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ON POTENTIALLY NILPOTENT DOUBLE STAR SIGN PATTERNS

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Abstract. A matrix $\mathscr{A}$ whose entries come from the set $\{+,-,0\}$ is called a *sign pattern matrix*, or *sign pattern*. A sign pattern is said to be potentially nilpotent if it has a nilpotent realization. In this paper, the characterization problem for some potentially nilpotent double star sign patterns is discussed. A class of double star sign patterns, denoted by $\mathcal{DSSP}(m,2)$, is introduced. We determine all potentially nilpotent sign patterns in $\mathcal{DSSP}(3,2)$ and $\mathcal{DSSP}(5,2)$, and prove that one sign pattern in $\mathcal{DSSP}(3,2)$ is potentially stable.

Keywords: sign pattern, double star, potentially nilpotent, potentially stable

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1. Introduction

A matrix $\mathscr{A}$ whose entries come from the set $\{+,-,0\}$ is called a *sign pattern matrix*, or *sign pattern*. Let $\mathscr{A}=[a_{ij}]$ and $\mathscr{B}=[b_{ij}]$ be $n\times n$ sign patterns. If $b_{ij}=a_{ij}$ whenever $a_{ij}\neq 0$, then $\mathscr{B}$ is a superpattern of $\mathscr{A}$ and $\mathscr{A}$ is a subpattern of $\mathscr{B}$. If $\mathscr{A}$ is a sign pattern and A is a real matrix for which each entry has the same sign as the corresponding entry of $\mathscr{A}$, then A is said to be a *realization* of $\mathscr{A}$, and we write $A\in Q(\mathscr{A})$. The *inertia* of a square matrix A is the ordered triple $i(A)=(i_+(A),i_-(A),i_0(A))$, in which $i_+(A)$, $i_-(A)$ and $i_0(A)$ are the numbers of eigenvalues of A with positive, negative and zero real parts, respectively. The *inertia* of sign pattern $\mathscr{A}$ is the set of ordered triples $i(\mathscr{A})=\{i(A)\colon A\in Q(\mathscr{A})\}$. A square matrix A is *stable* if $i(A)=(0,n,0)$. A pattern $\mathscr{A}$ is *potentially stable* if $(0,n,0)\in i(\mathscr{A})$. A pattern $\mathscr{A}$ is *inertially arbitrary* if $(r,s,t)\in i(\mathscr{A})$ for every

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nonnegative triple (r,s,t) with $r+s+t=n$. In particular, an $n\times n$ $(n\geqslant 2)$ sign pattern matrix $\mathscr{A}$ is *spectrally arbitrary* if given any monic polynomial $r(x)$ of degree n with real coefficients, there exists $A\in Q(\mathscr{A})$ such that the characteristic polynomial of A is $r(x)$.

In the following discussion, we need some notations about the cycles in a sign pattern, since every real matrix associated with it has the same qualitative cycle structure, and such cycle structure is crucial in studying the eigenvalues. The *directed graph* $D(\mathscr{A})$ associated with a pattern $\mathscr{A}=(a_{ij})$ is the directed graph with vertex set $\{1,2,\ldots,n\}$ and arc (i,j) if $a_{ij}\neq 0$, for all $1\leqslant i,j\leqslant n$. A *simple k-cycle* γ of length $|\gamma|=k$ of $\mathscr{A}$ is a sequence of k arcs $(i_1,i_2),(i_2,i_3)\ldots(i_k,i_1)$ in $D(\mathscr{A})$ such that the vertices $i_1,i_2,\ldots,i_k$ are distinct. We denote the cycle γ by $(i_1,i_2,\ldots,i_k,i_1)$. Write $\prod\limits_{\gamma}=a_{i_1i_2}a_{i_2i_3}\ldots a_{i_ki_1}$, the *cycle product* of $\mathscr{A}$ associated with a simple k-cycle γ.

It is well known that the determinant of an $n\times n$ matrix A is the sum of all possible terms of the form

$$(-1)^{|\gamma_1|-1}\prod_{\gamma_1}(-1)^{|\gamma_2|-1}\prod_{\gamma_2}\ldots(-1)^{|\gamma_p|-1}\prod_{\gamma_p} \tag{1.1}$$

where $\gamma_1,\ldots,\gamma_p$ are disjoint simple cycles the sum of whose lengths is equal to n. Similarly, $E_k(A)$, the sum of all $k\times k$ principal minors of A, is equal to the sum of all terms of the form (1.1) where $\gamma_1,\ldots,\gamma_p$ are disjoint simple cycles whose length sum equals k. The computation of the characteristic polynomial of A is then expressed in terms of its cycle products as follows:

$$P_A(x)=x^n+\sum_{k=1}^{n}(-1)^kE_k(A)x^{n-k}. \tag{1.2}$$

An $n\times n$ sign pattern $\mathscr{A}$ is said to be *potentially nilpotent* if there exists $A\in Q(\mathscr{A})$ such that $P_A(x)=x^n$. From (1.2), this is equivalent to finding $A\in\mathscr{A}$ such that $E_k(A)=0$ for all $1\leqslant k\leqslant n$. Clearly, if $\mathscr{A}$ is spectrally arbitrary, then $\mathscr{A}$ is potentially nilpotent, but not conversely. Two sign patterns $\mathscr{A}$ and $\mathscr{B}$ are *equivalent* if $\mathscr{B}$ may be obtained from $\mathscr{A}$ by some combination of negation, transposition, permutation similarity, and signature similarity. Note that if $\mathscr{A}$ and $\mathscr{B}$ are equivalent, then $\mathscr{A}$ is potentially nilpotent if and only if $\mathscr{B}$ is potentially nilpotent. In [6], Lina Yeh characterized some star sign patterns and linear tree sign patterns that are potentially nilpotent. In [3], Eschenbach and Li obtained a number of qualitative necessary or sufficient conditions for a sign pattern to be potentially nilpotent. In [5], all star sign patterns that are potentially nilpotent are characterized. In this paper, we introduce one class of double star sign patterns which is denoted by $\mathcal{DSSP}(m,2)$

and determine all potentially nilpotent sign patterns in $\mathcal{DSSP}(3,2)$ and $\mathcal{DSSP}(5,2)$. Moreover, we prove that one sign pattern in $\mathcal{DSSP}(3,2)$ is potentially stable.

2. Double star sign patterns

A sign pattern $\mathscr{A} = [a_{ij}]$ is *combinatorially symmetric* if $a_{ij} \neq 0$ whenever $a_{ji} \neq 0$. The graph $G(\mathscr{A})$ of a combinatorially symmetric sign pattern $\mathscr{A}$ of order n has vertices $1, 2, \ldots, n$ and an edge joining vertices i and j if and only if $a_{ij} \neq 0$. Note that loops are allowed. A *star* is the graph with vertex set $\{1, 2, \ldots, n\}$ and an edge joining a fixed center vertex i and each leaf vertex j for all $j \neq i$ (and no other edges). A *double star* is the graph obtained from two stars by joining the two centers of stars with an edge. The notion of center and leaf vertices of the two stars is preserved for the resulting double star. A combinatorially symmetric sign pattern $\mathscr{A}$ is a *double star sign pattern* if the graph obtained from $G(\mathscr{A})$ by deleting all loops is a double star.

For $n \geqslant 4$ and $2 \leqslant m \leqslant n-2$, consider the double star matrix

$$A = \begin{pmatrix} a_1 & b_2 & \ldots & b_m & b_{m+1} & & & \\ c_2 & a_2 & & & & & & \\ \vdots & & \ddots & & & & & \\ c_m & & & a_m & & & & \\ c_{m+1} & & & & a_{m+1} & b_{m+2} & \ldots & b_n \\ & & & & c_{m+2} & a_{m+2} & & \\ & & & & \vdots & & \ddots & \\ & & & & c_n & & & a_n \end{pmatrix}.$$

The entries $b_2, \ldots, b_n, c_2, \ldots, c_n$ are nonzero real numbers. The entries not specified in the matrix are all zeros. Let A_k ($A_{\overline{k}}$) denote the principal submatrix of A lying in the first k rows and columns (lying in the last $n-k$ rows and columns). Expanding the determinant $\det(xI - A)$ along the first row, we obtain

Proposition 2.1. *The characteristic polynomial of A (as above) is*

$$P_A(x) = P_{A_m}(x) P_{A_{\overline{m}}}(x) - b_{m+1}c_{m+1} \prod_{\substack{j=2 \\ j \neq m+1}}^{n} (x - a_j),$$

where

$$P_{A_m}(x) = \left(\prod_{j=1}^{m} (x - a_j) - \sum_{i=2}^{m} \left(b_i c_i \prod_{\substack{j=2 \\ j \neq i}}^{m} (x - a_j) \right) \right)$$

and

$$P_{A_{\overline{m}}}(x) = \left(\prod_{j=m+1}^{n} (x-a_j) - \sum_{i=m+2}^{n} \left(b_i c_i \prod_{\substack{j=m+2 \\ j\neq i}}^{n} (x-a_j) \right) \right).$$

From Proposition 2.1, the off-diagonal entries b_i and c_i enter into the characteristic polynomial of A only as a product $b_i c_i$. It is therefore sufficient to consider only matrices with $b_i = 1$ for $2 \leqslant i \leqslant n$ and for double star sign patterns it is sufficient to take entries $(1, i)$ and $(m+1, j)$ for $2 \leqslant i \leqslant m+1$ and $m+2 \leqslant j \leqslant n$ as $+$.

Proposition 2.2. *Let $n \geqslant 4$. If $\mathscr{A} = [a_{ij}]$ is an inertially arbitrary double star sign pattern, then $a_{ii} = 0$ for at most one leaf vertex i.*

P r o o f. Let $n \geqslant 4$, $2 \leqslant m \leqslant n-2$ and let $\mathscr{A} = [a_{ij}]$ be an inertially arbitrary double star sign pattern. Let the real matrix

$$A = \begin{pmatrix} a_1 & 1 & \ldots & 1 & 1 & & & \\ b_2 & a_2 & & & & & & \\ \vdots & & \ddots & & & & & \\ b_m & & & a_m & & & & \\ b_{m+1} & & & & a_{m+1} & 1 & \ldots & 1 \\ & & & & b_{m+2} & a_{m+2} & & \\ & & & & \vdots & & \ddots & \\ & & & & b_n & & & a_n \end{pmatrix} \tag{2.1}$$

be a realization of $\mathscr{A}$. Then,

$$\det(A) = \left(\prod_{j=1}^{m} a_j - \sum_{i=2}^{m} \left(b_i \prod_{\substack{j=2 \\ j\neq i}}^{m} a_j \right) \right) \left(\prod_{j=m+1}^{n} a_j - \sum_{i=m+2}^{n} \left(b_i \prod_{\substack{j=m+2 \\ j\neq i}}^{n} a_j \right) \right) - b_{m+1} \prod_{\substack{j=2 \\ j\neq m+1}}^{n} a_j.$$

By the definition of a double star sign pattern, $b_i \neq 0$ for all i. Suppose that $a_i = 0$ for exactly two i's with one in $\{2, \ldots, m\}$ and another in $\{m+2, \ldots, n\}$. Then $\det(A) \neq 0$ and A has only non-zero eigenvalues. The eigenvalues of A with zero real parts must be conjugate pairs of purely imaginary numbers. Therefore $i_0(A)$ is even for any real matrix $A \in Q(\mathscr{A})$. Hence $\mathscr{A}$ is not inertially arbitrary, giving a contradiction.

Now suppose that $a_i = 0$ for more than one i with $2 \leqslant i \leqslant m$ or $m+2 \leqslant i \leqslant n$. Then $\det(A) = 0$. But this implies that $i_0(A) \geqslant 1$ for all $A \in Q(\mathscr{A})$, again giving a contradiction. $\square$

Corollary 2.1. *Let $n \geqslant 4$. If $\mathscr{A} = [a_{ij}]$ is a spectrally arbitrary double star sign pattern, then $a_{ii} = 0$ for at most one leaf vertex i.*

Proposition 2.3. *Let $n \geqslant 4$ and A be a real matrix as in* (2.1). *If A is nilpotent, then the non-zero entries among $a_2, \ldots, a_m$ are distinct and so are the non-zero entries among $a_{m+2}, \ldots, a_n$.*

Proof. By Proposition 2.1, the characteristic polynomial of A is

$$P_A(x) = \left(\prod_{j=1}^{m}(x-a_j) - \sum_{i=2}^{m}\left(b_i \prod_{\substack{j=2\\ j\neq i}}^{m}(x-a_j)\right)\right)$$
$$\times\left(\prod_{j=m+1}^{n}(x-a_j) - \sum_{i=m+2}^{n}\left(b_i \prod_{\substack{j=m+2\\ j\neq i}}^{n}(x-a_j)\right)\right) - b_{m+1}\prod_{\substack{j=2\\ j\neq m+1}}^{n}(x-a_j).$$

If $a_i = a_j$ for some $2 \leqslant i \neq j \leqslant m$, then a_i is a zero of $P_A(x) = x^n$, and consequently $a_i = 0$. Thus, if $a_i \neq 0$, then $a_i \neq a_j$ for all $2 \leqslant i \neq j \leqslant m$. Similarly, we may prove that if $a_i \neq 0$, then $a_i \neq a_j$ for all $m+2 \leqslant i \neq j \leqslant n$. □

Proposition 2.4. *Let A be a real matrix as in* (2.1) *with zero main diagonal. Then A is nilpotent if and only if either $\sum\limits_{i=2}^{m} b_i = 0$ and $\sum\limits_{i=m+1}^{n} b_i = 0$, or $\sum\limits_{i=m+2}^{n} b_i = 0$ and $\sum\limits_{i=2}^{m+1} b_i = 0$.*

Proof. Consider the matrix A in (2.1) with $a_i = 0$ for $1 \leqslant i \leqslant n$. The characteristic polynomial of A is $P_A(x) = x^n + E_2(B)x^{n-2} + E_4(B)x^{n-4}$. Thus A being nilpotent is equivalent to

$$-\sum_{i=2}^{n} b_i = E_2(B) = 0, \tag{2.2}$$

and

$$\sum_{i=2}^{m} b_i \sum_{i=m+2}^{n} b_i = E_4(B) = 0. \tag{2.3}$$

From (2.3) we have either $\sum\limits_{i=2}^{m} b_i = 0$ or $\sum\limits_{i=m+2}^{n} b_i = 0$. Taking (2.2) into consideration, we see that A is nilpotent if and only if $\sum\limits_{i=2}^{m} b_i = 0$ and $\sum\limits_{i=m+1}^{n} b_i = 0$, or $\sum\limits_{i=m+2}^{n} b_i = 0$ and $\sum\limits_{i=2}^{m+1} b_i = 0$. □

Proposition 2.5. *Let A be a real matrix as in* (2.1) *with the only nonzero entries a_1, a_{m+1} on the main diagonal. Then A is nilpotent if and only if $a_1 = -a_{m+1}$, $b_{m+1} = -a_1^2$ and $\sum\limits_{i=2}^{m} b_i = \sum\limits_{j=m+2}^{n} b_j = 0$.*

P r o o f. For the matrix A in (2.1), if $a_i = 0$ for $2 \leqslant i \leqslant m$ and $m+2 \leqslant i \leqslant n$, then $P_A(x) = x^n - E_1(A)x^{n-1} + E_2(A)x^{n-2} - E_3(A)x^{n-3} + E_4(A)x^{n-4}$. The matrix A is nilpotent if and only if

$$a_1 + a_{m+1} = E_1(A) = 0, \tag{2.4}$$

$$a_1 a_{m+1} - \sum_{i=2}^{n} b_i = E_2(A) = 0, \tag{2.5}$$

$$-a_1 \sum_{i=m+2}^{n} b_i - a_{m+1} \sum_{j=2}^{m} b_j = E_3(A) = 0, \tag{2.6}$$

and

$$\sum_{i=2}^{m} b_i \sum_{j=m+2}^{n} b_j = E_4(A) = 0. \tag{2.7}$$

From (2.7), $\sum\limits_{i=2}^{m} b_i = 0$ or $\sum\limits_{j=m+2}^{n} b_j = 0$. So, by (2.6), we have $\sum\limits_{i=2}^{m} b_i = \sum\limits_{j=m+2}^{n} b_j = 0$. From (2.4) and (2.5), we have $a_1 = -a_{m+1}$ and $b_{m+1} = -a_1^2$. $\square$

Now consider the following double star matrix

$$\begin{pmatrix} 0 & 1 & \dots & 1 & 1 & \\ b_2 & a_2 & & & & \\ \vdots & & \ddots & & & \\ b_m & & & a_m & & \\ b_{m+1} & & & & 0 & 1 \\ & & & & b_{m+2} & 0 \end{pmatrix}, \tag{2.8}$$

where $m \geqslant 3$. The sign pattern $\mathscr{A}$ determined by a real matrix A with form (2.8) is denoted by $\mathcal{DSSP}(m, 2)$. In the following we shall give a characterization for all potentially nilpotent sign patterns in $\mathcal{DSSP}(m, 2)$ for some small values of m.

Theorem 2.1. *Let $\mathscr{A} = [a_{ij}] \in \mathcal{DSSP}(3, 2)$. Then $\mathscr{A}$ is potentially nilpotent if and only if $a_{22}a_{33} = -$, $a_{21}a_{31} = +$ and $(a_{22}, a_{21}, a_{41}, a_{54})$ is one of the forms (up to equivalence)* $(+,-,+,-)$, $(+,+,-,+)$, $(+,-,+,+)$.

P r o o f. Suppose $A \in Q(\mathscr{A})$ is parameterized as in (2.8). Then A is nilpotent if and only if

$$\begin{cases} a_2 + a_3 = 0, \\ a_2a_3 - b_2 - b_3 - b_4 - b_5 = 0, \\ -a_2(b_3 + b_4 + b_5) - a_3(b_2 + b_4 + b_5) = 0, \\ -a_2a_3(b_4 + b_5) + b_5(b_2 + b_3) = 0, \\ (a_2b_3 + a_3b_2)b_5 = 0. \end{cases}$$

By simplification, we have

$$\begin{aligned} &a_2 = -a_3, && b_4 + b_5 = a_2a_3 - b_2 - b_3, \\ &b_5(b_2 + b_3) = a_2a_3(b_4 + b_5), && a_2b_3 + a_3b_2 = 0, \end{aligned}$$

that is,

$$a_2 = -a_3, \quad b_2 = b_3 = \frac{a_2^4}{2(b_5 - a_2^2)}, \quad b_4 = \frac{b_5^2}{a_2^2 - b_5}.$$

We distinguish the following cases.

Case 1: $b_5 > 0$.

Subclass 1: $b_5 > a_2^2$. Then $b_2 > 0$ and $b_4 < 0$.

Subclass 2: $b_5 < a_2^2$. Then $b_2 < 0$ and $b_4 > 0$.

Case 2: If $b_5 < 0$, then $b_2 < 0$ and $b_4 > 0$.

Therefore, if $\mathscr{A}$ is potentially nilpotent, then $a_{22}a_{33} = -$, $a_{21}a_{31} = +$ and $(a_{22}, a_{21}, a_{41}, a_{54})$ must be one of the forms $(+,-,+,-), (+,+,-,+), (+,-,+,+)$.

Conversely, for a sign pattern $\mathscr{A} = [a_{ij}] \in \mathcal{DSSP}(3,2)$, if $a_{22}a_{33} = -$, $a_{21}a_{31} = +$ and $(a_{22}, a_{21}, a_{41}, a_{54})$ is one of the forms $(+,-,+,-)$, $(+,+,-,+)$, $(+,-,+,+)$, choosing real matrices $A \in Q(\mathscr{A})$ as in (2.8) satisfying $a_2 = -a_3$, $b_2 = b_3$, and $(a_2, b_2, b_4, b_5) = (1, \frac{1}{2}, -4, 2), (1, -1, \frac{1}{2}, \frac{1}{2}), (1, -\frac{1}{4}, \frac{1}{2}, -1)$, respectively, it is easy to verify that A is nilpotent, respectively. $\square$

We next introduce some more notation. Let $X = \{x_j : j \in J\}$ be a set of variables with finite index set J. Let $\Sigma_i(X)$ be the sum of the products of the elements of the i-element subsets of X for each i with $1 \leqslant i \leqslant |J|$. For each $j \in J$, let $X_j = X \setminus \{x_j\}$. If $j \notin J$, we assume $X_j = X$. For example,

$$\Sigma_1(X_j) = \sum_{k \in J\setminus\{j\}} x_k, \qquad \Sigma_2(X_j) = \sum_{\substack{k,l \in J\setminus\{j\} \\ k<l}} x_k x_l.$$

Theorem 2.2. *No sign pattern $\mathscr{A} \in \mathcal{DSSP}(4,2)$ is potentially nilpotent.*

Proof. Suppose the matrix A is in $Q(\mathscr{A})$. Then we can assume that A has the form

$$A = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & \\ b_2 & a_2 & & & & \\ b_3 & & a_3 & & & \\ b_4 & & & a_4 & & \\ b_5 & & & & 0 & 1 \\ & & & & b_6 & 0 \end{pmatrix}.$$

Let $X = \{a_2, a_3, a_4\}$. Then we have

$$\begin{cases} E_1(A) = \Sigma_1(X), \\ E_2(A) = \Sigma_2(X) - \sum\limits_{i=2}^{6} b_i, \\ E_3(A) = \Sigma_3(X) - \sum\limits_{i=2}^{6} b_i \Sigma_1(X_i), \\ E_4(A) = -\sum\limits_{i=2}^{6} b_i \Sigma_2(X_i) + b_6 \sum\limits_{i=2}^{4} b_i, \\ E_5(A) = -\sum\limits_{i=5}^{6} b_i \Sigma_3(X) + b_6 \sum\limits_{i=2}^{4} b_i \Sigma_1(X_i), \\ E_6(A) = b_6 \sum\limits_{i=2}^{4} b_i \Sigma_2(X_i). \end{cases}$$

If A is nilpotent, then $E_k(A) = 0$ for all k. By $E_1(A) = 0$, $E_3(A) = 0$ and $E_5(A) = 0$, we have

$$\begin{cases} \Sigma_3(X) = \sum\limits_{i=2}^{4} b_i \Sigma_1(X_i), \\ \dfrac{b_5 + b_6}{b_6} \Sigma_3(X) = \sum\limits_{i=2}^{4} b_i \Sigma_1(X_i). \end{cases}$$

So $(b_5/b_6) a_2 a_3 a_4 = 0$, which is a contradiction and thus the asserted conclusion follows. $\square$

Theorem 2.3. *Let $\mathscr{A} = [a_{ij}] \in \mathcal{DSSP}(5,2)$. Then $\mathscr{A}$ is potentially nilpotent if and only if, up to equivalence, $a_{22}a_{44} = - = a_{33}a_{55}$, $a_{21}a_{41} = + = a_{31}a_{51}$ and $(a_{22}, a_{33}, a_{21}, a_{31}, a_{61}, a_{76})$ is one of the forms: $(-,-,+,-,-,+)$, $(-,-,-,-,+,+)$, $(-,-,-,+,+,-)$.*

P r o o f. Consider the matrix $A \in Q(\mathscr{A})$ as in (2.8). Let $X = \{a_2, a_3, a_4, a_5\}$. That A is nilpotent is equivalent to

$$\Sigma_1(X) = E_1(A) = 0, \tag{2.9}$$

$$\Sigma_2(X) - \sum_{i=2}^{7} b_i = E_2(A) = 0, \tag{2.10}$$

$$\Sigma_3(X) - \sum_{i=2}^{7} b_i \Sigma_1(X_i) = E_3(A) = 0, \tag{2.11}$$

$$\Sigma_4(X) - \sum_{i=2}^{7} b_i \Sigma_2(X_i) + b_7 \sum_{i=2}^{5} b_i = E_4(A) = 0, \tag{2.12}$$

$$-\sum_{i=2}^{7} b_i \Sigma_3(X_i) + b_7 \sum_{i=2}^{5} b_i \Sigma_1(X_i) = E_5(A) = 0, \tag{2.13}$$

$$-(b_6 + b_7)\Sigma_4(X) + b_7 \sum_{i=2}^{5} b_i \Sigma_2(X_i) = E_6(A) = 0, \tag{2.14}$$

and

$$b_7 \sum_{i=2}^{5} b_i \Sigma_3(X_i) = E_7(A) = 0. \tag{2.15}$$

From (2.19), we have

$$\sum_{i=2}^{5} a_i = 0. \tag{2.16}$$

From (2.15), we have $\sum\limits_{i=2}^{5} b_i \Sigma_3(X_i) = 0$, that is,

$$\sum_{i=2}^{5} \frac{b_i}{a_i} = 0. \tag{2.17}$$

From (2.13) and (2.11), we obtain that $b_7 \sum\limits_{i=2}^{5} b_i \Sigma_1(X_i) = (b_6 + b_7)\Sigma_3(X)$ and $\Sigma_3(X) = \sum\limits_{i=2}^{5} b_i \Sigma_1(X_i)$. Combining these two equations, we have

$$\sum_{i=2}^{5} b_i a_i = 0, \tag{2.18}$$

$$\sum_{i=2}^{5} \frac{1}{a_i} = 0. \tag{2.19}$$

Since the off-diagonal terms in a double star matrix enter into its characteristic polynomial only as products, $-\mathscr{A}$ is equivalent to the sign pattern obtained from $\mathscr{A}$ by taking the negative of the main diagonal. So we may consider only two cases:

Case (1): $(a_2, a_3, a_4, a_5) = (-, +, +, +)$. From (2.16), we have $a_3 + a_4 + a_5 = -a_2$. But from (2.19), we have $1/a_3 + 1/a_4 + 1/a_5 = 1/(a_3 + a_4 + a_5)$, which is a contradiction.

Case (2): $(a_2, a_3, a_4, a_5) = (-, -, +, +)$. We shall prove that there are two pairs of oppositely signed numbers among a_2, a_3, a_4, a_5. From (2.9), we have $a_2 + a_4 = -(a_3 + a_5)$. Let $a_2 = -x$, $a_3 = -y$, $\mu = a_2 + a_4$, where $y > \mu > -x$, $y > 0$, $x > 0$. Then $a_4 = x + \mu$, $a_5 = y - \mu$. By (2.19), we have $-1/x - 1/y + 1/(x + \mu) + 1/(y - \mu) = 0$, and so $(x + \mu)(y - \mu) = xy$, that is, $\mu = 0$ or $\mu = y - x$. Without loss of generality, we always assume that $a_2 = -a_4$, $a_3 = -a_5$. Thus from (2.17) and (2.18), we have

$$\begin{cases} \dfrac{1}{a_2}(b_2 - b_4) + \dfrac{1}{a_3}(b_3 - b_5) = 0, \\ a_2(b_2 - b_4) + a_3(b_3 - b_5) = 0. \end{cases}$$

Consequently, either $b_2 = b_4$ and $b_3 = b_5$ or $a_2 = a_3$ and $b_2 + b_3 = b_4 + b_5$.

Let us distinguish two cases:

Case 1: $a_2 = a_3$ and $b_2 + b_3 = b_4 + b_5$. In this case, $X = \{a_2, a_3, a_4, a_5\} = \{a_2, a_2, -a_2, -a_2\}$ and

$$\begin{gathered} \Sigma_1(X) = 0, \quad \Sigma_2(X) = -2a_2^2, \quad \Sigma_3(X) = 0, \quad \Sigma_4(X) = a_2^4, \\ \Sigma_1(X_2) = -a_2 = \Sigma_1(X_3), \quad \Sigma_1(X_4) = a_2 = \Sigma_1(X_5), \\ \Sigma_2(X_2) = \Sigma_2(X_3) = \Sigma_2(X_4) = \Sigma_2(X_5) = -a_2^2, \\ \Sigma_3(X_2) = a_2^3 = \Sigma_3(X_3), \quad \Sigma_3(X_4) = -a_2^3 = \Sigma_2(X_5). \end{gathered}$$

Solving the equations (2.10), (2.12) and (2.14), we get the following equations

$$\begin{cases} 2a_2^2 + 2(b_2 + b_3) + b_6 + b_7 = 0 \ \text{(by (2.10))}, \\ a_2^4 + 2a_2^2(b_2 + b_3) + 2a_2^2(b_6 + b_7) + 2b_7(b_2 + b_3) = 0 \ \text{(by (2.12))}, \\ a_2^2(b_6 + b_7) + 2b_7(b_2 + b_3) = 0 \ \text{(by (2.14))}. \end{cases}$$

By solving the above equations, we come to a contradiction that $a_2 = 0$.

Case 2: $b_2 = b_4$ and $b_3 = b_5$. In this case it is sufficient to solve the six entries $a_2, a_3, b_2, b_3, b_6, b_7$ to insure the nilpotent matrix A. Now $X = \{a_2, a_3, a_4, a_5\} = \{a_2, a_3, -a_2, -a_3\}$ and

$$\begin{gathered} \Sigma_1(X) = 0, \quad \Sigma_2(X) = -a_2^2 - a_3^2, \quad \Sigma_3(X) = 0, \quad \Sigma_4(X) = a_2^2 a_3^2, \\ \Sigma_1(X_2) = -a_2, \quad \Sigma_1(X_3) = -a_3, \quad \Sigma_1(X_4) = a_2, \quad \Sigma_1(X_5) = a_3, \\ \Sigma_2(X_2) = -a_3^2, \quad \Sigma_2(X_3) = -a_2^2, \quad \Sigma_2(X_4) = -a_3^2, \quad \Sigma_2(X_5) = -a_2^2, \\ \Sigma_3(X_2) = a_2 a_3^2, \quad \Sigma_3(X_3) = a_2^2 a_3, \quad \Sigma_3(X_4) = -a_2 a_3^2, \quad \Sigma_2(X_5) = -a_2^2 a_3. \end{gathered}$$

By a straightforward calculation of the equations (2.10), (2.12) and (2.14), we get the following linear system of equations

$$\begin{cases} a_2^2 + a_3^2 + 2(b_2 + b_3) + b_6 + b_7 = 0 \ \text{(by (2.10))}, \\ a_2^2 a_3^2 + 2(b_2 a_3^2 + b_3 a_2^2) + (a_2^2 + a_3^2)(b_6 + b_7) + 2b_7(b_2 + b_3) = 0 \ \text{(by (2.12))}, \\ a_2^2 a_3^2 (b_6 + b_7) + 2b_7(b_2 a_3^2 + b_3 a_2^2) = 0 \ \text{(by 2.14))}. \end{cases}$$

It is easy to see that the system is underdetermined. In this case, the first three components b_2, b_3, b_6 can be solved in terms of the last three components a_2, a_3, b_7, that is

$$b_2 = \frac{a_2^6}{2(b_7 - a_2^2)(a_2^2 - a_3^2)},$$
$$b_3 = \frac{-a_3^6}{2(b_7 - a_3^2)(a_2^2 - a_3^2)},$$
$$b_6 = \frac{-b_7^3}{(b_7 - a_2^2)(b_7 - a_3^2)}.$$

Without loss of generality, we always assume $a_2^2 > a_3^2$.

Subcase 2.1: $b_7 > 0$.

Subcase 2.1.1: $b_7 > a_2^2 \Rightarrow b_6 < 0, b_2 > 0, b_3 < 0$.

Subcase 2.1.2: $a_3^2 < b_7 < a_2^2 \Rightarrow b_6 > 0, b_2 < 0, b_3 < 0$.

Subcase 2.1.3: $b_7 < a_3^2 \Rightarrow b_6 < 0, b_2 < 0, b_3 > 0$.

Subcase 2.2: $b_7 < 0 \Rightarrow b_6 > 0, b_2 < 0, b_3 > 0$.

Note that the sign pattern in Subcase 2.1.1 is permutation similar to that in Subcase 2.1.3.

Therefore, if $\mathscr{A}$ is potentially nilpotent, then $a_{22}a_{44} = - = a_{33}a_{55}$, $a_{21}a_{41} = + = a_{31}a_{51}$ and $(a_{22}, a_{33}, a_{21}, a_{31}, a_{61}, a_{76})$ must be one of the forms: $(-,-,+,-,-,+)$, $(-,-,-,-,+,+)$, $(-,-,-,+,+,-)$.

The same result may be obtained if $a_2^2 < a_3^2$.

Conversely, for a sign pattern $\mathscr{A} = [a_{ij}] \in \mathcal{DSSP}(5,2)$, if $a_{22}a_{44} = - = a_{33}a_{55}$, $a_{21}a_{41} = + = a_{31}a_{51}$ and $(a_{22}, a_{33}, a_{21}, a_{31}, a_{61}, a_{76})$ is one of the forms $(-,-,+,-,-,+)$, $(-,-,-,-,+,+)$, $(-,-,-,+,+,-)$, choosing real matrices $A \in Q(\mathscr{A})$ as in (2.8) satisfying $a_2 = -a_4$, $a_3 = -a_5$, $b_2 = b_4$, $b_3 = b_5$, and $(a_2, a_3, b_2, b_3, b_6, b_7) = (-2, -1, \frac{32}{3}, -\frac{1}{24}, -\frac{125}{4}, 5)$, $(-2, -1, -\frac{64}{15}, -\frac{1}{3}, \frac{27}{10}, \frac{3}{2})$, $(-2, -1, -\frac{32}{15}, \frac{1}{12}, \frac{1}{10}, -1)$, respectively, it is not difficult to verify that A is nilpotent, respectively. □

One can see that with the increasing values of m, the work of giving the proofs becomes more and more arduous. Up to now it is not clear how the main results of the paper can be extended to larger size matrices, that is to say, where $m > 5$. A new method might be needed for these cases.

Remark 2.1. It is known that by the method in [2], which is sometimes called the Nilpotent-Jacobian method, if a nilpotent realization of a sign pattern $\mathscr{A}$ can be found (this is usually not an easy task), for which the Jacobian is nonzero, then every superpattern of $\mathscr{A}$ is spectrally arbitrary. Up to now we conclude that some sign patterns in $\mathcal{DSSP}(m,2)$ for $m=3,5$ are potentially nilpotent. It is natural to ask whether these sign patterns are spectrally arbitrary. Unfortunately, the answer is negative.

In other words, we have

Theorem 2.4. *No sign pattern $\mathscr{A}$ in $\mathcal{DSSP}(3,2)$ or $\mathcal{DSSP}(5,2)$ is spectrally arbitrary.*

Proof. Consider sign patterns in $\mathcal{DSSP}(3,2)$. Without loss of generality, it may be assumed that each realization of sign pattern $\mathscr{A}$ in $\mathcal{DSSP}(3,2)$ has the form of A in (2.8). The characteristic polynomial of A is

$$\begin{aligned} P_A(x) = x^5 &- (a_2+a_3)x^4 + (a_2a_3 - b_2 - b_3 - b_4 - b_5)x^3 \\ &+ (a_2(b_3+b_4+b_5) + a_3(b_2+b_4+b_5))x^2 \\ &+ (b_5(b_2+b_3) - a_2a_3(b_4+b_5))x - (a_2b_3+a_3b_2)b_5. \end{aligned}$$

If $a_2+a_3=0$ and $(a_2b_3+a_3b_2)b_5=0$, then $a_2(b_3+b_4+b_5)+a_3(b_2+b_4+b_5)=0$, so $P_A(x)$ cannot equal $x^5+\alpha x^2$ for any nonzero α. Thus $\mathscr{A}$ is not spectrally arbitrary. Consequently, $\mathcal{DSSP}(3,2)$ does not contain sign patterns that are spectrally arbitrary. Similarly, it is easy to verify that $\mathcal{DSSP}(5,2)$ contains no sign pattern that is spectrally arbitrary. $\square$

However, for the sign pattern

$$\mathscr{A} = \begin{pmatrix} 0 & + & + & + & \\ - & - & & & \\ - & & + & & \\ + & & & 0 & + \\ & & & - & 0 \end{pmatrix}$$

in $\mathcal{DSSP}(3,2)$, it can be proved that $\mathscr{A}$ is potentially stable. In [1], Bone presented some constructions of potentially stable sign patterns, one of which is the following:

Construction 1'. Given an $(n-1)\times(n-1)$ matrix A^-, we construct an $n\times n$ matrix A as follows:
(i) let A^- appear as a principal submatrix of A;
(ii) A contains a negative n-cycle.

Theorem 2.5. *The sign pattern $\mathscr{A}$ (as above) is potentially stable.*

Proof. We know that $\mathscr{A}_4$ is potentially stable from the result in [4]. It is easy to see that A contains a negative 5-cycle. $\square$

Example 2.1. For the matrix

$$A = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 \\ -3.87265 & -2.31427 & 0 & 0 & 0 \\ -9.917 & 0 & 0.123349 & 0 & 0 \\ 0.79188 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -0.720757 & 0 \end{pmatrix} \in Q(\mathscr{A}),$$

the spectrum of A is $\{-0.227038 + 3.43587\mathrm{i}, -0.227038 - 3.43587\mathrm{i}, -1.73634, -0.000253292 + 0.887001\mathrm{i}, -0.000253292 - 0.887001\mathrm{i}\}$. So A is a stable matrix and $\mathscr{A}$ is potentially stable.

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