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THE k-DOMATIC NUMBER OF A GRAPH

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Abstract. Let k be a positive integer, and let G be a simple graph with vertex set $V(G)$. A *k-dominating set* of the graph G is a subset D of $V(G)$ such that every vertex of $V(G)-D$ is adjacent to at least k vertices in D. A *k-domatic partition* of G is a partition of $V(G)$ into k-dominating sets. The maximum number of dominating sets in a k-domatic partition of G is called the *k-domatic number* $d_k(G)$.

In this paper, we present upper and lower bounds for the k-domatic number, and we establish Nordhaus-Gaddum-type results. Some of our results extend those for the classical domatic number $d(G) = d_1(G)$.

Keywords: domination, k-domination, k-domatic number

MSC 2010: 05C69

1. Terminology and introduction

We consider finite, undirected and simple graphs G with vertex set $V(G)$. The number of vertices $|V(G)|$ of a graph G is called the *order* of G and is denoted by $n = n(G)$.

The *open neighborhood* $N(v) = N_G(v)$ of a vertex v consists of the vertices adjacent to v and $d(v) = d_G(v) = |N(v)|$ is the *degree* of v. The *closed neighborhood* of a vertex v is defined by $N[v] = N_G[v] = N(v) \cup \{v\}$. The *maximum degree* and the *minimum degree* of a graph G are denoted by $\Delta(G) = \Delta$ and $\delta(G) = \delta$, respectively. A graph G with $\delta(G) = \Delta(G)$ is called *regular*. The complement of a graph G is denoted by $\overline{G}$. We write K_n for the *complete graph* of order n.

Let k be a positive integer. A subset $D \subseteq V(G)$ is a *k-dominating set* of the graph G if $|N_G(v) \cap D| \geqslant k$ for every $v \in V(G) - D$. The *k-domination number* $\gamma_k(G)$ is the minimum cardinality among the k-dominating sets of G. Note that the 1-domination number $\gamma_1(G)$ is the classical *domination number* $\gamma(G)$. A *k-domatic partition* of G is a partition of $V(G)$ into k-dominating sets. The maximum number of dominating sets in a k-domatic partition of G is called the *k-domatic number* $d_k(G)$. The 1-domatic number $d_1(G)$ is the usual *domatic number* $d(G)$.

The k-domination number was first studied by Fink and Jacobson [2], [3], and Cockayne and Hedetniemi [1] introduced the concept of the domatic number $d(G)$ of a graph G. For more information on the domatic number and their variants, we refer the reader to the survey article by Zelinka [7]. The following theorem provides a lower bound for the k-domination number in terms of order and maximum degree.

Theorem 1.1 (Fink and Jacobson [2] 1985)**.** *For any graph G,*

$$\gamma_k(G) \geqslant \frac{kn(G)}{k+\Delta(G)}.$$

For a comprehensive treatment of domination in graphs, see the monographs by Haynes, Hedetniemi and Slater [4], [5].

2. Bounds for the k-domatic number

We begin this section with some straightforward observations which are useful for further investigations.

Proposition 2.1. *If $k > p \geqslant 1$ are integers, then $d_p(G) \geqslant d_k(G)$ for any graph G.*

Proof. Let $D_1, D_2, \ldots, D_t$ be a k-domatic partition of G such that $t = d_k(G)$. Then $D_1, D_2, \ldots, D_t$ is also a p-domatic partition of G and thus $d_p(G) \geqslant d_k(G)$. □

Proposition 2.2. *If G is a graph of order n, then $d_k(G) \leqslant n/\gamma_k(G)$.*

Proof. If $k \geqslant n$, then $\gamma_k(G) = n$ and the desired bound is valid. Thus assume now that $k < n$, and let $D_1, D_2, \ldots, D_t$ be a k-domatic partition of G such that $t = d_k(G)$. Then $|D_i| \geqslant \gamma_k(G)$ for each $i \in \{1, 2, \ldots, k\}$. Hence

$$n = \sum_{i=1}^{t} |D_i| \geqslant t\gamma_k(G) = d_k(G)\gamma_k(G),$$

and the desired bound for $d_k(G)$ follows. □

Since $\gamma_k(G) \geqslant \min\{k, n(G)\}$ for any graph G, Proposition 2.2 implies the next bound immediately.

Corollary 2.3. *If G is a graph of order n, then $d_k(G) \leqslant n/k$.*

Corollary 2.4. *If G is a graph of order n, then*

$$d_k(G) + \gamma_k(G) \leqslant d_k(G) + \frac{n}{d_k(G)} \leqslant n + 1. \tag{1}$$

Proof. Proposition 2.2 yields the first inequality in (1). The other inequality follows from the fact that $1 \leqslant d_k(G) \leqslant n$. □

Example 2.5. Let $\overline{H}$ be the disjoint union of p copies of the complete graph K_k. Then H is a graph of order $n(H) = kp$, k-domatic number $d_k(H) = p$ and k-domination number $\gamma_k(H) = k$. Thus

$$d_k(H) + \gamma_k(H) = p + k = d_k(H) + \frac{n(H)}{d_k(H)}.$$

This example shows that Proposition 2.2, Corollary 2.3 and the first inequality in (1) are the best possible.

Theorem 2.6. *Let G be a graph of order n. Then $d_k(G) + \gamma_k(G) = n + 1$ if and only if $\Delta(G) < k$ or $G = K_n$ when $k = 1$.*

Proof. If $\Delta(G) < k$ or $G = K_n$ when $k = 1$, trivially $d_k(G) + \gamma_k(G) = n + 1$.

Conversely, assume that $\Delta(G) \geqslant k$ and $G \neq K_n$ when $k = 1$. Then $n \geqslant 3$ and $\gamma_k(G) \leqslant n - 1$.

If $\gamma_k(G) \geqslant 2$, then Proposition 2.2 implies

$$d_k(G) + \gamma_k(G) \leqslant \gamma_k(G) + \frac{n}{\gamma_k(G)}.$$

If we define $x = \gamma_k(G)$ and $g(x) = x + n/x$ for $x > 0$, then, because of $2 \leqslant \gamma_k(G) \leqslant n - 1$, we have to determine the maximum of the function g in the interval $I\colon\ 2 \leqslant x \leqslant n - 1$. It is straightforward to verify that

$$\begin{aligned}\max_{x \in I}\{g(x)\} &= \max\{g(2), g(n-1)\} = \max\left\{2 + \frac{n}{2}, n - 1 + \frac{n}{n-1}\right\}\\ &= n - 1 + \frac{n}{n-1} < n + 1,\end{aligned}$$

and we obtain $d_k(G) + \gamma_k(G) \leqslant n$ when $\gamma_k(G) \geqslant 2$.

The case that remains is $k = 1$ and $\gamma(G) = 1$. Since $G \neq K_n$, it follows that $d(G) \leqslant n - 1$ and thus $d(G) + \gamma(G) \leqslant n$. □

Corollary 2.7 (Cockayne and Hedetniemi [1] 1977)**.** *For any graph G with n vertices, $d(G) + \gamma(G) \leqslant n + 1$, with equality if and only if $G = K_n$ or $\overline{K_n}$.*

Corollary 2.8. *Let G be a graph of order n, and let $k \geqslant 2$ be an integer. If $d_k(G) \geqslant 2$, then*

$$d_k(G) + \gamma_k(G) \leqslant 2 + \frac{n}{2}.$$

Proof. Since $k \geqslant 2$ and $d_k(G) \geqslant 2$, it follows from Corollary 2.3 that $2 \leqslant d_k(G) \leqslant n/k \leqslant n/2$. Applying the first inequality in (1), we obtain

$$d_k(G) + \gamma_k(G) \leqslant d_k(G) + \frac{n}{d_k(G)} \leqslant 2 + \frac{n}{2}.$$

□

Corollary 2.8 is no longer true for $k = 1$. For example, if H is the complete graph of order $n \geqslant 5$ minus one edge, then $\gamma(H) = 1$ and $d(H) = n - 1$ and therefore $d(H) + \gamma(H) = n > 2 + n/2$.

Theorem 2.9. *For any graph G,*

$$d_k(G) \leqslant \frac{\delta(G)}{k} + 1.$$

Proof. Let $u \in V(G)$ be such that $d_G(u) = \delta(G)$, and let $D_1, D_2, \ldots, D_t$ be a k-domatic partition of G such that $t = d_k(G)$. Then either $u \in D_i$ or $|N_G(u) \cap D_i| \geqslant k$ for each $i \in \{1, 2, \ldots, t\}$. Since $D_1, D_2, \ldots, D_t$ is a partition of $V(G)$, we obtain the desired bound. □

The special case $k = 1$ of Theorem 2.9 can be found in the article by Cockayne and Hedetniemi [1].

For the graph H in Example 2.5 we have $n(H) = kp$, $d_k(H) = p$ and $\delta(H) = n - k = k(p - 1)$. Consequently,

$$d_k(H) = p = \frac{k(p-1)}{k} + 1 = \frac{\delta(H)}{k} + 1,$$

and therefore Theorem 2.9 is the best possible.

The next result is an extension of a lower bound for the classical domatic number given by Zelinka [6].

Theorem 2.10. *For any graph G of order n and minimum degree δ,*

$$d_k(G) \geqslant \left\lfloor \frac{n}{k(n-\delta)} \right\rfloor.$$

Proof. If $k > \delta$, then

$$k(n-\delta) \geqslant (\delta+1)(n-\delta) = n + \delta(n-\delta-1) \geqslant n$$

and the desired bound is obvious.

Assume next that $k \leqslant \delta$. Since the desired bound is trivial in the case $k(n-\delta) > n$, we assume in the sequel that $k(n-\delta) \leqslant n$. Let $D \subseteq V(G)$ be any subset with $|D| \geqslant k(n-\delta)$. It follows that

$$|D| \geqslant k(n-\delta) = n - \delta + (k-1)(n-\delta) \geqslant n - \delta + (k-1)$$

and therefore $|V(G) - D| \leqslant \delta - k + 1$. If $v \in V(G) - D$, then $|N_G[v]| \geqslant \delta + 1$ and $|V(G) - D| \leqslant \delta - k + 1$ imply that $|N_G(v) \cap D| \geqslant k$. Hence D is a k-dominating set of G. Thus one can take any $\lfloor n/(k(n-\delta)) \rfloor$ disjoint subsets, each of cardinality $k(n-\delta)$. All these subsets are k-dominating sets of G, and so Theorem 2.10 follows. □

If H is the complete graph of order $n(H) = kp$, then $d_k(H) = p$ and $\delta(H) = n(H) - 1$ and thus

$$d_k(H) = p = \frac{kp}{k} = \frac{n(H)}{k(n(H) - \delta(H))}.$$

Therefore the lower bound on $d_k(G)$ in Theorem 2.10 is sharp.

3. Nordhaus-Gaddum-type results

Theorem 3.1. *For every graph G of order n,*

$$d_k(G) + d_k(\overline{G}) \leqslant \frac{n-1}{k} + 2, \tag{2}$$

and equality in (2) *implies that G is a regular graph.*

Proof. Because of $\delta(G) + \delta(\overline{G}) \leqslant n - 1$, it follows from Theorem 2.9 that

$$d_k(G) + d_k(\overline{G}) \leqslant \frac{\delta(G)}{k} + 1 + \frac{\delta(\overline{G})}{k} + 1 = \frac{\delta(G) + \delta(\overline{G})}{k} + 2 \leqslant \frac{n-1}{k} + 2,$$

and (2) is proved. If G is not regular, then $\delta(G) + \delta(\overline{G}) \leqslant n - 2$, and we obtain analogously a better bound $d_k(G) + d_k(\overline{G}) \leqslant (n-2)/k + 2$. □

As an immediate corollary of Theorem 3.1, we have the following Nordhaus-Gaddum-type result which was established in [1].

Corollary 3.2 (Cockayne and Hedetniemi [1] 1977). *For every graph G having n vertices, $d(G) + d(\overline{G}) \leqslant n + 1$.*

Theorem 3.3. *Let G be a graph of order $n \geqslant 2$ such that*

$$d_k(G) + d_k(\overline{G}) = \frac{n-1}{k} + 2. \tag{3}$$

If we assume, without loss of generality, that $d_k(G) \geqslant d_k(\overline{G})$, then

$$d_k(G) = \frac{n}{r} \tag{4}$$

for an integer $r \in \{k, k+1, \ldots, 2k-1\}$.

If $k = 1$, then G is isomorphic to the complete graph K_n.

If $k \geqslant 2$, then $k + 1 \leqslant r \leqslant 2k - 1$ and $n < kr^2/(r-k)$.

Proof. If $k \geqslant n$, then equality (3) is impossible, and hence we assume in the sequel that $k \leqslant n-1$. The hypothesis $d_k(G) \geqslant d_k(\overline{G})$ and (3) lead to

$$d_k(G) \geqslant \frac{n+2k-1}{2k}. \tag{5}$$

Let $D_1, D_2, \ldots, D_t$ be a k-domatic partition of G such that $t = d_k(G)$ and $r = |D_1| \leqslant |D_2| \leqslant \ldots \leqslant |D_t|$. Clearly, $r \geqslant k$, and if $r \geqslant 2k$, then (5) yields the contradiction $n \geqslant rt \geqslant 2kd_k(G) \geqslant n + 2k - 1$.

Assume next that $k \leqslant r \leqslant 2k-1$. We notice that

$$n \geqslant rd_k(G). \tag{6}$$

In addition, since D_1 is a k-dominating set of G, we deduce that

$$\sum_{v \in D_1} d_G(v) \geqslant k(n-r)$$

and thus $\Delta(G) \geqslant k(n-r)/r$ and so

$$\delta(\overline{G}) = n - \Delta(G) - 1 \leqslant n - 1 - \frac{k(n-r)}{r} = \frac{n(r-k) + r(k-1)}{r}.$$

Applying Theorem 2.9, we then obtain

$$d_k(\overline{G}) \leqslant \frac{n(r-k) + r(k-1)}{rk} + 1 = \frac{n(r-k) + r(2k-1)}{rk}.$$

Now (3) leads to

$$d_k(G) = \frac{n+2k-1}{k} - d_k(\overline{G}) \geqslant \frac{r(n+2k-1)-(n(r-k)+r(2k-1))}{rk} = \frac{n}{r}.$$

Using this inequality and (6), we arrive at the identity (4).

If $k=1$, then it follows from $k \leqslant r \leqslant 2k-1$ that $r=1$, and therefore (4) implies $d_1(G) = d(G) = n$. However, this is only possible when G is isomorphic to the complete graph K_n.

Assume next that $k \geqslant 2$.

Assume that $r = k$. We deduce that each vertex $v \in V(G) - D_1$ is adjacent to each vertex of D_1 and thus $\Delta(G) \geqslant n-k$ and so $\delta(\overline{G}) \leqslant k-1$. In view of Theorem 2.9, we obtain $d_k(\overline{G}) = 1$, and hence (3) and Corollary 2.3 yield the contradiction

$$\frac{n-1}{k} + 2 = d_k(G) + d_k(\overline{G}) \leqslant \frac{n}{k} + 1.$$

Assume that $k+1 \leqslant r \leqslant 2k-1$. First we note that (4) implies that $|D_i| = r$ for every $i \in \{1,2,\ldots,t\}$. Since $D_1, D_2, \ldots, D_t$ are k-dominating sets of G, each vertex $v \in D_i$ is adjacent to at most $r-k$ vertices in D_j in the graph $\overline{G}$ for $i \neq j$.

Next, let F be any minimum k-dominating set in $\overline{G}$. If $D_i \cap F = \emptyset$ for any $i \in \{1,2,\ldots,t\}$, then the last observation shows that $|F| \geqslant (kr)/(r-k)$. In the other case when $D_i \cap F \neq \emptyset$ for every $i \in \{1,2,\ldots,t\}$, we obviously have $|F| \geqslant t = d_k(G)$. This leads to

$$\gamma_k(\overline{G}) \geqslant \min\Big\{d_k(G), \frac{kr}{r-k}\Big\}. \tag{7}$$

If we suppose on the contrary that $n \geqslant kr^2/(r-k)$, then (4) implies

$$d_k(G) = \frac{n}{r} \geqslant \frac{kr}{r-k},$$

and thus it follows from (7) that $\gamma_k(\overline{G}) \geqslant kr/(r-k)$. Combining this with (3), (4) and Proposition 2.2, we arrive at the contradiction

$$\frac{n-1}{k} + 2 = d_k(G) + d_k(\overline{G}) \leqslant \frac{n}{r} + \frac{n}{\gamma_k(\overline{G})} \leqslant \frac{n}{r} + \frac{n(r-k)}{kr} = \frac{n}{k}.$$

Altogether we have shown that $k+1 \leqslant r \leqslant 2k-1$ and $n < kr^2/(r-k)$ in the case $k \geqslant 2$, and the proof of Theorem 3.3 is complete. □

Since $d(K_n) + d(\overline{K_n}) = n+1$, the next well-known result is an immediate consequence of Theorem 3.3.

Corollary 3.4 (Cockayne and Hedetniemi [1] 1977). *If G is a graph of order n, then $d(G) + d(\overline{G}) = n + 1$ if and only if $G = K_n$ or $\overline{K_n}$.*

Corollary 3.5. *Let $k \geqslant 2$ be an integer. Then there is only a finite number of graphs G of order n such that*

$$d_k(G) + d_k(\overline{G}) = \frac{n-1}{k} + 2.$$

Proof. If $k \geqslant 2$ is a fixed integer, then the hypothesis and Theorem 3.3 lead to $n < kr^2/(r-k)$ with $k+1 \leqslant r \leqslant 2k-1$. This implies that

$$n < \frac{kr^2}{r-k} \leqslant k(2k-1)^2,$$

and the proof is complete. □

Next we investigate the cases $k = 2$ and $k = 3$ in Theorem 3.3 more precisely.

Theorem 3.6. *If G is a graph of order $n \geqslant 3$ such that*

$$d_2(G) + d_2(\overline{G}) = \frac{n-1}{2} + 2, \tag{8}$$

then $n = 9$ and G is 4-regular.

Proof. We assume, without loss of generality, that $d_2(G) \geqslant d_2(\overline{G})$. Let $D_1, D_2, \ldots, D_t$ be a 2-domatic partition of G such that $t = d_2(G)$ and $r = |D_1| \leqslant |D_2| \leqslant \ldots \leqslant |D_t|$. Applying (8) and Theorem 3.3, we deduce that $r = 3$, $3d_2(G) = n$ and $n < 18$ is odd. Since $n = 3$ is not possible, there remain two cases $n = 9$ and $n = 15$.

If $n = 15$, then (7) implies that $\gamma_2(\overline{G}) \geqslant 5$ and thus Proposition 2.2 leads to $d_2(\overline{G}) \leqslant 3$. Combining this with $d_2(G) = 5$, we obtain $d_2(G) + d_2(\overline{G}) \leqslant 8$, a contradiction to the hypothesis (8).

Assume that $n = 9$. First we note that, in view of Theorem 3.1, G and $\overline{G}$ are regular graphs. According to (4), we have $d_2(G) = 3$. Theorem 1.1 implies that

$$3 = r \geqslant \gamma_2(G) \geqslant \frac{2n}{2 + \delta(G)}$$

and thus $\delta(G) \geqslant 4$. This yields $\delta(\overline{G}) \leqslant 4$. If we suppose that $\delta(\overline{G}) \leqslant 3$, then Theorem 2.9 leads to $d_2(\overline{G}) \leqslant 2$. Thus

$$d_2(G) + d_2(\overline{G}) \leqslant 5,$$

a contradiction to (8). Hence $\overline{G}$ and G are 4-regular graphs, and the proof is complete. □

Example 3.7. If H is the 4-regular graph of order 9 in Figure 1, then $\{u_1, u_2, u_3\}$, $\{v_1, v_2, v_3\}$ and $\{w_1, w_2, w_3\}$ are 2-dominating sets of H. Therefore $d_2(H) \geqslant 3$.

In Figure 2 we have sketched the graph $\overline{H}$, and we observe that $\{u_1, v_1, w_2\}$, $\{u_2, v_3, w_3\}$ and $\{u_3, v_2, w_1\}$ are 2-dominating sets of $\overline{H}$. Combining this with (2), we deduce that $d_2(H) + d_2(\overline{H}) = 6 = \frac{1}{2}(n-1) + 2$.

This example demonstrates that there exists at least one 4-regular graph of order 9 such that the identity (8) holds.

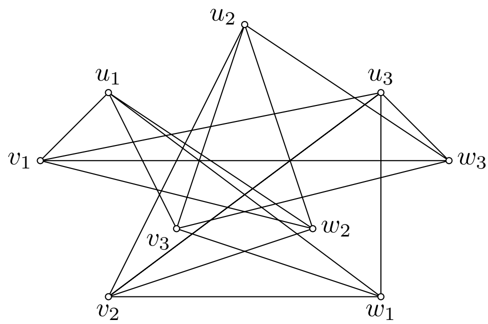

Figure 1. Graph H

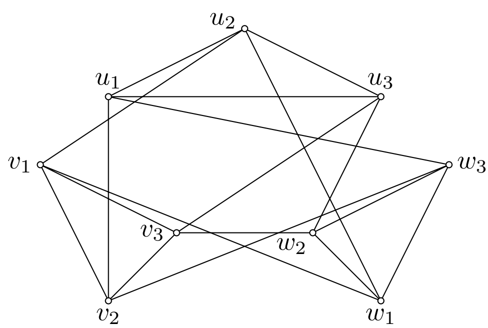

Figure 2. Graph $\overline{H}$

Theorem 3.8. *If G is a graph of order $n \geqslant 4$ such that*

$$(9) \qquad d_3(G) + d_3(\overline{G}) = \frac{n-1}{3} + 2,$$

then $n = 25$ and G is 12-regular or $n = 28$ and G or $\overline{G}$ is 9-regular.

Proof. We assume, without loss of generality, that $d_3(G) \geqslant d_3(\overline{G})$. Let $D_1, D_2, \ldots, D_t$ be a 3-domatic partition of G such that $t = d_3(G)$ and $r = |D_1| \leqslant$

$|D_2| \leqslant \ldots \leqslant |D_t|$. Applying Theorem 3.3, we deduce that $r = 4$ or $r = 5$. In view of Theorem 3.1, G and $\overline{G}$ are regular graphs.

Case 1: Assume that $r = 4$. Then it follows from Theorem 3.3 that $4d_3(G) = n$ and $n < 48$. As $3 \mid (n-1)$, we deduce that $n = 4(3j-2)$ for an integer $j \geqslant 1$. Since $n = 4$ is not possible, there remain three cases $n = 16$, $n = 28$ and $n = 40$.

Subcase 1.1: Assume that $n = 40$. Then (7) implies that $\gamma_3(\overline{G}) \geqslant 10$ and thus Proposition 2.2 leads to $d_3(\overline{G}) \leqslant 4$. Combining this with $d_3(G) = 10$, we obtain $d_3(G) + d_3(\overline{G}) \leqslant 14$, a contradiction to the hypothesis (9).

Subcase 1.2: Assume that $n = 16$. According to (4), we have $d_3(G) = 4$. Using Theorem 1.1, we obtain

$$4 = r \geqslant \gamma_3(G) \geqslant \frac{3n}{3 + \delta(G)}$$

and thus $\delta(G) \geqslant 9$. This yields $\delta(\overline{G}) \leqslant 6$, and hence Theorem 1.1 leads to $\gamma_3(\overline{G}) \geqslant 6$. Now it follows from Proposition 2.2 that $d_3(\overline{G}) \leqslant 2$, and we arrive at the contradiction $d_3(G) + d_3(\overline{G}) \leqslant 6$.

Subcase 1.3: Assume that $n = 28$. According to (4), we have $d_3(G) = 7$. Theorem 1.1 implies that

$$4 = r \geqslant \gamma_3(G) \geqslant \frac{3n}{3 + \delta(G)}$$

and thus $\delta(G) \geqslant 18$. This yields $\delta(\overline{G}) \leqslant 9$. If we suppose that $\delta(\overline{G}) \leqslant 8$, then Theorem 2.9 leads to $d_3(\overline{G}) \leqslant 3$. Thus

$$d_3(G) + d_3(\overline{G}) \leqslant 10,$$

a contradiction to (9). Hence $\overline{G}$ is 9-regular and G is 18-regular.

Case 2: Assume that $r = 5$. Then it follows from Theorem 3.3 that $5d_3(G) = n$ and $n < 37$. As $3 \mid (n-1)$, we deduce that $n = 5(3j-1)$ for an integer $j \geqslant 1$ and thus $n = 10$ or $n = 25$.

Subcase 2.1: Assume that $n = 10$. Then (4) implies that $d_3(G) = 2$. Using Theorem 1.1, we obtain

$$5 = r \geqslant \gamma_3(G) \geqslant \frac{3n}{3 + \delta(G)}$$

and thus $\delta(G) \geqslant 3$. This yields $\delta(\overline{G}) \leqslant 6$, and hence Theorem 1.1 leads to $\gamma_3(\overline{G}) \geqslant 4$. Now it follows from Proposition 2.2 that $d_3(\overline{G}) \leqslant 2$, and we arrive at the contradiction $d_3(G) + d_3(\overline{G}) \leqslant 4$.

Subcase 2.2: Assume that $n = 25$. Then (4) implies that $d_3(G) = 5$. Using Theorem 1.1, we obtain

$$5 = r \geqslant \gamma_3(G) \geqslant \frac{3n}{3 + \delta(G)}$$

and thus $\delta(G) \geqslant 12$. This yields $\delta(\overline{G}) \leqslant 12$. If we suppose that $\delta(\overline{G}) \leqslant 11$, then Theorem 2.9 leads to $d_3(\overline{G}) \leqslant 4$. Thus

$$d_3(G) + d_3(\overline{G}) \leqslant 9,$$

a contradiction to (9). Hence both $\overline{G}$ and G are 12-regular graphs, and the proof is complete. □

Example 3.9. The following adjacency matrix represents a 12-regular graph H with vertex set $\{1, 2, \ldots, 25\}$.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
1	0	0	0	0	0	1	0	1	0	1	1	1	0	1	0	1	0	1	0	1	1	1	0	1	0
2	0	0	0	0	0	1	1	0	1	0	0	1	1	0	1	1	1	0	1	0	0	1	1	0	1
3	0	0	0	0	0	0	1	1	0	1	1	0	1	1	0	0	1	1	0	1	1	0	1	1	0
4	0	0	0	0	0	1	0	1	1	0	0	1	0	1	1	1	0	1	1	0	0	1	0	1	1
5	0	0	0	0	0	0	1	0	1	1	1	0	1	0	1	0	1	0	1	1	1	0	1	0	1
6	1	1	0	1	0	0	0	0	0	0	1	0	1	0	1	1	1	0	1	0	1	0	1	0	1
7	0	1	1	0	1	0	0	0	0	0	1	1	0	1	0	0	1	1	0	1	1	1	0	1	0
8	1	0	1	1	0	0	0	0	0	0	0	1	1	0	1	1	0	1	1	0	0	1	1	0	1
9	0	1	0	1	1	0	0	0	0	0	1	0	1	1	0	0	1	0	1	1	1	0	1	1	0
10	1	0	1	0	1	0	0	0	0	0	0	1	0	1	1	1	0	1	0	1	0	1	0	1	1
11	1	0	1	0	1	1	1	0	1	0	0	0	0	0	0	1	0	1	0	1	1	1	0	1	0
12	1	1	0	1	0	0	1	1	0	1	0	0	0	0	0	1	1	0	1	0	0	1	1	0	1
13	0	1	1	0	1	1	0	1	1	0	0	0	0	0	0	0	1	1	0	1	1	0	1	1	0
14	1	0	1	1	0	0	1	0	1	1	0	0	0	0	0	1	0	1	1	0	0	1	0	1	1
15	0	1	0	1	1	1	0	1	0	1	0	0	0	0	0	0	1	0	1	1	1	0	1	0	1
16	1	1	0	1	0	1	0	1	0	1	1	1	0	1	0	0	0	0	0	0	1	0	1	0	1
17	0	1	1	0	1	1	1	0	1	0	0	1	1	0	1	0	0	0	0	0	1	1	0	1	0
18	1	0	1	1	0	0	1	1	0	1	1	0	1	1	0	0	0	0	0	0	0	1	1	0	1
19	0	1	0	1	1	1	0	1	1	0	0	1	0	1	1	0	0	0	0	0	1	0	1	1	0
20	1	0	1	0	1	0	1	0	1	1	1	0	1	0	1	0	0	0	0	0	0	1	0	1	1
21	1	0	1	0	1	1	1	0	1	0	1	0	1	0	1	1	1	0	1	0	0	0	0	0	0
22	1	1	0	1	0	0	1	1	0	1	1	1	0	1	0	0	1	1	0	1	0	0	0	0	0
23	0	1	1	0	1	1	0	1	1	0	0	1	1	0	1	1	0	1	1	0	0	0	0	0	0
24	1	0	1	1	0	0	1	0	1	1	1	0	1	1	0	0	1	0	1	1	0	0	0	0	0
25	0	1	0	1	1	1	0	1	0	1	0	1	0	1	1	1	0	1	0	1	0	0	0	0	0

This adjacency matrix shows easily that $D_1 = \{1, 2, 3, 4, 5\}$, $D_2 = \{6, 7, 8, 9, 10\}$, $D_3 = \{11, 12, 13, 14, 15\}$, $D_4 = \{16, 17, 18, 19, 20\}$ and $D_5 = \{21, 22, 23, 24, 25\}$ are 3-dominating sets of H. Therefore $d_3(H) \geqslant 5$. In addition, it is straightforward to verify that $F_1 = \{1, 6, 11, 16, 21\}$, $F_2 = \{2, 7, 12, 17, 22\}$, $F_3 = \{3, 8, 13, 18, 23\}$, $F_4 = \{4, 9, 14, 19, 24\}$ and $F_5 = \{5, 10, 15, 20, 25\}$ are 3-domiting sets of $\overline{H}$ and thus $d_3(\overline{H}) \geqslant 5$. Combining this with (2), we arrive at $d_3(H) + d_3(\overline{H}) = 10 = \frac{1}{3}(n(H) - 1) + 2$.

This example shows that there exists at least one 12-regular graph of order 25 such that the identity (9) holds.

Whether there exist regular graphs of order $n = 28$ with equality in (9) remains still open.

Following the idea of Example 3.9, for each $k \geqslant 4$ we have constructed $2k(k-1)$-regular graphs H of order $(2k-1)^2$ such that

$$d_k(H) + d_k(\overline{H}) = \frac{n(H)-1}{k} + 2 = 4k - 2.$$

While this work was printed, we discovered an article of B. Zelinka [8], where he introduced the k-domatic number as the k-ply domatic number. In Zelinka's article one can find Proposition 2.1 and Theorem 2.9 of our work.

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