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ON STRONGLY (P)-CYCLIC ACTS

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Abstract. By a regular act we mean an act such that all its cyclic subacts are projective. In this paper we introduce strong (P)-cyclic property of acts over monoids which is an extension of regularity and give a classification of monoids by this property of their right (Rees factor) acts.

Keywords: strongly (P)-cyclic, right PCP, Rees factor act

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1. Introduction

Throughout this paper S will denote a monoid. We refer the reader to [1] and [3] for basic results, definitions and terminology relating to semigroups and acts over monoids and to [4], [5] for definitions and results on flatness which are used here.

A monoid S is called *right (left) reversible* if for every $s, s' \in S$ there exist $u, v \in S$ such that $us = vs'(su = s'v)$. A monoid S is said to be *left collapsible* if for any $p, q \in S$ there exists $r \in S$ such that $rp = rq$. An element s of a monoid S is callled left e-cancellable for an idempotent $e \in S$ if $s = se$ and $\ker \lambda_s \leqslant \ker \lambda_e$. By ([3, III, 10.15]), this is equivalent to saying that $\ker \lambda_s = \ker \lambda_e$.

A right ideal K of a monoid S is called *left stabilizing* if for every $k \in K$, there exists $l \in K$ such that $lk = k$, and it is called *left annihilating* if

$$(\forall t \in S)(\forall x, y \in S \setminus K)(xt, yt \in K \Rightarrow xt = yt).$$

If for all $s, t \in S \setminus K$ and for all homomorphisms $f\colon {}_S(Ss \cup St) \to {}_SS$

$$f(s), f(t) \in K \Rightarrow f(s) = f(t),$$

then K is called *strongly left annihilating.*

A right S-act A satisfies Condition (P) if for all $a, a' \in A$, $s, s' \in S$, $as = a's'$ implies that there exist $a'' \in A$, $u, v \in S$ such that $a = a''u$, $a' = a''v$ and $us = vs'$. A monoids S is called right PCP if all principal right ideals of S satisfy Condition (P).

A right S-act A is called *(strongly) faithful* if for $s, t \in S$ the equality $as = at$ for (some) all $a \in A$ implies $s = t$.

A right S-act A is called *simple* if it contains no subacts other than A itself.

We use the following abbreviations:

weak homoflatness = (WP)

principal weak homoflatness = (PWP)

weak flatness = WF

principal weak flatness = PWF

2. Classification by strong (P)-cyclic property of right acts

In this section we give a classification of monoids when acts with other properties imply strong (P)-cyclic property and vice versa. We also give a classification of monoids when all their acts are strongly (P)-cyclic.

We recall that an element a of a right S-act A is called *act-regular* if there exists a homomorphism $f\colon aS \to S$ such that $af(a) = a$, and A is called a *regular act* if every $a \in A$ is act-regular. It can be seen by ([3, III, 19.3]) that A is a regular act if and only if for every $a \in A$ the cyclic subact aS is projective.

Theorem 2.1. *Let S be a monoid and A a right S-act. Then A is regular if and only if for every $a \in A$ there exists $z \in S$ such that* $\ker \lambda_a = \ker \lambda_z$ *and zS is projective.*

P r o o f. By ([3, III, 19.2]), ([3, III, 19.3]) and ([3, III, 17.8]), it is obvious. □

Definition 2.1. A right S-act A is called *strongly (P)-cyclic* if for every $a \in A$ there exists $z \in S$ such that $\ker \lambda_a = \ker \lambda_z$ and zS satisfies Condition (P).

It can be seen that if a right S-act A is strongly (P)-cyclic, then for every $a \in A$ there exists $z \in S$ such that $aS \cong zS$. Since zS satisfies Condition (P), aS also satisfies Condition (P). Thus every cyclic subact of A satisfies Condition (P). However, note that the converse is not true in general, for if S is a non trivial group and $\Theta_S = \{\theta\}$ is the one element act then, since S is right reversible, by ([3, III, 13.7]) Θ_S satisfies Condition (P), but since for every $z \in S$, $\ker \lambda_z = \Delta_S \neq S \times S = \ker \lambda_\theta$, then Θ_S is not strongly (P)-cyclic.

It is obvious that every regular right act is strongly (P)-cyclic, but the converse is not true, for if $S = S_1 \cup S_2$, where $S_1 = \{1, e_1, e_2, \ldots, \}$ is an infinite semigroup with the multiplication defined by $e_k \cdot e_l = e_{\max\{k,l\}}$, $S_2 = \{x, x^2, x^3, \ldots\}$ is an infinite monogenic semigroup and the multiplication in S is defined by $s \cdot x^n = x^n \cdot s = x^n$ for every $s \in S_1$ and every natural number n, then S is a right PCP monoid, but it is not a right PP monoid, that is, S_S is a strongly (P)-cyclic right S-act which is not regular.

Now we establish some general properties.

Theorem 2.2. *Let S be a monoid. Then:*

(1) *Θ_S is strongly (P)-cyclic if and only if S contains a left zero element.*

(2) *S_S is strongly (P)-cyclic if and only if S is right PCP.*

(3) *If $\{A_i\}_{i \in I}$ is a family of subacts of A_S, then $\bigcup_{i \in I} A_i$ is strongly (P)-cyclic if and only if for every $i \in I$, A_i is strongly (P)-cyclic.*

(4) *Every subact of a strongly (P)-cyclic right S-act is strongly (P)-cyclic.*

Proof. (1) Suppose $\Theta_S = \{\theta\}$ is strongly (P)-cyclic. Then by definition there exists $z \in S$ such that $\ker \lambda_\theta = \ker \lambda_z$. Since $\ker \lambda_\theta = S \times S$, z is a left zero element.

Conversely, suppose that S contains a left zero element z. Then $\ker \lambda_\theta = \ker \lambda_z = S \times S$. Also, S is right reversible, hence by ([3, III, 13.7]), $zS = \{z\}$ satisfies Condition (P).

The proofs of other parts are straightforward. □

Note that freeness does not imply strong (P)-cyclic property, for if $S = \{0, 1, x\}$ where $x^2 = 0$, then S_S as a right S-act is free, but S_S is not strongly (P)-cyclic, otherwise $xS = \{0, x\}$ as a cyclic subact of S_S would satisfy Condition (P) and so $x \cdot x = x \cdot 0$ would imply that there exist u, v in S such that $x = xu = xv$ and $ux = v0$, which is not true.

Now we characterize monoids over which freeness and projectivity of (finitely generated, cyclic) acts imply strong (P)-cyclic property of acts.

Theorem 2.3. *For any monoid S the following statements are equivalent:*

(1) *All projective right S-acts are strongly (P)-cyclic.*

(2) *All projective finitely generated right S-acts are strongly (P)-cyclic.*

(3) *All projective cyclic right S-acts are strongly (P)-cyclic.*

(4) *All projective generators right S-acts are strongly (P)-cyclic.*

(5) *All projective generators finitely generated right S-acts are strongly (P)-cyclic.*

(6) *All projective generators cyclic right S-acts are strongly (P)-cyclic.*

(7) *All free right S-acts are strongly (P)-cyclic.*

(8) *All free finitely generated right S-acts are strongly (P)-cyclic.*

(9) *All free cyclic right S-acts are strongly (P)-cyclic.*

(10) *S is right PCP.*

(11) $(\forall s,t,z \in S)\ (zs = zt \Rightarrow (\exists u,v \in S)\ (z = zu = zv \wedge us = vt))$.

P r o o f. Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3), (4) $\Rightarrow$ (5) $\Rightarrow$ (6), (7) $\Rightarrow$ (8) $\Rightarrow$ (9), (3) $\Rightarrow$ (6) $\Rightarrow$ (9) and (1) $\Rightarrow$ (4) $\Rightarrow$ (7) are obvious.

(9) $\Rightarrow$ (10). By ([3, I, 5.13]), S_S is a free cyclic right S-act and so by assumption it is strongly (P)-cyclic, thus by (2) of Theorem 2.2, S is right PCP.

(10) $\Leftrightarrow$ (11). By ([3, III, 13.10]), it is obvious.

(10) $\Rightarrow$ (1). Suppose that A is a projective right S-act. Then by ([3, III, 17.8]), $A = \coprod_{i\in I} A_i$, where $A_i \cong e_iS$ for some $e_i \in E(S)$. Thus for every $i \in I$, A_i is strongly (P)-cyclic. Since by assumption S_S is strongly (P)-cyclic, hence by (4) of Theorem 2.2, e_iS is strongly (P)-cyclic. Thus A_i is strongly (P)-cyclic and so by (3) of Theorem 2.2, A is strongly (P)-cyclic as required. □

Note that cofreeness does not imply strong (P)-cyclic property, otherwise every act would be strongly (P)-cyclic, as by ([3, II, 4.3]), every act can be embedded into a cofree act and also by (4) of Theorem 2.2, every subact of a strongly (P)-cyclic act is strongly (P)-cyclic. Now if we consider the monoid $S = \{0,1,x\}$ with $x^2 = 0$, then as we saw before Theorem 2.3, S_S as a right S-act is not strongly (P)-cyclic and so we have a contradiction. Now it is obvious that divisibility does not imply strong (P)-cyclic property, either. Note also that S_S is a cyclic faithful act and so faithfulness of cyclic acts does not imply strong (P)-cyclic property, either.

Theorem 2.4. *For any monoid S the following statements are equivalent:*

(1) *All right S-acts are strongly (P)-cyclic.*

(2) *All finitely generated right S-acts are strongly (P)-cyclic.*

(3) *All cyclic right S-acts are strongly (P)-cyclic.*

(4) *All monocyclic right S-acts are strongly (P)-cyclic.*

(5) *All divisible right S-acts are strongly (P)-cyclic.*

(6) *All principally weakly injective right S-acts are strongly (P)-cyclic.*

(7) *All fg-weakly injective right S-acts are strongly (P)-cyclic.*

(8) *All weakly injective right S-acts are strongly (P)-cyclic.*

(9) *All injective right S-acts are strongly (P)-cyclic.*

(10) *All cofree right S-acts are strongly (P)-cyclic.*

(11) *All faithful right S-acts are strongly (P)-cyclic.*

(12) *All faithful finitely generated right S-acts are strongly (P)-cyclic.*

(13) *All faithful right S-acts generated by at most two elements are strongly (P)-cyclic.*

(14) *$S = \{1\}$ or $S = \{0,1\}$.*

P r o o f. Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4), (5) $\Rightarrow$ (6) $\Rightarrow$ (7) $\Rightarrow$ (8) $\Rightarrow$ (9) $\Rightarrow$ (10), (11) $\Rightarrow$ (12) $\Rightarrow$ (13), (1) $\Rightarrow$ (5), and (1) $\Rightarrow$ (11) are obvious.

(4) $\Rightarrow$ (14). By assumption all monocyclic right S-acts satisfy condition (P) and so by ([3, IV, 9.9]), $S = G$ or $S = G^0$, where G is a group. Now we show in both cases that $|G| = 1$. If $S = G$ and $|G| > 1$, then for every $s \in G \setminus \{1\}$, $S/\varrho(s,1)$ is strongly (P)-cyclic and so there exists $z \in G$ such that $\ker \lambda_{[1]_{\varrho(s,1)}} = \ker \lambda_z$. Since $(s,1) \in \varrho(s,1)$, we have $[1]_{\varrho(s,1)}1 = [1]_{\varrho(s,1)}s$, that is, $(1,s) \in \ker \lambda_{[1]_{\varrho(s,1)}} = \ker \lambda_z$. Thus $z = zs$ and so $s = 1$, which is a contradiction. If $S = G^0$ and $|G| > 1$, then by assumption for every $s \in G \setminus \{1\}$, $S/\varrho(s,1)$ is strongly (P)-cyclic and so there exists $z \in G^0$ such that $\ker \lambda_{[1]_{\varrho(s,1)}} = \ker \lambda_z$. If $z \in G$, then $\ker \lambda_z = \ker \lambda_1 = \Delta_S$ and so $(1,s) \in \Delta_S$, that is, $s = 1$, which is a contradiction. If $z = 0$, then $\ker \lambda_{[1]_{\varrho(s,1)}} = \ker \lambda_0 = G^0 \times G^0$ and so $(0,1) \in \ker \lambda_{[1]_{\varrho(s,1)}}$, that is, $[0]_{\varrho(s,1)} = [1]_{\varrho(s,1)}$. Thus $(0,1) \in \varrho(s,1)$ and so by ([3, I, 4.37]), there exist $s_1, s_2, \ldots, s_n, t_1, t_2, \ldots, t_n, y_1, y_2, \ldots, y_n \in S$ such that for every $i \in \{1, 2, \ldots, n\}$, $\{s_i, t_i\} = \{s, 1\}$,

$$\begin{gathered} 0 = s_1 y_1, \quad t_2 y_2 = s_3 y_3, \quad \ldots, \quad t_n y_n = 1, \\ t_1 y_1 = s_2 y_2, \quad t_3 y_3 = s_4 y_4, \quad \ldots \end{gathered}$$

From $0 = s_1 y_1$ we have $y_1 = 0$ and so $0 = s_2 y_2$, which implies that $y_2 = 0$. By continuing this procedure we obtain contradiction. Thus $|G| = 1$ and so either $S = \{1\}$ or $S = \{0,1\}$ as required.

(14) $\Rightarrow$ (1). If $S = \{1\}$ or $S = \{0,1\}$, then by ([3, IV, 14.4]), all right S-acts are regular and so all right S-acts are strongly (P)-cyclic as required.

(10) $\Rightarrow$ (1). Suppose that A is a right S-act. By ([3, II, 4.3]), A can be embedded into a cofree right S-act. Since A is a subact of a cofree right S-act, by assumption A is a subact of a strongly (P)-cyclic right S-act and so by (4) of Theorem 2.2, A is strongly (P)-cyclic.

(13) $\Rightarrow$ (3). Suppose that aS is a cyclic right S-act and $B_S = aS \coprod S$. Since S is faithful, B_S is also faithful and so by assumption B_S is strongly (P)-cyclic. Since aS is a subact of B_S, by (4) of Theorem 2.2, aS is also strongly (P)-cyclic. Thus every cyclic right S-act is strongly (P)-cyclic. □

Now from Theorem 2.4 and ([3, IV, 14.4]) it is easy to show that all right S-acts are regular: it suffices to show that all monocyclic right S-acts are strongly (P)-cyclic or equivalently, if there exists a right S-act which is not regular, then there exists a monocyclic right S-act which is not strongly (P)-cyclic.

Lemma 2.1. *Let S be a monoid, zS a strongly (P)-cyclic right ideal of S and I_S a right ideal of S such that $I_S \subset zS$. Then $A_S = zS \coprod^{I_S} zS$ is strongly (P)-cyclic.*

P r o o f. We know that $A_S = (z,x)S \mathbin{\dot\cup} I_S \mathbin{\dot\cup} (z,\varrho y)S$, where $B_S = (z,x)S \mathbin{\dot\cup} I_S \cong zS \cong (z,y)S \mathbin{\dot\cup} I_S = C_S$. Since by assumption zS is strongly (P)-cyclic and $A_S = B_S \cup C_S$, hence by (3) of Theorem 2.2, A_S is also strongly (P)-cyclic as required. $\square$

Now we show that strong (P)-cyclic property does not imply torsion freeness in general. Let $S = (\mathbb{N}, \cdot)$, where $\mathbb{N}$ is the set of natural numbers and $A_{\mathbb{N}} = \mathbb{N} \coprod^{2\mathbb{N}} \mathbb{N}$. Then by Lemma 2.1, $A_{\mathbb{N}}$ is a strongly (P)-cyclic right S-act, since $\mathbb{N}_{\mathbb{N}}$ is strongly (P)-cyclic and $2\mathbb{N}$ is an ideal of $\mathbb{N}$ such that $2\mathbb{N} \subset \mathbb{N}$. But $A_{\mathbb{N}}$ is not torsion free, since $2 = (1,x)2 = (1,y)2$, but $(1,x) \neq (1,y)$.

Now it is obvious that strong (P)-cyclic property does not imply other properties which imply torsion freeness, hence it is natural to ask for monoids S over which strong (P)-cyclic property of acts imply torsion freeness and other properties which implies torsion freeness.

Lemma 2.2. *Let S be a monoid. If there exists a strongly (P)-cyclic right S-act, then there exists the greatest strongly (P)-cyclic right ideal T of S.*

P r o o f. By assumption there exists a strongly (P)-cyclic right S-act A. Thus for every $a \in A$ there exists $z \in S$ such that $aS \cong zS$. Since aS as a subact of A is strongly (P)-cyclic, zS is also strongly (P)-cyclic and so we have at least one strongly (P)-cyclic right ideal of S. Now the union of all strongly (P)-cyclic right ideals of S is the greatest right ideal T of S, which by (3) of Theorem 2.2 is strongly (P)-cyclic. $\square$

In the following theorems we suppose that there exists at least a strongly (P)-cyclic right S-act and T is the greatest strongly (P)-cyclic right ideal of S.

Theorem 2.5. *Let S be a monoid. Then all strongly (P)-cyclic right S-acts are torsion free if and only if for every $z \in T$ and every right cancellable element c of S there exists an element $l \in S$ such that $z = zcl$.*

P r o o f. Suppose that all strongly (P)-cyclic right S-acts are torsion free and let $z \in T$, $c \in S$, where c is right cancellable. We claim that $zS = zcS$, otherwise $zcS \subset zS$ and so by Lemma 2.1, $A_S = zS \coprod^{zcS} zS$ is strongly (P)-cyclic, since $zS \subseteq T$ and T is strongly (P)-cyclic. Thus by assumption A_S is torsion free. Since $zc = (z,x)c = (z,y)c$, we have $(z,x) = (z,y)$, which is a contradiction. Thus $zS = zcS$ and so there exists $l \in S$ such that $z = zcl$.

Conversely, suppose that A is a strongly (P)-cyclic right S-act, $ac = bc$ for $a, b \in A$ and a right cancellable element c of S. Then there exist $z_1, z_2 \in S$ such that $\ker \lambda_a = \ker \lambda_{z_1}$ and $\ker \lambda_b = \ker \lambda_{z_2}$ and so $aS \cong z_1 S$ and $bS \cong z_2 S$. Since A is strongly (P)-cyclic, hence by (4) of Theorem 2.2, aS and bS are strongly (P)-cyclic. Thus z_1S and z_2S are also strongly (P)-cyclic. Since $z_1S \cup z_2S \subseteq T$, by assumption there exists $l \in S$ such that $z_1 = z_1cl$. Thus

$$z_1 = z_1cl \Rightarrow (1, cl) \in \ker \lambda_{z_1} = \ker \lambda_a \Rightarrow a = acl.$$

Thus

$$ac = aclc \Rightarrow bc = bclc \Rightarrow (c, clc) \in \ker \lambda_b = \ker \lambda_{z_2} \Rightarrow z_2c = z_2clc$$
$$\Rightarrow z_2 = z_2cl \Rightarrow (1, cl) \in \ker \lambda_{z_2} = \ker \lambda_b \Rightarrow b = bcl = acl = a.$$

Thus A is torsion free as required. □

Lemma 2.3. *Let S be a monoid and A a right S-act. If all cyclic subacts of A are simple, then for every $a, a' \in A$, either $aS \cap a'S = \emptyset$ or $aS = a'S$.*

Proof. Suppose $a, a' \in A$ and let $x \in aS \cap a'S$. Then $xS \subseteq aS$ and $xS \subseteq a'S$. Since aS and $a'S$ are simple, we have $aS = xS = a'S$. □

Theorem 2.6. *For any monoid S the following statements are equivalent:*
(1) *All strongly (P)-cyclic right S-acts satisfy Condition (P).*
(2) *All strongly (P)-cyclic right S-acts satisfy Condition (WP).*
(3) *All strongly (P)-cyclic right S-acts satisfy Condition (PWP).*
(4) *For every $z \in T$, zS is a minimal right ideal of S.*

Proof. Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are obvious.

(3) $\Rightarrow$ (4). Let $z \in T$. We claim that zS is a minimal right ideal of S, otherwise there exists a right ideal I of S such that $I \subset zS$. Then by Lemma 2.1, $A_S = zS \coprod^{I_S} zS$ is strongly (P)-cyclic and so A_S satisfies Condition (PWP). Now let $zu \in I$. Then by the definition of A_S, $zu = (z, x)u = (z, y)u$ and so there exist $a \in A_S$, $w_1, w_2 \in S$ such that $(z, x) = aw_1$, $(z, y) = aw_2$ and $w_1u = w_2u$. Now $(z, x) = aw_1$ implies that $a = (t, x)$ for some $t \in zS \setminus I$, similarly $a = (t', y)$ for some $t' \in zS \setminus I$ and so we have a contradiction.

(4) $\Rightarrow$ (1). Suppose that A is a strongly (P)-cyclic right S-act and let $a \in A$. Then by definition there exists $z \in S$ such that $aS \cong zS$. Since by (4) of Theorem 2.2, aS is strongly (P)-cyclic, zS is strongly (P)-cyclic. Since T is the greatest strongly (P)-cyclic right ideal of S, we have $zS \subseteq T$ and so $z \in T$. Thus by assumption zS is a minimal right ideal of S and so aS is simple. Now suppose that $as = a't$, for $a, a' \in A$ and $s, t \in S$. Since $as = a't$, hence $aS \cap a'S \neq \emptyset$ and so by Lemma 2.3, $aS = a'S$. Thus $a' = as_1$ for some $s_1 \in S$ and so $as = as_1t$. Since A is strongly (P)-cyclic, aS satisfies Condition (P) and so there exist $s_2, u, v \in S$ such that $a = as_2u$, $as_1 = as_2v$ and $us = vt$. Now if $a'' = as_2$, then $a = a''u$, $a' = as_1 = as_2v = a''v$ and $us = vt$. Thus A satisfies condition (P) as required. □

Theorem 2.7. *Let S be a monoid. Then all strongly (P)-cyclic right S-acts are strongly flat if and only if for every $z \in T$, zS is a strongly flat minimal right ideal of S.*

Proof. Suppose all strongly (P)-cyclic right S-acts are strongly flat and let $z \in T$. Then by Theorem 2.6, zS is a minimal right ideal of S. Since T is strongly (P)-cyclic and zS is a subact of T, by (4) of Theorem 2.2, zS is also strongly (P)-cyclic and so by assumption it is strongly flat.

Conversely, suppose that A is a strongly (P)-cyclic right S-act. Since zS is a minimal right ideal of S for $z \in T$, zS is simple and so for every $a \in A$, aS is simple, as by definition $aS \cong zS$, for some $z \in T$. Thus by Lemma 2.3 for every $a, a' \in A$ either $aS \cap a'S = \emptyset$ or $aS = a'S$. Hence there exists $A' \subseteq A$ such that $A = \dot{\bigcup\limits_{a \in A'}} aS$. On the other hand, aS is strongly flat for every $a \in A'$, as $aS \cong zS$ and by assumption zS is strongly flat. Thus by ([3, III, 9.3]), A is strongly flat as required. □

Theorem 2.8. *Let S be a monoid. Then all strongly (P)-cyclic right S-acts are projective if and only if for every $z \in T$, zS is a projective minimal right ideal of S.*

Proof. Suppose that all strongly (P)-cyclic right S-acts are projective and let $z \in T$. Then by Theorem 2.6, zS is a minimal right ideal of S. Since T is strongly (P)-cyclic and zS is a subact of T, by (4) of Theorem 2.2, zS is also strongly (P)-cyclic and so by assumption it is projective.

Conversely, suppose that A is a strongly (P)-cyclic right S-act. Then by definition for every $a \in A$ there exists $z \in S$ such that $aS \cong zS$. Since by assumption zS is projective, by ([3, III, 17.16]), there exists $e \in E(S)$ such that $\ker \lambda_z = \ker \lambda_e$ and so $zS \cong eS$. Thus for every $a \in A$ there exists $e \in E(S)$ such that $aS \cong eS$. As we saw in the proof of Theorem 2.7, there exists a subset A' of A such that $A = \dot{\bigcup\limits_{a \in A'}} aS$. Thus by ([3, III, 17.8]), A is projective. □

Theorem 2.9. *Let S be a monoid. Then all cyclic strongly (P)-cyclic right S-acts are projective generators if and only if for every $z \in T$ there exists $e \in E(T)$ such that $\ker \lambda_z = \ker \lambda_e$ and $e\mathcal{J}1$.*

Proof. Suppose that all cyclic strongly (P)-cyclic right S-acts are projective generators and let $z \in T$. Then zS as a subact of T is strongly (P)-cyclic and so by assumption it is a projective generator. Thus by ([3, III, 18.8]) there exists $e \in E(S)$ such that $\ker \lambda_z = \ker \lambda_e$ and $e\mathcal{J}1$. Since zS is strongly (P)-cyclic and $zS \cong eS$, eS is strongly (P)-cyclic and so $e \in E(T)$.

Conversely, suppose that aS is a strongly (P)-cyclic right S-act. By definition there exists $z \in S$ such that $\ker \lambda_a = \ker \lambda_z$ and so $aS \cong zS$. Thus zS is strongly (P)-cyclic and so $z \in T$. Hence by assumption there exists $e \in E(T)$ such that $\ker \lambda_z = \ker \lambda_e$ and $e\mathcal{J}1$. Thus by ([3, III, 18.8]), zS and hence aS are projective generators. $\square$

Theorem 2.10. *For any monoid S the following statements are equivalent:*
(1) *All strongly (P)-cyclic right S-acts are free.*
(2) *All strongly (P)-cyclic finitely generated right S-acts are free.*
(3) *All strongly (P)-cyclic right S-acts are projective generators.*
(4) *S is a group.*

Proof. Implications (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (3) are obvious.

(2) $\Rightarrow$ (4). Suppose that A is a strongly (P)-cyclic finitely generated right S-act. Then for every $a \in A$, aS is strongly (P)-cyclic and so by assumption aS is free. Thus $aS \cong S$ and so for every $t \in S$ there exists $u \in S$ such that $tS \cong auS$. Since aS is strongly (P)-cyclic, auS is strongly (P)-cyclic and so tS is strongly (P)-cyclic. Thus by assumption tS is free and so tS satisfies Condition (P), that is, S is strongly (P)-cyclic. Now if there exists $t \in S$ such that $tS \neq S$, then by Lemma 2.1, $B_S = S\coprod^{tS} S$ is strongly (P)-cyclic. Since also $B_S = (1,x)S \dot{\cup} tS \dot{\cup} (1,y)S$ is generated by $(1,x)$ and $(1,y)$, by assumption B_S is free and so B_S satisfies Condition (P). Since $t = (1,x)t = (1,y)t$, there exist $b \in B_S$ and $u, v \in S$ such that $(1,x) = bu$, $(1,y) = bv$ and $ut = vt$. Now $(1,x) = bu$ implies that there exists $s \in S \setminus tS$ such that $b = (s,x)$, similarly, there exists $s' \in S \setminus tS$ such that $b = (s',y)$, which is a contradiction. Hence for every $t \in S$, $tS = S$ and so S is a group.

(3) $\Rightarrow$ (4). By assumption all strongly (P)-cyclic right S-acts satisfy Condition (P) and so by Theorem 2.6, for every $z \in T$, zS is a minimal right ideal of S. Also zS as a subact of T is strongly (P)-cyclic and so by assumption zS is a projective generator. Thus by ([3, II, 3.16]), there exists an epimorphism $f\colon\ zS \to S_S$ and so there exists $x \in S$ such that $f(zx) = 1$. Now we show that f is a monomorphism. To this end we suppose that $f(zl) = f(zk)$, where $l, k \in S$. Since zS is simple, we have $zxS = zS$ and so $zl = zxl'$ and $zk = zxk'$ for some $l', k' \in S$. Thus $f(zl) = f(zxl') = f(zk) = f(zxk')$ and hence $f(zx)l' = f(zx)k'$. But $f(zx) = 1$ and so $l' = k'$. Consequently, $zl = zk$, that is, f is one to one and so it is an isomorphism. Thus $zS \cong S$ and so S is simple, as zS is simple. Thus S is a group.

(4) $\Rightarrow$ (1). Suppose that A is a strongly (P)-cyclic right S-act. Then by assumption for every $a \in A$ there exists $g \in S$ such that $\ker \lambda_a = \ker \lambda_g$. On the other hand $\ker \lambda_g = \ker \lambda_1$, since S is a group. Thus $\ker \lambda_a = \ker \lambda_1$ and so $aS \cong S$, that is, every cyclic subact of A is free. Now we suppose $a, a' \in A$ and $aS \cap a'S \neq \emptyset$.

Then there exist $t, t' \in S$ such that $at = a't'$. Since S is a group, $a = a't't^{-1}$ and so $aS \subseteq a'S$. Similarly, $a'S \subseteq aS$ and so $aS = a'S$. Thus there exists $A' \subseteq A$ such that $A = \dot{\bigcup\limits_{a \in A'}} aS$ and $aS \cong S$ for every $a \in A'$. Hence by ([3, I, 5.13]), A is free as required. □

Theorem 2.11. *For any monoid S the following statements are equivalent:*

(1) *There exists a cyclic strongly (P)-cyclic right S-act and all cyclic strongly (P)-cyclic right S-acts are free.*

(2) *All principal right ideals of S are free.*

(3) *For every $z \in S$ there exists $e \in E(S)$ such that $\ker \lambda_z = \ker \lambda_e$ and $e\mathcal{D}1$.*

Proof. (1) ⇒ (2). Suppose that aS is a cyclic strongly (P)-cyclic right S-act. By assumption aS is free and so $aS \cong S$. Thus for every $t \in S$ there exists $u \in S$ such that $tS \cong auS$, since every cyclic subact of aS is isomorphic to a cyclic subact of S. Since aS is strongly (P)-cyclic, auS is strongly (P)-cyclic. Thus by assumption auS is free and since $tS \cong auS$, we conclude that tS is also free.

(2) ⇒ (3). By ([3, I, 5.20]), it is obvious.

(3) ⇒ (1). By assumption and ([3, I, 5.20]), all principal right ideals of S are free and so all principal right ideals satisfy Condition (P). Thus S_S is a cyclic strongly (P)-cyclic right S-act and so there exists a cyclic strongly (P)-cyclic right S-act. Now we suppose aS is strongly (P)-cyclic. Then by definition there exists $z \in S$ such that $\ker \lambda_a = \ker \lambda_z$ and so $aS \cong zS$. On the other hand, by assumption there exists $e \in E(S)$ such that $e\mathcal{D}1$ and $\ker \lambda_z = \ker \lambda_e$. Thus $zS \cong eS$ and also by ([3, III, 17.17]), eS is free. Since $aS \cong zS$, then aS is free as required. □

Theorem 2.12. *For any monoid S the following statements are equivalent:*

(1) *All strongly (P)-cyclic right S-acts are divisible.*

(2) *All strongly (P)-cyclic finitely generated right S-acts are divisible.*

(3) *All cyclic strongly (P)-cyclic right S-acts are divisible.*

(4) *For every $z \in T$, zS is a divisible right ideal of S.*

Proof. Implications (1) ⇒ (2) ⇒ (3) are obvious.

(3) ⇒ (4). Let $z \in T$. Then zS as a subact of T is strongly (P)-cyclic and so by assumption it is divisible.

(4) ⇒ (1). Suppose that A is a strongly (P)-cyclic right S-act. Then by definition, for every $a \in A$ there exists $z \in S$ such that $\ker \lambda_a = \ker \lambda_z$ and so $aS \cong zS$. Since aS as a subact of A is strongly (P)-cyclic, zS is strongly (P)-cyclic and so $z \in T$. Thus by assumption zS is divisible and so aS is divisible, that is, for every left

cancellable element $c \in S$, $aSc = aS$. But

$$Ac = \left(\bigcup_{a \in A} aS\right)c = \bigcup_{a \in A} aSc = \bigcup_{a \in A} aS = A$$

and so A is divisible as required. □

Theorem 2.13. *For any monoid S the following statements are equivalent:*

(1) *All strongly (P)-cyclic right S-acts are principally weakly injective.*

(2) *All strongly (P)-cyclic finitely generated right S-acts are principally weakly injective.*

(3) *All cyclic strongly (P)-cyclic right S-acts are principally weakly injective.*

(4) *For every $z \in T$, zS is a principally weakly injective right ideal of S.*

Proof. Implications (1) ⇒ (2) ⇒ (3) are obvious.

(3) ⇒ (4). Suppose $z \in T$. Then zS as a subact of T is strongly (P)-cyclic and so by assumption it is principally weakly injective.

(4) ⇒ (1). Suppose that A is a strongly (P)-cyclic right S-act. Then by definition, for every $a \in A$ there exists $z \in S$ such that $\ker \lambda_a = \ker \lambda_z$ and so $aS \cong zS$. Since aS as a subact of A is strongly (P)-cyclic, zS is strongly (P)-cyclic and so $z \in T$. Thus by assumption zS is principally weakly injective and so aS is principally weakly injective, hence by ([3, III, 3.4]), $A = \bigcup_{a \in A} aS$ is principally weakly injective as required. □

Theorem 2.14. *Let S be a monoid. Then all strongly (P)-cyclic right S-acts are strongly faithful if and only if S is left cancellative.*

Proof. Suppose that A is a strongly (P)-cyclic right S-act and that for every $s, t, z \in S$, $zs = zt$. Let $a \in A$. Then $(az)s = (az)t$. Since A is strongly faithful, $s = t$ and so S is left cancellable.

Conversely, suppose that A is a strongly (P)-cyclic right S-act and that for $a \in A$, $s, t \in S$, $as = at$. By definition there exists $z \in S$ such that $\ker \lambda_a = \ker \lambda_z$. Then $as = at$ implies that $(s, t) \in \ker \lambda_a = \ker \lambda_z$ and so $zs = zt$. Since S is left cancellative, hence $s = t$ and so A is strongly faithful as required. □

Theorem 2.15. *Let S be a monoid. Then all strongly (P)-cyclic right S-acts are faithful if and only if for every $z \in T$, zS is a faithful right ideal of S.*

Proof. Let $z \in T$. Then zS as a subact of T is strongly (P)-cyclic and so by assumption it is faithful.

Conversely, suppose that A is a strongly (P)-cyclic right S-act and let $s, t \in S$, $s \neq t$, $a \in A$. Then there exists $z \in S$ such that $\ker \lambda_a = \ker \lambda_z$ and so $aS \cong zS$.

Since A is strongly (P)-cyclic, by (4) of Theorem 2.2, aS is also strongly (P)-cyclic. Thus zS is strongly (P)-cyclic and hence $z \in T$. But by assumption zS is faithful and so there exists $u \in S$ such that $zus \neq zut$. Since $\ker \lambda_a = \ker \lambda_z$, hence $(au)s \neq (au)t$ and so A is faithful as required. □

As we mentioned after Definition 2.1, every regular right act is strongly (P)-cyclic, but the converse is not true. Now it is natural to look for monoids over which strong (P)-cyclic property of acts implies regularity.

Theorem 2.16. *Let S be a monoid. Then all strongly (P)-cyclic right S-acts are regular if and only if for all $z \in T$ there exists $e \in E(T)$ such that $\ker \lambda_z = \ker \lambda_e$.*

Proof. Suppose that all strongly (P)-cyclic right S-acts are regular and let $z \in T$. Since T is strongly (P)-cyclic and zS is a subact of T, by (4) of Theorem 2.2, zS is also strongly (P)-cyclic and so by assumption zS is regular. Thus by ([3, III, 19.3]), zS is projective and so by ([3, III, 17.16]), z is left e-cancellable for some idempotent $e \in S$, that is, there exists $e \in E(S)$ such that $\ker \lambda_z = \ker \lambda_e$. Thus $zS \cong eS$ and so eS is strongly (P)-cyclic. Hence $eS \subseteq T$ and so $e \in E(T)$.

Conversely, suppose that A is a strongly (P)-cyclic right S-act and let $a \in A$. Then there exists $z \in S$ such that $\ker \lambda_z = \ker \lambda_a$ and so $aS \cong zS$. But by (4) of Theorem 2.2, aS is strongly (P)-cyclic and so zS is also strongly (P)-cyclic. Thus $zS \subseteq T$. Since $z \in T$, by assumption there exists $e \in E(T)$ such that $\ker \lambda_z = \ker \lambda_e$. But by ([3, III, 17.16]), zS is projective and so aS is also projective. Thus by ([3, III, 19.3]), A is regular. □

3. Classification by strong (P)-cyclic property of right Rees factor acts

In this section we give a classification of monoids such that flatness properties of Rees factor acts imply strong (P)-cyclic property and vice versa.

Theorem 3.1. *Let S be a monoid and K_S a right ideal of S. Then S/K_S is strongly (P)-cyclic if and only if $|K_S| = 1$ and S is right PCP, or $K_S = S$ and S contains a left zero.*

Proof. Suppose that S/K_S is strongly (P)-cyclic for the right ideal K_S of S. Then there are two cases as follows:

Case 1. $K_S = S$. Then $S/K_S \cong \Theta_S$ is strongly (P)-cyclic and so by (1) of Theorem 2.2, S contains a left zero element.

C a s e 2. K_S is a proper right ideal of S. Since by assumption S/K_S is strongly (P)-cyclic, S/K_S satisfies Condition (P). Thus by ([3, III, 13.9]), $|K_S| = 1$ and so $S/K_S \cong S_S$. Since S/K_S is strongly (P)-cyclic, S_S is strongly (P)-cyclic and so by (2) of Theorem 2.2, S is right PCP as required.

Conversely, suppose $|K_S| = 1$ and S is right PCP. Then $S/K_S \cong S_S$ and so by (2) of Theorem 2.2, S/K_S is strongly (P)-cyclic.

If $K_S = S$ and S contains a left zero, then $S/K_S \cong \Theta_S$ and so by (1) of Theorem 2.2, S/K_S is strongly (P)-cyclic. □

Theorem 3.2. *Let S be a monoid and let U be a property of S-acts implied by freeness. Then the following statements are equivalent:*

(1) *All right Rees factor S-acts having property U are strongly (P)-cyclic.*

(2) *All right Rees factor S-acts having property U satisfy Condition (P) and if S contains a left zero, then S is right PCP, and if Θ_S has property U, then S contains a left zero.*

P r o o f. (1) ⇒ (2). If all right Rees factor S-acts having property U are strongly (P)-cyclic, then all right Rees factor S-acts having property U satisfy Condition (P).

Now suppose that S contains a left zero element z. If $K_S = zS = \{z\}$, then $S/K_S \cong S_S$ and so S/K_S is free, since S_S is free. Thus S/K_S has property U and so by assumption S/K_S is strongly (P)-cyclic. Thus S_S is strongly (P)-cyclic and so by (2) of Theorem 2.2, S is right PCP.

If $\Theta_S \cong S/S_S$ has property U, then by assumption Θ_S is strongly (P)-cyclic and so by (1) of Theorem 2.2, S contains a left zero element.

(2) ⇒ (1). Suppose that S/K_S has property U for the right ideal K_S of S. Then there are two cases as follows:

C a s e 1. $K_S = S$. Then $S/K_S \cong \Theta_S$ and so by assumption S contains a left zero. Thus by (1) of Theorem 2.2, S/K_S is strongly (P)-cyclic.

C a s e 2. K_S is a proper right ideal of S. Since by assumption S/K_S satisfies Condition (P), by ([3, III, 13.9]), $|K_S| = 1$ and so $K_S = zS = \{z\}$ for some $z \in S$. Thus z is a left zero and so by assumption S is right PCP. Hence by (2) of Theorem 2.2, $S/K_S \cong S_S$ is strongly (P)-cyclic. □

Corollary 3.1. *For any monoid S the following statements are equivalent:*

(1) *All projective right Rees factor S-acts are strongly (P)-cyclic.*

(2) *All projective generators right Rees factor S-acts are strongly (P)-cyclic.*

(3) *All free right Rees factor S-acts are strongly (P)-cyclic.*

(4) *S has no left zero, or S is right PCP.*

P r o o f. Implications (1) ⇒ (2) ⇒ (3) are obvious.

(3) $\Rightarrow$ (4). Suppose that S contains a left zero element. Then by Theorem 3.2, S is right PCP.

(4) $\Rightarrow$ (1). By Theorem 3.2, it suffices to show that if Θ_S is projective, then S contains a left zero and this is true by ([3, III, 17.2]). □

Corollary 3.2. *For any monoid S the following statements are equivalent:*

(1) *All strongly flat right Rees factor S-acts are strongly (P)-cyclic.*

(2) *S is not left collapsible or S contains a left zero and S is right PCP.*

P r o o f. (1) $\Rightarrow$ (2). If S is left collapsible, then by ([3, III, 14.3]), Θ_S satisfies Condition (E) and so it is strongly flat. Thus by (2) of Theorem 3.2, S contains a left zero and also S is right PCP.

The converse is true by Theorem 3.2 and ([3, III, 14.3]). □

Corollary 3.3. *For any monoid S the following statements are equivalent:*

(1) *All right Rees factor S-acts satisfying Condition (P) are strongly (P)-cyclic.*

(2) *S is not right reversible or S is right PCP and contains a left zero.*

P r o o f. (1) $\Rightarrow$ (2). If S is right reversible, then by ([3, III, 13.7]), Θ_S satisfies Condition (P) and so by (2) of Theorem 3.2, S is right PCP and contains a left zero.

The converse is true by Theorem 3.2 and ([3, III, 13.7]). □

Corollary 3.4. *For any monoid S the following statements are equivalent:*

(1) *All right Rees factor S-acts satisfying Condition (WP) are strongly (P)-cyclic.*

(2) *S is not right reversible or S is right PCP, contains a left zero and no nontrivial right ideal of S is left stabilizing and strongly left annihilating.*

P r o o f. (1) $\Rightarrow$ (2). If S is right reversible, then by ([4, Theorem 2.14]), Θ_S satisfies Condition (WP) and so by Theorem 3.2, S contains a left zero. Again by Theorem 3.2, S is right PCP. On the other hand, by Theorem 3.2, all right Rees factor S-acts satisfying Condition (WP) satisfy Condition (P) and so by ([4, Proposition 3.26]), no nontrivial right ideal of S is left stabilizing and strongly left annihilating.

The converse is true by Theorem 3.2 and ([4, Proposition 3.26]). □

Corollary 3.5. *For any monoid S the following statements are equivalent:*

(1) *All right Rees factor S-acts satisfying Condition (PWP) are strongly (P)-cyclic.*

(2) *S is right PCP, contains a left zero and no nontrivial right ideal of S is left stabilizing and left annihilating.*

P r o o f. (1) $\Rightarrow$ (2). Since Θ_S satisfies Condition (PWP), then by Theorem 3.2, S contains a left zero. Again by Theorem 3.2, all principal right ideals of S satisfy Condition (P). On the other hand, by Theorem 3.2, all right Rees factor S-acts satisfying Condition (PWP) satisfy Condition (P) and so by ([4, Corollary 3.27]), no nontrivial right ideal of S is left stabilizing and left annihilating.

The converse is true by Theorem 3.2 and ([4, Corollary 3.27]). □

Corollary 3.6. *For any monoid S the following statements are equivalent:*

(1) *All flat right Rees factor S-acts are strongly (P)-cyclic.*

(2) *S is not right reversible or S is right PCP, contains a left zero and no proper right ideal K_S of S with $|K_S| \geqslant 2$ is left stabilizing.*

P r o o f. (1) $\Rightarrow$ (2). If S is right reversible, then by ([3, III, 12.2]), Θ_S is flat and so by Theorem 3.2, S contains a left zero. Again by Theorem 3.2, S is right PCP. On the other hand, by Theorem 3.2, all flat right Rees factor S-acts satisfy Condition (P) and so by ([3, IV, 9.2]), no proper right ideal K_S of S with $|K_S| \geqslant 2$ is left stabilizing.

The converse is true by Theorem 3.2 and ([3, IV, 9.2]). □

Note that Corollary 3.6 is also true if we substitute WF for flat, since by ([3, III, 12.17]), for Rees factor acts flatness and weak flatness coincide.

Corollary 3.7. *For any monoid S the following statements are equivalent:*

(1) *All PWF right Rees factor S-acts are strongly (P)-cyclic.*

(2) *S is right PCP, contains a left zero, and no proper right ideal K_S of S with $|K_S| \geqslant 2$ is left stabilizing.*

P r o o f. (1) $\Rightarrow$ (2). Since Θ_S is principally weakly flat, by Theorem 3.2, S contains a left zero. Again by Theorem 3.2, S is right PCP. On the other hand, by Theorem 3.2, all PWF right Rees factor S-acts satisfy Condition (P) and so by ([3, IV, 9.7]), no proper right ideal K_S of S with $|K_S| \geqslant 2$ is left stabilizing.

The converse is true by Theorem 3.2 and ([3, IV, 9.7]). □

Corollary 3.8. *For any monoid S the following statements are equivalent:*

(1) *All torsion free right Rees factor S-acts are strongly (P)-cyclic.*

(2) *S is right PCP, contains a left zero and S is either right cancellative, or right cancellative with a zero adjoined.*

Proof. (1) $\Rightarrow$ (2). Since Θ_S is torsion free, hence by Theorem 3.2, S is right PCP and contains a left zero. Also by Theorem 3.2, all torsion free right Rees factor S-acts satisfy Condition (P). Since S contains a left zero, S is right reversible and so by ([3, IV, 9.8]), S is right cancellative or right cancellative with a zero adjoined.

The converse is true by Theorem 3.2 and ([3, IV, 9.8]). □

Theorem 3.3. *Let S be a monoid and let U be a property of S-acts implied by freeness. Then all strongly (P)-cyclic right Rees factor S-acts have property U if and only if S has no left zero or Θ_S has property U.*

Proof. Suppose that S contains a left zero. Then by (1) of Theorem 2.2, $\Theta_S \cong S/S_S$ is strongly (P)-cyclic and so by assumption Θ_S has property U.

Conversely, Suppose that S/K_S is strongly (P)-cyclic for the right ideal K_S of S. Then there are two cases as follows:

Case 1. $K_S = S$. Then $S/K_S \cong \Theta_S$ and so by (1) of Theorem 2.2, S contains a left zero. Hence by assumption $S/K_S \cong \Theta_S$ has property U.

Case 2. K_S is a proper right ideal of S. Since by assumption S/K_S satisfies Condition (P), by ([3, III, 13.9]) we have $|K_S| = 1$. Thus $S/K_S \cong S_S$ has property U, since S_S is free. □

Corollary 3.9. *Let S be a monoid. Then all strongly (P)-cyclic right Rees factor S-acts are free if and only if S has no left zero or $S = \{1\}$.*

Proof. It follows from Theorem 3.3 and ([3, I, 5.23]). □

Corollary 3.10. *Let S be a monoid. Then all strongly (P)-cyclic right Rees factor S-acts are projective.*

Proof. It follows from Theorem 3.3 and ([3, III, 17.2]). □

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