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DEGREE SEQUENCES OF GRAPHS CONTAINING A CYCLE WITH PRESCRIBED LENGTH

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Abstract. Let $r \geqslant 3$, $n \geqslant r$ and $\pi = (d_1, d_2, \ldots, d_n)$ be a non-increasing sequence of nonnegative integers. If π has a realization G with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ such that $d_G(v_i) = d_i$ for $i = 1, 2, \ldots, n$ and $v_1 v_2 \ldots v_r v_1$ is a cycle of length r in G, then π is said to be potentially C_r''-graphic. In this paper, we give a characterization for π to be potentially C_r''-graphic.

Keywords: graph, degree sequence, potentially C_r-graphic sequence

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1. Introduction

A non-increasing sequence $\pi = (d_1, d_2, \ldots, d_n)$ of nonnegative integers is said to be *graphic* if it is the degree sequence of a simple graph G on n vertices, and such a graph G is referred to as a *realization* of π. The following well-known result due to Erdős and Gallai [2] which gave a characterization for π to be graphic.

Theorem 1.1 (Erdős and Gallai [2]). *Let $\pi = (d_1, d_2, \ldots, d_n)$ be a non-increasing sequence of nonnegative integers, where $\sum_{i=1}^{n} d_i$ is even. Then π is graphic if and only if*

$$\sum_{i=1}^{t} d_i \leqslant t(t-1) + \sum_{i=t+1}^{n} \min\{t, d_i\}$$

for each t, $1 \leqslant t \leqslant n$.

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A non-increasing sequence $\pi = (d_1, d_2, \ldots, d_n)$ of nonnegative integers is said to be *potentially K_{r+1}-graphic* if there is a realization of π containing K_{r+1} as a subgraph, where K_{r+1} is the complete graph on $r+1$ vertices. If π has a realization in which the $r+1$ vertices of largest degree induce a clique, then π is *potentially A_{r+1}-graphic*. In [7], Rao proved that π is potentially A_{r+1}-graphic if and only if π is potentially K_{r+1}-graphic. In [8], Rao gave a characterization (Theorem 1.2) for π to be potentially A_{r+1}-graphic. This is a generalization of Erdős-Gallai characterization for π to be graphic (which corresponds to $r = 0$).

Theorem 1.2 (Rao [8]). *Let $n \geqslant r+1$ and $\pi = (d_1, d_2, \ldots, d_n)$ be a non-increasing sequence of nonnegative integers, where $d_{r+1} \geqslant r$ and $\sum\limits_{i=1}^{n} d_i$ is even. Then π is potentially A_{r+1}-graphic if and only if*

$$\sum_{i=1}^{s} d_i + \sum_{i=1}^{t} d_{r+1+i} \leqslant (s+t)(s+t-1) + \sum_{i=s+1}^{r+1} \min\{s+t, d_i - r + s\} + \sum_{i=r+t+2}^{n} \min\{s+t, d_i\}$$

for any s and t, $0 \leqslant s \leqslant r+1$ and $0 \leqslant t \leqslant n-r-1$.

The original proof of Theorem 1.2 remains unpublished, but Kézdy and Lehel in [5] have given a different proof using network flows.

A non-increasing sequence $\pi = (d_1, d_2, \ldots, d_n)$ of nonnegative integers is said to be *potentially C_r-graphic* if there is a realization of π containing C_r as a subgraph, where C_r is the cycle of length r. If π has a realization containing C_r on the $|V(C_r)|$ highest degree vertices in π, then π is said to be *potentially C_r'-graphic*. Furthermore, if π has a realization G with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ such that $d_G(v_i) = d_i$ for $i = 1, 2, \ldots, n$ and $v_1 v_2 \ldots v_r v_1$ is a C_r, then π is said to be *potentially C_r''-graphic*. It follows from a result in [4] that π is potentially C_r'-graphic if and only if π is potentially C_r-graphic. An extremal problem on potentially C_r-graphic sequences was investigated by Lai [6]. In this paper, we shall give a characterization for π to be potentially C_r''-graphic. In other words, we will prove the following

Theorem 1.3. *Let $r \geqslant 3$, $n \geqslant r$ and $\pi = (d_1, d_2, \ldots, d_n)$ be a non-increasing sequence of nonnegative integers, where $d_r \geqslant 2$ and $\sum\limits_{i=1}^{n} d_i$ is even. Then π is potentially*

C_r''*-graphic if and only if*

$$\sum_{i=1}^{p} d_i + \sum_{i=r+1}^{r+q} d_i \leqslant (p+q)(p+q-1) + \min\{p+q, d_{p+1}-1\} + \sum_{i=p+2}^{r-1} \min\{p+q, d_i-2\} + \min\{p+q, d_r-1\} + \sum_{i=r+q+1}^{n} \min\{p+q, d_i\}$$

for any p and q, $0 \leqslant p \leqslant r$ and $0 \leqslant q \leqslant n-r$.

Remark. If $p=0$, the above inequality means that

$$\sum_{i=r+1}^{r+q} d_i \leqslant q(q-1) + \sum_{i=1}^{r} \min\{q, d_i-2\} + \sum_{i=r+q+1}^{n} \min\{q, d_i\}.$$

2. The proof of Theorem 1.3

In order to prove Theorem 1.3, we shall use a simple version of a general result of Fulkerson, Hoffman and Mcandrew [3] (see also [1] and [5]). Let H be a simple graph on vertex set $V(H) = \{v_1, v_2, \ldots, v_n\}$. We say that H satisfies the odd-cycle condition, if between any two disjoint odd cycles there is an edge.

Theorem 2.1 (Fulkerson, Hoffman and Mcandrew [3]). *Assume that $H = (V(H), E(H))$ satisfies the odd-cycle condition, where $V(H) = \{v_1, v_2, \ldots, v_n\}$. There exists a subgraph $G \subseteq H$ such that every vertex v_i has degree d_i, if and only if*

(i) $\sum\limits_{i=1}^{n} d_i$ *is even,*

(ii) *for every $A, B \subseteq V(H)$ such that $A \cap B = \emptyset$, we have*

$$\sum_{v_i \in A} d_i \leqslant |\{(v_i, v_j)\colon\ v_iv_j \in E(H),\ v_i \in A,\ v_j \in V(H) \setminus B\}| + \sum_{v_i \in B} d_i.$$

The following observation is obvious.

Observation 2.1. Let $\pi = (d_1, d_2, \ldots, d_n)$, where $d_1 \geqslant d_2 \geqslant \ldots \geqslant d_n$. Take $i_1, i_2, \ldots, i_p \in \{1, 2, \ldots, n\}$ such that $i_1 < i_2 < \ldots < i_p$ and $i_1 > 1$, $i_2 > 2$, ..., $i_p > p$. If $d_{i_1} + d_{i_2} + \ldots + d_{i_p} = d_1 + d_2 + \ldots + d_p$, then $d_1 = d_2 = \ldots = d_{i_p}$.

Proof of Theorem 1.3. To prove the necessity, we let G be a graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ such that $d_G(v_i) = d_i$ for $i = 1, 2, \ldots, n$ and $v_1v_2 \ldots v_rv_1$ is a C_r in G. Then, $\sum\limits_{i=1}^{p} d_i + \sum\limits_{i=r+1}^{r+q} d_i$ is the sum of the number of edges from v_h to $\{v_1, \ldots, v_p, v_{r+1}, \ldots, v_{r+q}\}$ the summation being taken over $h = 1, 2, \ldots, n$. Now the contribution of v_h to this sum is at most $p+q-1$ if $h \in \{1, \ldots, p, r+1, \ldots, r+q\}$, at most $\min\{p+q, d_h-1\}$ if $h = p+1$, at most $\min\{p+q, d_h-2\}$ if $h \in \{p+2, \ldots, r-1\}$, at most $\min\{p+q, d_h-1\}$ if $h = r$ and at most $\min\{p+q, d_h\}$ if $h \in \{r+q+1, \ldots, n\}$. Thus the necessity is proved.

We now prove the sufficiency. Denote $L(p,q) = \sum\limits_{i=1}^{p}(d_i - 2) + \sum\limits_{i=r+1}^{r+q} d_i$ and

$$\begin{aligned} R(p,q) = (p+q)(p+q-1) - 2p + \min\{p+q, d_{p+1} - 1\} \\ + \sum_{i=p+2}^{r-1} \min\{p+q, d_i - 2\} + \min\{p+q, d_r - 1\} \\ + \sum_{i=r+q+1}^{n} \min\{p+q, d_i\}. \end{aligned}$$

Assume that $r \geqslant 3$, $n \geqslant r$ and $\pi = (d_1, d_2, \ldots, d_n)$ is a non-increasing sequence of nonnegative integers such that $d_r \geqslant 2$, $\sum\limits_{i=1}^{n} d_i$ is even and $L(p,q) \leqslant R(p,q)$ for any p and q, $0 \leqslant p \leqslant r$ and $0 \leqslant q \leqslant n-r$.

Let $\pi'_r = (d'_1, \ldots, d'_r, d'_{r+1}, \ldots, d'_n)$, where $d'_i = d_i - 2$ for $1 \leqslant i \leqslant r$ and $d'_i = d_i$ for $r+1 \leqslant i \leqslant n$, and let H be the graph obtained from K_n with vertex set $V(K_n) = \{v_1, v_2, \ldots, v_n\}$ by deleting edges $v_1v_2, v_2v_3, \ldots, v_{r-1}v_r, v_rv_1$. It is easy to see that π is potentially C''_r-graphic if and only if H has a subgraph G with the degree sequence π'_r such that every vertex v_i has degree d'_i. Observe that between any two disjoint cycles of H there is an edge. Therefore, H satisfies the odd-cycle condition and we may apply Theorem 2.1.

Let $K = \{v_1, v_2, \ldots, v_r\}$ and $A, B \subseteq V(H)$ such that $A \cap B = \emptyset$. Let $A_1 = A \cap K$, $A_2 = A \setminus K$, $B_1 = B \cap K$, $B_2 = B \setminus K$, $C = K \setminus (A_1 \cup B_1)$, $D = \{v_{r+1}, \ldots, v_n\} \setminus (A_2 \cup B_2)$ and set $p = |A_1|$, $q = |A_2|$, $b_1 = |B_1|$, $b_2 = |B_2|$. For convenience, we denote

$$\begin{aligned} L'(A,B) &= \sum_{v_i \in A} d'_i = \sum_{v_i \in A_1} (d_i - 2) + \sum_{v_i \in A_2} d_i, \\ R'(A,B) &= |\{(v_i, v_j)\colon\ v_iv_j \in E(H),\ v_i \in A,\ v_j \in V(H) \setminus B\}| + \sum_{v_i \in B} d'_i \\ &= |\{(v_i, v_j)\colon\ v_iv_j \in E(H),\ v_i \in A,\ v_j \in V(H) \setminus B\}| \\ &\quad + \sum_{v_i \in B_1} (d_i - 2) + \sum_{v_i \in B_2} d_i, \end{aligned}$$

$$F(A,B) = \sum_{v_i \in C} (p+q) + \sum_{v_i \in B_1} (d_i - 2) + \sum_{v_i \in D} (p+q) + \sum_{v_i \in B_2} d_i,$$

$$W(A,B) = \sum_{i=p+1}^{r-b_1} (p+q) + \sum_{i=r+1-b_1}^{r} (d_i - 2) + \sum_{i=r+q+1}^{n-b_2} (p+q) + \sum_{i=n+1-b_2}^{n} d_i.$$

Clearly, $L'(A,B) \leqslant L(p,q)$. We now prove that $L'(A,B) \leqslant R'(A,B)$.

If $b_1 = 0$, then $B_1 = \emptyset$ and $|C| = r-p$. Since $|\{(v_i, v_j)\colon\ v_iv_j \in E(H),\ v_i \in A,\ v_j \in V(H) \setminus B\}|$ is the number of counting the edges of H between A and $V(H) \setminus (A \cup B)$ and double counting the edges induced by A, we get

$$\begin{aligned} R'(A,B) &\geqslant (p+q)(p+q-1) - 2p + F(A,B) \\ &\geqslant (p+q)(p+q-1) - 2p + W(A,B) \geqslant R(p,q) \geqslant L(p,q). \end{aligned}$$

If $b_1 \geqslant 1$ and $|C| = 0$, then $b_1 = r - p$. Thus

$$\begin{aligned} R'(A,B) &\geqslant (p+q)(p+q-1) - 2p + 2 + F(A,B) \\ &\geqslant (p+q)(p+q-1) - 2p + 2 + W(A,B) \geqslant R(p,q) \geqslant L(p,q). \end{aligned}$$

If $b_1 \geqslant 1$ and $|C| = 1$, then $b_1 = r - p - 1$. Thus

$$\begin{aligned} R'(A,B) &\geqslant (p+q)(p+q-1) - 2p + 1 + F(A,B) \\ &\geqslant (p+q)(p+q-1) - 2p + 1 + W(A,B) \geqslant R(p,q) \geqslant L(p,q). \end{aligned}$$

We assume that $b_1 \geqslant 1$ and $|C| = r - p - b_1 \geqslant 2$. Then $p \leqslant r - 3$. If $v_1 \in A_1$ and $v_r \in B_1$, then

$$\begin{aligned} R'(A,B) &\geqslant (p+q)(p+q-1) - 2p + 1 + F(A,B) \\ &\geqslant (p+q)(p+q-1) - 2p + 1 + W(A,B) \geqslant R(p,q) \geqslant L(p,q). \end{aligned}$$

If $v_1 \in A_1$ and $v_r \notin B_1$, then

$$\begin{aligned} R'(A,B) &\geqslant (p+q)(p+q-1) - 2p + F(A,B) \\ &\geqslant (p+q)(p+q-1) - 2p + \sum_{i=p+1}^{r-b_1-1} (p+q) \\ &\quad + \sum_{i=r-b_1}^{r-1} (d_i - 2) + (p+q) + \sum_{i=r+q+1}^{n-b_2} (p+q) + \sum_{i=n+1-b_2}^{n} d_i \\ &\geqslant R(p,q) \geqslant L(p,q). \end{aligned}$$

If $L'(A,B) < L(p,q)$, then

$$\begin{aligned} R'(A,B) &\geqslant (p+q)(p+q-1) - 2p + 1 + F(A,B) - 1 \\ &\geqslant (p+q)(p+q-1) - 2p + 1 + W(A,B) - 1 \\ &\geqslant R(p,q) - 1 \geqslant L(p,q) - 1 \geqslant L'(A,B). \end{aligned}$$

We further assume that $v_1 \notin A_1$ and $L'(A,B) = L(p,q)$. Then $\sum\limits_{v_i \in A_1} (d_i - 2) = \sum\limits_{i=1}^{p} (d_i - 2)$ and $\sum\limits_{v_i \in A_2} d_i = \sum\limits_{i=r+1}^{r+q} d_i$. By Observation 2.1, we have that $d_1 - 2 = d_2 - 2 = \ldots = d_{p+1} - 2$. We now consider the following two cases.

Case 1: $q \geqslant 1$. In this case, if $p+q \geqslant d_{p+2} - 1$, then

$$\begin{aligned} R'(A,B) &\geqslant (p+q)(p+q-1) - 2p + F(A,B) \\ &\geqslant (p+q)(p+q-1) - 2p + W(A,B) \\ &\geqslant (p+q)(p+q-1) - 2p + \min\{p+q, d_{p+1} - 1\} \\ &\quad + \sum_{i=p+2}^{r} \min\{p+q, d_i - 2\} + 1 + \sum_{i=r+q+1}^{n} \min\{p+q, d_i\} = R(p,q) \geqslant L(p,q). \end{aligned}$$

If $L(p,q) < R(p,q)$, then

$$\begin{aligned} R'(A,B) &\geqslant (p+q)(p+q-1) - 2p + F(A,B) \\ &\geqslant (p+q)(p+q-1) - 2p + 1 + W(A,B) - 1 \\ &\geqslant R(p,q) - 1 \geqslant L(p,q). \end{aligned}$$

If $p+q \leqslant d_{p+2} - 2$ and $L(p,q) = R(p,q)$, then by $L(p+1,q-1) \leqslant R(p+1,q-1)$, we have that $L(p+1,q-1) - L(p,q) \leqslant R(p+1,q-1) - R(p,q)$, that is, $d_{p+1} - d_{r+q} \leqslant 0$. Hence $d_{p+1} = d_{r+q}$. Thus

$$\begin{aligned} R'(A,B) &\geqslant (p+q)(p+q-1) - 2p + F(A,B) \\ &\geqslant (p+q)(p+q-1) - 2p + W(A,B) \geqslant R(p,q) \geqslant L(p,q). \end{aligned}$$

Case 2: $q = 0$. In this case, if $L(p,0) < R(p,0)$ or $R(p,0) \leqslant p(p-1) - 2p + W(A,B)$, then $R'(A,B) \geqslant L(p,0)$ is clear.

If $L(p,0) = R(p,0) > p(p-1) - 2p + W(A,B)$, then we only have $L(p,0) = R(p,0) = p(p-1) - 2p + W(A,B) + 1$, and $p \leqslant d_i - 2$ for $p+1 \leqslant i \leqslant r - b_1$, $p \geqslant d_i - 2$ for $r+1-b_1 \leqslant i \leqslant r-1$, $p \geqslant d_r - 1$, $p \leqslant d_i$ for $r+1 \leqslant i \leqslant n - b_2$ and $p \geqslant d_i$ for $n+1-b_2 \leqslant i \leqslant n$. On the one hand, it follows from $d_1 - 2 = d_2 - 2 = \ldots = d_{p+1} - 2$ that

$$L(p,0) = p(d_1 - 2) = p(p-1) - 2p + py + z,$$

and hence

$$L(p+1,0) = (p+1)(d_1-2) = (p+1)\Big(p-3+y+\frac{z}{p}\Big),$$

where $y = n-p-b_1-b_2$ and $z = \sum\limits_{i=r+1-b_1}^{r-1}(d_i-2)+(d_r-1)+\sum\limits_{i=n+1-b_2}^{n} d_i$. On the other hand, it is easy to see that

$$\begin{aligned}
L(p+1,0) &\leqslant R(p+1,0)\\
&= (p+1)p-2(p+1)+\min\{p+1,d_{p+2}-1\}+\sum_{i=p+3}^{r-1}\min\{p+1,d_i-2\}\\
&\quad+\min\{p+1,d_r-1\}+\sum_{i=r+1}^{n}\min\{p+1,d_i\}\\
&\leqslant (p+1)p-2(p+1)+(p+1)(y-1)+z\\
&= (p+1)\Big(p-3+y+\frac{z}{p+1}\Big)\\
&< (p+1)\Big(p-3+y+\frac{z}{p}\Big) = L(p+1,0), \quad \text{a contradiction.}
\end{aligned}$$

□

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