{p+1}} f(\xi_{p+1,k})(x_{p+1,k} - x_{p+1,k-1})\right| \leqslant v_e.$$

Therefore f is Henstock-Kurzweil integrable on $\langle a|b\rangle$ and its value is I. □

Definition 2.5. Let X be a vector lattice with unit.

For $D \in \mathcal{CO}_X$ we consider the following condition:

(S) $[a \wedge b, a \vee b] \cap [b \wedge c, b \vee c] \cap [c \wedge a, c \vee a] \cap D \neq \emptyset$ for any $a, b, c \in D$.

Theorem 2.3. *Let X be a vector lattice with unit satisfying the principal projection property, Y a complete vector lattice and $a,b,c \in D \in \mathcal{CO}_X$. Suppose that D satisfies* (S).

If $f \in (\mathrm{HK})(\langle \mathbf{a}|\mathbf{b}\rangle, \mathbf{Y}) \cap (\mathrm{HK})(\langle \mathbf{b}|\mathbf{c}\rangle, \mathbf{Y}) \cap (\mathrm{HK})(\langle \mathbf{c}|\mathbf{a}\rangle, \mathbf{Y})$, *then*

$$o\text{-}(HK)\int_a^b f(x)\,\mathrm{d}x + o\text{-}(HK)\int_b^c f(x)\,\mathrm{d}x + o\text{-}(HK)\int_c^a f(x)\,\mathrm{d}x = 0.$$

Proof. By assumption there exist $\{v_{\langle a|b\rangle,e}\}, \{v_{\langle b|c\rangle,e}\}, \{v_{\langle c|a\rangle,e}\} \in \mathscr{U}_Y^s(\mathscr{K}_X, \geqslant)$ such that for any $e \in \mathscr{K}_X$ there exist gauges $\delta_{\langle a|b\rangle}$, $\delta_{\langle b|c\rangle}$ and $\delta_{\langle c|a\rangle}$ such that for any $\delta_{\langle a|b\rangle}$-fine division $\{(\langle x_{\langle a|b\rangle,k-1}|x_{\langle a|b\rangle,k}\rangle, \xi_{\langle a|b\rangle,k})\colon\ k=1,\dots,K_{\langle a|b\rangle}\}$ of $\langle a|b\rangle$

$$\Bigl|\sum_{k=1}^{K_{\langle a|b\rangle}} f(\xi_{\langle a|b\rangle,k})(x_{\langle a|b\rangle,k} - x_{\langle a|b\rangle,k-1}) - I(f;a,b)\Bigr| \leqslant v_{\langle a|b\rangle,e},$$

for any $\delta_{\langle b|c\rangle}$-fine division $\{(\langle x_{\langle b|c\rangle,k-1}|x_{\langle b|c\rangle,k}\rangle, \xi_{\langle b|c\rangle,k})\colon\ k=1,\dots,K_{\langle b|c\rangle}\}$ of $\langle b|c\rangle$

$$\Bigl|\sum_{k=1}^{K_{\langle b|c\rangle}} f(\xi_{\langle b|c\rangle,k})(x_{\langle b|c\rangle,k} - x_{\langle b|c\rangle,k-1}) - I(f;b,c)\Bigr| \leqslant v_{\langle b|c\rangle,e}$$

and for any $\delta_{\langle c|a\rangle}$-fine division $\{(\langle x_{\langle c|a\rangle,k-1}|x_{\langle c|a\rangle,k}\rangle, \xi_{\langle c|a\rangle,k})\colon\ k=1,\dots,K_{\langle c|a\rangle}\}$ of $\langle c|a\rangle$

$$\Bigl|\sum_{k=1}^{K_{\langle c|a\rangle}} f(\xi_{\langle c|a\rangle,k})(x_{\langle c|a\rangle,k} - x_{\langle c|a\rangle,k-1}) - I(f;c,a)\Bigr| \leqslant v_{\langle c|a\rangle,e}.$$

Let $\delta(\cdot,\cdot) = \delta_{\langle a|b\rangle}(\cdot,\cdot) \wedge \delta_{\langle b|c\rangle}(\cdot,\cdot) \wedge \delta_{\langle c|a\rangle}(\cdot,\cdot)$. Then the above formulae are true simultaneously. By (S) there exists $d \in [a\wedge b, a\vee b] \cap [b\wedge c, b\vee c] \cap [c\wedge a, c\vee a] \cap D$. By Theorem 2.1 there exist a δ-fine division $\{(\langle x_{\langle a|d\rangle,k-1}|x_{\langle a|d\rangle,k}\rangle, \xi_{\langle a|d\rangle,k})\colon\ k=1,\dots,K_{\langle a|d\rangle}\}$ of $\langle a|d\rangle$, a δ-fine division $\{(\langle x_{\langle b|d\rangle,k-1}|x_{\langle b|d\rangle,k}\rangle, \xi_{\langle b|d\rangle,k})\colon\ k=1,\dots,K_{\langle b|d\rangle}\}$ of $\langle b|d\rangle$ and a δ-fine division $\{(\langle x_{\langle c|d\rangle,k-1}|x_{\langle c|d\rangle,k}\rangle, \xi_{\langle c|d\rangle,k})\colon\ k=1,\dots,K_{\langle c|d\rangle}\}$ of $\langle c|d\rangle$. By Remark 2.2 $\{(\langle x_{\langle a|d\rangle,k}|x_{\langle a|d\rangle,k-1}\rangle,\ \xi_{\langle a|d\rangle,k})\colon\ k=K_{\langle a|d\rangle},\dots,1\}$, $\{(\langle x_{\langle b|d\rangle,k}|x_{\langle b|d\rangle,k-1}\rangle,$ $\xi_{\langle b|d\rangle,k})\colon\ k=K_{\langle b|d\rangle},\dots,1\}$ and $\{(\langle x_{\langle c|d\rangle,k}|x_{\langle c|d\rangle,k-1}\rangle,\ \xi_{\langle c|d\rangle,k})\colon\ k=K_{\langle c|d\rangle},\dots,1\}$ are δ-fine divisions of $\langle d|a\rangle$, $\langle d|b\rangle$ and $\langle d|c\rangle$, respectively. Since $d \in [a\wedge b, a\vee b] \cap D$, it holds that

$$\begin{aligned} 0\wedge(b-a) &\leqslant d-a \leqslant 0\vee(b-a),\\ 0\wedge(b-a) &\leqslant b-d \leqslant 0\vee(b-a). \end{aligned}$$

Therefore if $b-a \in \overline{Q}(x)$ and $d \neq a,b$, then by [12, Lemma 3.5] we have that $d-a, b-d \in \overline{Q}(x)$. By Remark 2.3 the division connected $\{(\langle x_{\langle a|d\rangle,k-1}|x_{\langle a|d\rangle,k}\rangle, \xi_{\langle a|d\rangle,k})\colon$

$k = 1, \ldots, K_{\langle a|d\rangle}\}$ and $\{(\langle x_{\langle b|d\rangle,k}|x_{\langle b|d\rangle,k-1}\rangle,\ \xi_{\langle b|d\rangle,k})\colon\ k = K_{\langle b|d\rangle}, \ldots, 1\}$ is a δ-fine division of $\langle a|b\rangle$, the division connected $\{(\langle x_{\langle b|d\rangle,k-1}|x_{\langle b|d\rangle,k}|,\rangle\xi_{\langle b|d\rangle,k})\colon\ k = 1, \ldots, K_{\langle b|d\rangle}\}$ and $\{(\langle x_{\langle c|d\rangle,k}|x_{\langle c|d\rangle,k-1}\rangle, \xi_{\langle c|d\rangle,k})\colon\ k = K_{\langle c|d\rangle}, \ldots, 1\}$ is a δ-fine division of $\langle b|c\rangle$ and the division connected $\{(\langle x_{\langle c|d\rangle,k-1}|x_{\langle c|d\rangle,k}\rangle,\ \xi_{\langle c|d\rangle,k})\colon\ k = 1, \ldots, K_{\langle c|d\rangle}\}$ and $\{(\langle x_{\langle a|d\rangle,k}|x_{\langle a|d\rangle,k-1}\rangle,\ \xi_{\langle a|d\rangle,k})\colon\ k = K_{\langle a|d\rangle}, \ldots, 1\}$ is a δ-fine division of $\langle c|a\rangle$. By Lemma 2.1

$$\Big|\sum_{k=1}^{K_{\langle a|b\rangle}} f(\xi_{\langle a|b\rangle,k})(x_{\langle a|b\rangle,k} - x_{\langle a|b\rangle,k-1}) - \sum_{k=1}^{K_{\langle a|d\rangle}} f(\xi_{\langle a|d\rangle,k})(x_{\langle a|d\rangle,k} - x_{\langle a|d\rangle,k-1})$$
$$- \sum_{k=1}^{K_{\langle b|d\rangle}} f(\xi_{\langle b|d\rangle,k})(x_{\langle b|d\rangle,k-1} - x_{\langle b|d\rangle,k})\Big| \leqslant v_{\langle a|b\rangle,e},$$

$$\Big|\sum_{k=1}^{K_{\langle b|c\rangle}} f(\xi_{\langle b|c\rangle,k})(x_{\langle b|c\rangle,k} - x_{\langle b|c|,\rangle k-1}) - \sum_{k=1}^{K_{\langle b|d\rangle}} f(\xi_{\langle b|d\rangle,k})(x_{\langle b|d\rangle,k} - x_{\langle b|d\rangle,k-1})$$
$$- \sum_{k=1}^{K_{\langle c|d\rangle}} f(\xi_{\langle c|d\rangle,k})(x_{\langle c|d\rangle,k-1} - x_{\langle c|d\rangle,k})\Big| \leqslant v_{\langle b|c|,\rangle e},$$

$$\Big|\sum_{k=1}^{K_{\langle c|a\rangle}} f(\xi_{\langle c|a\rangle,k})(x_{\langle c|a\rangle,k} - x_{\langle c|a\rangle,k-1}) - \sum_{k=1}^{K_{\langle c|d\rangle}} f(\xi_{\langle c|d\rangle,k})(x_{\langle c|d\rangle,k} - x_{\langle c|d\rangle,k-1})$$
$$- \sum_{k=1}^{K_{\langle a|d\rangle}} f(\xi_{\langle a|d\rangle,k})(x_{\langle a|d\rangle,k-1} - x_{\langle a|d\rangle,k})\Big| \leqslant v_{\langle c|a\rangle,e}.$$

Therefore

$$\begin{aligned}
&|I(f;a,b) + I(f;b,c) + I(f;c,a)| \\
&\quad\leqslant \Big|I(f;a,b) - \sum_{k=1}^{K_{\langle a|b\rangle}} f(\xi_{\langle a|b\rangle,k})(x_{\langle a|b\rangle,k} - x_{\langle a|b\rangle,k-1})\Big| \\
&\qquad + \Big|I(f;b,c) - \sum_{k=1}^{K_{\langle b|c\rangle}} f(\xi_{\langle b|c\rangle,k})(x_{\langle b|c\rangle,k} - x_{\langle b|c\rangle,k-1})\Big| \\
&\qquad + \Big|I(f;c,a) - \sum_{k=1}^{K_{\langle c|a\rangle}} f(\xi_{\langle c|a\rangle,k})(x_{\langle c|a\rangle,k} - x_{\langle c|a\rangle,k-1})\Big| \\
&\qquad + \Big|\sum_{k=1}^{K_{\langle a|b\rangle}} f(\xi_{\langle a|b\rangle,k})(x_{\langle a|b\rangle,k} - x_{\langle a|b\rangle,k-1}) - \sum_{k=1}^{K_{\langle a|d\rangle}} f(\xi_{\langle a|d\rangle,k})(x_{\langle a|d\rangle,k} - x_{\langle a|d\rangle,k-1}) \\
&\qquad\quad - \sum_{k=1}^{K_{\langle b|d\rangle}} f(\xi_{\langle b|d\rangle,k})(x_{\langle b|d\rangle,k-1} - x_{\langle b|d\rangle,k})\Big|
\end{aligned}$$

$$
\begin{aligned}
&+\Bigg|\sum_{k=1}^{K_{\langle b|c\rangle}} f(\xi_{\langle b|c\rangle,k})(x_{\langle b|c\rangle,k}-x_{\langle b|c\rangle,k-1})-\sum_{k=1}^{K_{\langle b|d\rangle}} f(\xi_{\langle b|d\rangle,k})(x_{\langle b|d\rangle,k}-x_{\langle b|d\rangle,k-1})\\
&\quad-\sum_{k=1}^{K_{\langle c|d\rangle}} f(\xi_{\langle c|d\rangle,k})(x_{\langle c|d\rangle,k-1}-x_{\langle c|d\rangle,k})\Bigg|\\
&+\Bigg|\sum_{k=1}^{K_{\langle c|a\rangle}} f(\xi_{\langle c|a\rangle,k})(x_{\langle c|a\rangle,k}-x_{\langle c|a\rangle,k-1})-\sum_{k=1}^{K_{\langle c|d\rangle}} f(\xi_{\langle c|d\rangle,k})(x_{\langle c|d\rangle,k}-x_{\langle c|d\rangle,k-1})\\
&\quad-\sum_{k=1}^{K_{\langle a|d\rangle}} f(\xi_{\langle a|d\rangle,k})(x_{\langle a|d\rangle,k-1}-x_{\langle a|d\rangle,k})\Bigg|\\
&\leqslant 2(v_{\langle a|b\rangle,e}+v_{\langle b|c\rangle,e}+v_{\langle c|a\rangle,e}).
\end{aligned}
$$

Since e is arbitrary, it holds by [12, Remark 2.1] that $I(f;a,b)+I(f;b,c)+I(f;c,a)=0$. □

Remark 2.4. Let f be Henstock-Kurzweil integrable on D. Then by Theorem 2.3 for any primitives F_a, F_b of f and for any $c,d\in D$

$$
\begin{aligned}
F_a(c)-F_a(d)&=I(f;a,c)-I(f;a,d)\\
&=I(f;d,c)\\
&=I(f;b,c)-I(f;b,d)\\
&=F_b(c)-F_b(d).
\end{aligned}
$$

This means the Henstock-Kurzweil primitive is uniquely determind like the Denjoy one if the difference of constant values is disregarded. Hereafter the Henstock-Kurzweil primitive is denoted simply F when it is not considered distance of constant values.

Theorem 2.4. *Let X be a vector lattice with unit satisfying the principal projection property, Y a complete vector lattice, $a,b\in D\in\mathcal{CO}_X$ and $\langle c|d\rangle$ a subinterval of $\langle a|b\rangle$.*

If $f\in(\mathrm{HK})(\langle\mathbf{a}|\mathbf{b}\rangle,\mathbf{Y})$, then $f\in(\mathrm{HK})(\langle\mathbf{c}|\mathbf{d}\rangle,\mathbf{Y})$.

Proof. By assumption and Lemma 2.1 there exists $\{v_e\}\in\mathscr{U}_Y^s(\mathscr{K}_X,\geqslant)$ such that for any $e\in\mathscr{K}_X$ there exists a gauge δ such that for any δ-fine divisions $\{(\langle x_{1,k-1}|x_{1,k}\rangle,\xi_{1,k})\colon k=1,\dots,K_1\}$ and $\{(\langle x_{2,k-1}|x_{2,k}\rangle,\xi_{2,k})\colon k=1,\dots,K_2\}$ of $\langle a|b\rangle$

$$
\Bigg|\sum_{k=1}^{K_1} f(\xi_{1,k})(x_{1,k}-x_{1,k-1})-\sum_{k=1}^{K_2} f(\xi_{2,k})(x_{2,k}-x_{2,k-1})\Bigg|\leqslant v_e.
$$

Let $\{(\langle x_{3,k-1}|x_{3,k}\rangle,\xi_{3,k})\colon\ k=1,\ldots,K_3\}$ and $\{(\langle x_{4,k-1}|x_{4,k}\rangle,\xi_{4,k})\colon\ k=1,\ldots,K_4\}$ be δ-fine divisions of $\langle c|d\rangle$, $\{(\langle x_{5,k-1}|x_{5,k}\rangle,\xi_{5,k})\colon\ k=1,\ldots,K_5\}$ a δ-fine division of $\langle a|c\rangle$ and $\{(\langle x_{6,k-1}|x_{6,k}\rangle,\xi_{6,k})|k=1,\ldots,K_6\}$ a δ-fine division of $\langle d|b\rangle$. Then by Remark 2.3 the division connected $\{(\langle x_{5,k-1}|x_{5,k}\rangle,\xi_{5,k})\colon\ k=1,\ldots,K_5\}$, $\{(\langle x_{3,k-1}|x_{3,k}\rangle,\xi_{3,k})\colon\ k=1,\ldots,K_3\}$ and $\{(\langle x_{6,k-1}|x_{6,k}\rangle,\xi_{6,k})\colon\ k=1,\ldots,K_6\}$ is a δ-fine division of $\langle a|b\rangle$ and the division connected $\{(\langle x_{5,k-1}|x_{5,k}\rangle,\xi_{5,k})\colon\ k=1,\ldots,K_5\}$, $\{(\langle x_{4,k-1}|x_{4,k}\rangle,\xi_{4,k})\colon\ k=1,\ldots,K_4\}$ and $\{(\langle x_{6,k-1}|x_{6,k}\rangle,\xi_{6,k})\colon\ k=1,\ldots,K_6\}$ is also so. Therefore

$$\begin{aligned}\Bigl|\sum_{k=1}^{K_3}f(\xi_{3,k})(x_{3,k}-x_{3,k-1})-\sum_{k=1}^{K_4}f(\xi_{4,k})(x_{4,k}-x_{4,k-1})\Bigr|\\ =\Bigl|\sum_{k=1}^{K_5}f(\xi_{5,k})(x_{5,k}-x_{5,k-1})+\sum_{k=1}^{K_3}f(\xi_{3,k})(x_{3,k}-x_{3,k-1})\\ +\sum_{k=1}^{K_6}f(\xi_{6,k})(x_{6,k}-x_{6,k-1})\\ -\sum_{k=1}^{K_5}f(\xi_{5,k})(x_{5,k}-x_{5,k-1})-\sum_{k=1}^{K_4}f(\xi_{4,k})(x_{4,k}-x_{4,k-1})\\ -\sum_{k=1}^{K_6}f(\xi_{6,k})(x_{6,k}-x_{6,k-1})\Bigr|\leqslant v_e.\end{aligned}$$

Therefore by Lemma 2.1, f is Henstock-Kurzweil integrable on $\langle c|d\rangle$. □

3. Saks-Henstock lemma

The Riemann type integrals satisfy the Saks-Henstock lemma in addition to the above properties. This lemma is useful for discussing the relations between other integrals and to show convergence theorems.

Note that every complete vector lattice is isomorphic to the vector sublattice of

$$C_\infty(\Omega)=\left\{f\colon\ \Omega\longrightarrow[-\infty,+\infty]\ \middle|\ \begin{array}{l}f\text{ is continuous}\\ \text{and }\{\omega\colon\ |f(\omega)|=+\infty\}\text{ is nowhere dense.}\end{array}\right\},$$

where Ω is an extremally disconnected compact space [1], [3], [19], [20].

Theorem 3.1. *Let X be a vector lattice with unit, Y a complete vector lattice and $a, b \in D \in \mathcal{CO}_X$. Suppose that $f \in (\mathrm{HK})(\langle\mathbf{a}|\mathbf{b}\rangle, \mathbf{Y})$ and its primitive on $\langle a|b\rangle$ is F, that is, there exists $\{v_e\} \in \mathscr{U}_Y^s(\mathscr{K}_X, \geqslant)$ such that for any $e \in \mathscr{K}_X$ there exists a gauge δ such that for any δ-fine division $\{(\langle x_{k-1}|x_k\rangle, \xi_k)\colon\ k = 1, \ldots, K\}$ of $\langle a|b\rangle$*

$$\left|\sum_{k=1}^{K} f(\xi_k)(x_k - x_{k-1}) - (F(b) - F(a))\right| \leqslant v_e.$$

Then

(1) *For any subdivision $\{(\langle x_{k_p-1}|x_{k_p}\rangle, \xi_{k_p})\colon\ p = 1, \ldots, P\}$ of $\{(\langle x_{k-1}|x_k\rangle, \xi_k)\colon\ k = 1, \ldots, K\}$*

$$\left|\sum_{p=1}^{P}(f(\xi_{k_p})(x_{k_p} - x_{k_p-1}) - (F(x_{k_p}) - F(x_{k_p-1})))\right| \leqslant 2v_e.$$

(2)

$$\sum_{k=1}^{K} |f(\xi_k)(x_k - x_{k-1}) - (F(x_k) - F(x_{k-1}))| \leqslant 4v_e.$$

P r o o f. (1) Let $\{(\langle x_{k_q-1}|x_{k_q}\rangle, \xi_{k_q})\colon\ q = 1, \ldots, Q\}$ be the rest of $\{(\langle x_{k_p-1}|x_{k_p}\rangle, \xi_{k_p})\colon\ p = 1, \ldots, P\}$ in $\{(\langle x_{k-1}|x_k\rangle, \xi_k)\colon\ k = 1, \ldots, K\}$. Since by [12, Remark 3.7], $\langle x_{k_q-1}|x_{k_q}\rangle$ is a subinterval, by Theorem 2.4 f is Henstock-Kurzweil integrable on each $\langle x_{k_q-1}|x_{k_q}\rangle$. Therefore there exists a gauge δ_q, where without loss of generality it may be assumed that $\delta_q \leqslant \delta$, such that for any δ_q-fine division $\{(\langle x_{q,k-1}|x_{q,k}\rangle, \xi_{q,k})\colon$ $k = 1, \ldots, K_q\}$

$$\left|\sum_{k=1}^{K_q}(f(\xi_{q,k})(x_{q,k} - x_{q,k-1}) - (F(x_{k_q}) - F(x_{k_q-1})))\right| \leqslant \frac{1}{Q}v_e.$$

By Remark 2.3 the division connected $\{(\langle x_{k_p-1}|x_{k_p}\rangle, \xi_{k_p})\colon\ p = 1, \ldots, P\}$ and $\{(\langle x_{q,k-1}|x_{q,k}\rangle, \xi_{q,k})\colon\ k = 1, \ldots, K_q\}$ $(q = 1, \ldots, Q)$ is δ-fine. Let $\{(\langle y_{k-1}|y_k\rangle, \eta_k)\colon$ $k = 1, \ldots, K\}$ be the division. Then

$$\begin{aligned}
&\left|\sum_{p=1}^{P}(f(\xi_{k_p})(x_{k_p} - x_{k_p-1}) - (F(x_{k_p}) - F(x_{k_p-1})))\right| \\
&\qquad \leqslant \left|\sum_{k=1}^{K} f(\eta_k)(y_k - y_{k-1}) - (F(b) - F(a))\right| \\
&\qquad\qquad + \sum_{q=1}^{Q}\left|\sum_{k=1}^{K_q}(f(\xi_{q,k})(x_{q,k} - x_{q,k-1}) - (F(x_{k_q}) - F(x_{k_q-1})))\right| \leqslant 2v_e.
\end{aligned}$$

(2) It may be assumed that $Y \subset C_\infty(\Omega)$ for an extremally disconnected compact space Ω. For any $\omega \in \Omega$ since $(f(\xi_k)(x_k - x_{k-1}) - (F(x_k) - F(x_{k-1})))(\omega)$ takes on a positive, 0 or negative value, we have that

$$\begin{aligned}
&\left(\sum_{k=1}^{K} |f(\xi_k)(x_k - x_{k-1}) - (F(x_k) - F(x_{k-1}))|\right)(\omega) \\
&\quad = \left(\sum_{\text{positive part}} (f(\xi_k)(x_k - x_{k-1}) - (F(x_k) - F(x_{k-1})))\right)(\omega) \\
&\qquad + \left(\sum_{\text{negative part}} (-(f(\xi_k)(x_k - x_{k-1}) - (F(x_k) - F(x_{k-1}))))\right)(\omega).
\end{aligned}$$

Therefore by (1) all terms of the above right hand side are less than or equal to $2v_e(\omega)$ proving that

$$\left(\sum_{k=1}^{K} |f(\xi_k)(x_k - x_{k-1}) - (F(x_k) - F(x_{k-1}))|\right)(\omega) \leqslant 4v_e(\omega).$$

Since $\omega \in \Omega$ is arbitrary, it holds that

$$\sum_{k=1}^{K} |f(\xi_k)(x_k - x_{k-1}) - (F(x_k) - F(x_{k-1}))| \leqslant 4v_e.$$

□

4. Relation

Definition 4.1. Let X be a vector lattice and Y a complete vector lattice. For $\mathscr{L}(X, Y)$ we consider the following condition:

(CB) There exists $\{l_n \colon n = 1, 2, \ldots\} \subset \mathscr{L}(X, Y)$ satisfying the following conditions:

(CB1) $l_{n_1} \leqslant l_{n_2}$ if $n_1 < n_2$.

(CB2) For any $l \in \mathscr{L}(X, Y)$ there exists a natural number n such that $|l| \leqslant l_n$.

(CB3) There exists $\{\varepsilon_n\} \subset \mathscr{K}_{\mathbb{R}}$ such that

$$\sum_{n=1}^{\infty} \varepsilon_n l_n \in \mathscr{L}(X, Y).$$

Example 4.1. We consider an example such that $\mathscr{L}(X,Y)$ satisfies (CB). It is when $\mathscr{L}(X,Y)$ has an Archimedean unit u. Actually $\{nu\colon n=1,2,\ldots\}$ satisfies (CB1)(CB2) clearly. Let $\varepsilon_n=(n\cdot 2^n)^{-1}$. Then

$$\sum_{n=1}^{\infty}\varepsilon_n nu=u.$$

Therefore it satisfies (CB3).

Lemma 4.1. *Let $X=\mathbb{R}$, Y a complete vector lattice and $D\in\mathcal{CO}_X$. Suppose that $\mathscr{L}(X,Y)$ satisfies* (CB).

For $f\colon D\longrightarrow Y$ if $f(x)=0$ for almost every $x\in D$, then $f\in$ (HK)$(\mathbf{D},\mathbf{Y})$ *and*

$$o\text{-}(HK)\int f(x)\,\mathrm{d}x=0.$$

Proof. Take $\{l_n\colon n=1,2,\ldots\}\subset\mathscr{L}(X,Y)$ in (CB). Let $N=\{x\colon f(x)\neq 0\}$, $N_1=\{x\colon 0<|f(x)|\leqslant l_1\}$ and $N_n=\{x\colon l_{n-1}<|f(x)|\leqslant l_n\}$ $(n=2,3,\ldots)$. By assumption N and N_n are null sets and $N=\bigcup\limits_{n=1}^{\infty}N_n$. Take $\{\varepsilon_n\}\subset\mathscr{K}_{\mathbb{R}}$ in (CB3). Then for any $e\in\mathscr{K}_X$ there exists $\{[a_{n,n'},b_{n,n'}]\colon n'=1,2,\ldots\}$ such that

$$N_n\subset\bigcup_{n'=1}^{\infty}[a_{n,n'},b_{n,n'}]^e \text{ and } \sum_{n'=1}^{\infty}q([a_{n,n'},b_{n,n'}])=\sum_{n'=1}^{\infty}(b_{n,n'}-a_{n,n'})\leqslant\varepsilon_n e.$$

In the case of $\xi\in N$ there exist n and n' such that $\xi\in[a_{n,n'},b_{n,n'}]^e$. Therefore $\delta(\xi,\cdot)$ can satisfy $[\xi-\delta(\xi,e)e,\xi+\delta(\xi,e)e]\subset[a_{n,n'},b_{n,n'}]$. In the case of $\xi\notin N$ take $\delta(\xi,\cdot)$ arbitrary. Then for any δ-fine division $\{(\langle x_{k-1}|x_k\rangle,\xi_k)\colon k=1,\ldots,K\}$

$$\begin{aligned}\left|\sum_{k=1}^{K}f(\xi_k)(x_k-x_{k-1})\right| &\leqslant\sum_{n=1}^{\infty}\sum_{n'=1}^{\infty}l_n(b_{n,n'}-a_{n,n'})\\ &\leqslant\left(\sum_{n=1}^{\infty}\varepsilon_n l_n\right)(e).\end{aligned}$$

Therefore $f\in$ (HK)$(\mathbf{D},\mathbf{Y})$ and

$$o\text{-}(HK)\int f(x)\,\mathrm{d}x=0.$$

□

Theorem 4.1. *Let X be a vector lattice with unit satisfying the principal projection property, Y a complete vector lattice and $a, b \in D \in \mathcal{CO}_X$ with $a < b$ or $a > b$. Suppose that X satisfies* (M).

Then for any $f \in (\mathrm{D}^*)(\langle \mathbf{a}|\mathbf{b}\rangle, \mathbf{Y})$ there exists $g \in (\mathrm{HK})(\langle \mathbf{a}|\mathbf{b}\rangle, \mathbf{Y})$ such that $g(x) = f(x)$ for almost every $x \in \langle a|b\rangle$ and

$$o\text{-}(HK)\int_a^b g(x)\,\mathrm{d}x = o\text{-}(D^*)\int_a^b f(x)\,\mathrm{d}x.$$

Moreover, if $X = \mathbb{R}$ and $\mathscr{L}(X, Y)$ satisfies (CB), *then*

$$(\mathrm{D}^*)(\langle \mathbf{a}|\mathbf{b}\rangle, \mathbf{Y}) \subset (\mathrm{HK})(\langle \mathbf{a}|\mathbf{b}\rangle, \mathbf{Y}).$$

Proof. We prove the theorem in the case of $a < b$. Similarly it can be proved in the case of $a > b$. Then (NOL) implies that $x_k - x_{k-1} > 0$ for any δ-fine division. Let $f \in (\mathrm{D}^*)(\langle \mathbf{a}|\mathbf{b}\rangle, \mathbf{Y})$ and let F be the Denjoy primitive of f. Since $F \in \mathbf{ACG}^*(\mathbf{D}, \mathbf{Y})$, there exist $\{E_p\colon\ E_p \subset D, p = 1, 2, \ldots\}$ with $\bigcup\limits_{p=1}^{\infty} E_p = D$ and $\{v_e\} \in \mathscr{U}_Y^s(\mathscr{K}_X, \geqslant)$ such that for any natural number p and for any $e \in \mathscr{K}_X$ there exists $\delta_p \in \mathscr{K}_{\mathbb{R}}$ such that for any $x_{1,k}, x_{2,k} \in D$ with $x_{1,k} < x_{2,k}$ and $x_{1,k} \in E_p$ or $x_{2,k} \in E_p$ $(k = 1, \ldots, K)$

$$\text{if } \sum_{k=1}^{K} q([x_{1,k}, x_{2,k}]) \leqslant \delta_p, \text{ then } \sum_{k=1}^{K} \omega(F, [x_{1,k}, x_{2,k}]) \leqslant v_{\theta(e,p)e} \leqslant \frac{1}{2^p} v_e,$$

where $\theta(e, p)$ is from the proof of [12, Lemma 2.1]. Let

$$N = \{x\colon\ x \in D, \text{ there exists neither} o\text{-}DF(x) \text{ nor } o\text{-}DF(x) \neq f(x)\},$$
$$N_p = N \cap E_p$$

and

$$g(x) = \begin{cases} f(x) & \text{if } x \notin N, \\ 0 & \text{if } x \in N. \end{cases}$$

Then N and N_p are null sets and $\bigcup\limits_{p=1}^{\infty} N_p = N$. Therefore there exists $\{[a_{p,j}, b_{p,j}]\colon\ j = 1, 2, \ldots\}$ such that

$$N_p \subset \bigcup_{j=1}^{\infty} [a_{p,j}, b_{p,j}]^e \quad \text{and} \quad \sum_{j=1}^{\infty} q([a_{p,j}, b_{p,j}]) \leqslant \delta_p.$$

If $\xi \in N_p$, then there exists a natural number j such that $\xi \in [a_{p,j}, b_{p,j}]^e$. Let $\delta(\xi, e)$ satisfy $[\xi - \delta(\xi, e)e, \xi + \delta(\xi, e)e] \subset [a_{p,j}, b_{p,j}]$. By assumption for any $\xi \in \langle a|b\rangle \setminus N$

there exists $\{w_e\} \in \mathscr{U}^s_{\mathscr{L}(X,Y)}(\mathscr{K}_X, \geqslant)$ such that for any $e \in \mathscr{K}_X$ there exists $\delta^{\pm}_{\xi} \in \mathscr{K}_{\mathbb{R}}$ such that $|F(\xi \pm h) - F(\xi) \mp f(\xi)(h)| = |F(\xi \pm h) - F(\xi) \mp g(\xi)(h)| \leqslant w_e(h)$ for any $h \in X$ with $0 < h \leqslant \delta^{\pm}_{\xi} e$. Let $\delta(\xi, e) = \delta^{+}_{\xi} \wedge \delta^{-}_{\xi}$. Then for any δ-fine division $\{(\langle x_{k-1}|x_k\rangle, \xi_k)\colon\ k = 1, 2, \ldots, K\}$

$$\begin{aligned}
&\Big|\sum_{k=1}^{K} g(\xi_k)(x_k - x_{k-1}) - (F(b) - F(a))\Big| \\
&\qquad \leqslant \sum_{\xi_k \notin N} |g(\xi_k)(x_k - x_{k-1}) - (F(x_k) - F(x_{k-1}))| \\
&\qquad\qquad + \sum_{\xi_k \in N} |g(\xi_k)(x_k - x_{k-1})| + \sum_{\xi_k \in N} |F(x_k) - F(x_{k-1})| \\
&\qquad \leqslant \sum_{\xi_k \notin N} 2w_e(x_k - x_{k-1}) + \sum_{p=1}^{\infty} \sum_{\xi_k \in N_p} |F(x_k) - F(x_{k-1})| \\
&\qquad \leqslant 2w_e(b - a) + v_e.
\end{aligned}$$

Therefore by [12, Remark 2.1] it holds that $g \in (\mathrm{HK})(\langle \mathbf{a}|\mathbf{b}\rangle, \mathbf{Y})$ and its primitive is F. The other part of Theorem follows immediately by Lemma 4.1. □

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