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Czechoslovak Mathematical Journal, Vol. 59 (2009), No. 3, 649–654

Persistent URL: <http://dml.cz/dmlcz/140507>

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BOUNDEDLY EXPRESSIBLE SETS

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(Received January 11, 2008)

Abstract. For a given sequence a boundedly expressible set is introduced. Three criteria concerning the Hausdorff dimension of such sets are proved.

Keywords: expressible set, Hausdorff dimension.

MSC 2010: 11K55

1. INTRODUCTION

Following Erdős [2] we say that for a given sequence $\{a_n\}_{n=1}^{\infty}$ the set

$$X_B\{a_n\}_{n=1}^{\infty} := \left\{ x \in \mathbb{R} : \exists K \in \mathbb{N} \exists \{c_n\}_{n=1}^{\infty}, c_n \in \{1, \dots, K\}, x = \sum_{n=1}^{\infty} \frac{1}{a_n c_n} \right\}$$

is its *boundedly expressible set*. In [2] it is shown that if $\lim_{n \rightarrow \infty} a_n^{1/2^n} = \infty$ and $a_n \in \mathbb{N}$ for all $n \in \mathbb{N}$ then $X_B\{a_n\}_{n=1}^{\infty}$ does not contain any rational number. It appears to be the case that in general evaluating the Lebesgue measure or Hausdorff dimension of the set $X_B\{a_n\}_{n=1}^{\infty}$ is not easy. In this paper we give conditions on $\{a_n\}_{n=1}^{\infty}$ to ensure that the Hausdorff dimension of the set $X_B\{a_n\}_{n=1}^{\infty}$ is zero. We prove the following.

Theorem 1. *Let $\{b_n\}_{n=1}^{\infty}$ be a non-decreasing sequence of positive integers such that $b_n = O(2^{2^{n/2}})$. Then $\dim X_B\left\{b_n + 2^{2^{\lfloor \log_2 n \rfloor}}\right\}_{n=1}^{\infty} = 0$.*

It is unknown to the authors if there exists a sequence of real numbers such that its boundedly expressible set has Hausdorff dimension greater than zero and Lebesgue

The research was supported by the grant GAČR 201/07/0191 and by the Research Projects ME09017 and MSM6198898701 of the Czech Government.

measure zero. For sequences which converge to infinity slower than $\{2^{2^n}\}_{n=1}^\infty$ Hančl in [3] proved the following theorem.

Theorem 2. *Let $\{a_n\}_{n=1}^\infty$ be a sequence of positive real numbers and let K be a positive integer. Assume that $F: \mathbb{N} \rightarrow \mathbb{R}^+$ is a function such that $F(n) < n$,*

$$a_n < K2^{2^{n-F(n)}} \quad \text{and} \quad \sum_{n=1}^{\infty} 2^{-F(n)} < \infty.$$

Put

$$B := \frac{1}{8K[a_1 + 1] \left[2^4 \sum_{n=2}^{\infty} 2^{-F(n)} + 2 \right]}.$$

Then for every number $x \in (0, B]$ there is a sequence $\{c_n\}_{n=1}^\infty \subseteq \mathbb{N}$ such that $x = \sum_{n=1}^{\infty} 1/a_n c_n$.

2. MAIN RESULTS

We start with the theorem which is the basis for the other results.

Theorem 3. *Let ε be a positive real number. Assume that $\{a_n\}_{n=1}^\infty$ is a non-decreasing sequence of positive integers such that*

$$(1) \quad L := \liminf_{n \rightarrow \infty} \frac{2(1 + \varepsilon) \log T_n}{\varepsilon \log a_n} \in [0, 1),$$

where $T_n = \text{lcm}(a_1, \dots, a_{n-1})$, and that for every sufficiently large n

$$(2) \quad a_n \geq n^{1+\varepsilon}.$$

Then $\dim X_B \{a_n\}_{n=1}^\infty \leq L$.

Theorem 1 follows immediately from the following more general result.

Theorem 4. Let ε be a positive real number. Assume that $\{a_n\}_{n=1}^\infty$ is a non-decreasing sequence of positive integers such that

$$(3) \quad \liminf_{n \rightarrow \infty} \frac{n \log a_{n-1}}{\log a_n} = 0$$

and that for every sufficiently large n

$$(4) \quad a_n \geq n^{1+\varepsilon}.$$

Then $\dim X_B\{a_n\}_{n=1}^\infty = 0$.

Example 1. Set $A := 1.1$ and $a_n := \lceil 2^{A^{2^{\lfloor \log_2 n \rfloor}} + n} \rceil$. From $n < 1.82 \times 2^{A^n}$ we obtain for $K = 3$ and $F(n) = n \log_2 2/A$ that $a_n < K 2^{2^{n-F(n)}}$. Theorem 2 implies that for every $x \in (0, \frac{1}{768}]$ there is a sequence $\{c_n\}_{n=1}^\infty$ of positive integers such that $x = \sum_{n=1}^\infty 1/a_n c_n$. Theorem 4 implies that for almost every x such sequence $\{c_n\}_{n=1}^\infty$ must be unbounded.

In the case that the sequence $\{a_n\}_{n=1}^\infty$ is of the Cantor type we can use the following criterion.

Theorem 5. Let A, B, S and ε be a positive real numbers such that $S > 1$ and $(2/S)(A/B)((1 + \varepsilon)/\varepsilon) < 1$. Let $\{a_n\}_{n=1}^\infty$ be a sequence of positive integers such that a_n divides a_{n+1} for every $n \in \mathbb{N}$. Suppose that for every sufficiently large n

$$(5) \quad a_n \geq n^{1+\varepsilon}.$$

Assume that for infinitely many N

$$(6) \quad a_{N-1} \leq 2^{AS^{N-1}} \quad \text{and} \quad a_N \geq 2^{BS^N}.$$

Then $\dim X_B\{a_n\}_{n=1}^\infty \leq (2/S)(A/B)((1 + \varepsilon)/\varepsilon)$.

Let us note that Theorem 5 also holds in the case when $A > B$. The authors do not know how to find non-trivial lower bounds for X_B .

Corollary 1. Let A, B and S be positive real numbers such that $S > 1$ and $(2/S)(A/B) < 1$. Let $\{a_n\}_{n=1}^\infty$ be a sequence of positive integers such that a_n divides a_{n+1} for every n . Assume that $a_n \geq 2^n$ for every sufficiently large n . Suppose that $a_{N-1} \leq 2^{AS^{N-1}}$ and $a_N \geq 2^{BS^N}$ for infinitely many N . Then $\dim X_B\{a_n\}_{n=1}^\infty \leq (2/S)(A/B)$.

Example 2. Let $S > 2$. As an immediate consequence of Corollary 1 we obtain that $\dim X_B\{2^{[(4+(-1)^n)S^n]}\}_{n=1}^\infty \leq \frac{6}{5}/S$.

Corollary 2. Let $A > 0$ and $S > 2$. Let $\{a_n\}_{n=1}^\infty$ be a sequence of positive integers such that a_n divides a_{n+1} for every $n \in \mathbb{N}$. Assume that $a_k \geq 2^k$ for every sufficiently large k . Suppose that $a_N \leq 2^{AS^N}$ for infinitely many N and $a_M \geq 2^{AS^M}$ for infinitely many M . Then $\dim X_B\{a_n\}_{n=1}^\infty \leq 2/S$.

Corollary 3. Let $0 < B < A$ and $S > 2B/A$. Let $\{a_n\}_{n=1}^\infty$ be a sequence of positive integers such that a_n divides a_{n+1} for every n and $2^{BS^n} \leq a_n \leq 2^{AS^n}$ for every sufficiently large n . Then $\dim X_B\{a_n\}_{n=1}^\infty \leq (2/S)(A/B)$.

Example 3. Let $S > 2$. Corollary 3 implies that $\dim X_B\{2^{\lfloor S^n \rfloor}\}_{n=1}^\infty \leq 2/S$.

3. PROOFS

We need the following classical Jarník-Besicovitch Theorem which can be found for example in [1]. See also [4].

Theorem 6. Let $\alpha > 2$. Then the Hausdorff dimension of the set of all positive real numbers x such that for infinitely many pairs $(p, q) \in \mathbb{N}$

$$\left| x - \frac{p}{q} \right| < \frac{1}{q^\alpha}$$

is equal to $2/\alpha$. In other words,

$$\dim \bigcap_{N=1}^\infty \bigcup_{q=N}^\infty \bigcup_{p=1}^\infty \left(\frac{p}{q} - \frac{1}{q^\alpha}, \frac{p}{q} + \frac{1}{q^\alpha} \right) = \frac{2}{\alpha}.$$

Proof of Theorem 3. Let $\delta > 0$ be a sufficiently small real number. Let $K \in \mathbb{N}$ and let $\{c_n\}_{n=1}^\infty$ be a sequence of integers such that $c_n \in \{1, \dots, K\}$ for each $n \in \mathbb{N}$. Let N be a sufficiently large integer. Then from (2) and from the fact that the sequence $\{a_n\}_{n=1}^\infty$ is non-decreasing we obtain that

$$\begin{aligned} (7) \quad \sum_{n=N}^\infty \frac{1}{a_n c_n} &\leq \sum_{n=N}^\infty \frac{1}{a_n} \leq \sum_{n \leq a_N^{1/(1+\varepsilon)}} \frac{1}{a_n} + \sum_{n \geq a_N^{1/(1+\varepsilon)}} \frac{1}{a_n} \\ &\leq \frac{a_N^{1/(1+\varepsilon)}}{a_N} + \sum_{n \geq a_N^{1/(1+\varepsilon)}} \frac{1}{n^{1+\varepsilon}} \leq \frac{1}{a_N^{(1-\delta)\varepsilon/(1+\varepsilon)}}. \end{aligned}$$

Set $S_K := \text{lcm}(1, \dots, K)$,

$$q_N := \text{lcm}(a_1 c_1, \dots, a_{N-1} c_{N-1}) \quad \text{and} \quad p_N := q_N \sum_{n=1}^{N-1} \frac{1}{a_n c_n}.$$

We have

$$q_N \leq S_K T_N \leq T_N^{1+\delta}.$$

This, (1) and (7) imply that for infinitely many N

$$q_N^{2(1-\delta)/(1+\delta)(L+\delta)} \sum_{n=N}^{\infty} \frac{1}{a_n c_n} \leq \frac{T_N^{2(1-\delta)/(L+\delta)}}{a_N^{(1-\delta)\varepsilon/(1+\varepsilon)}} = \left(\frac{T_N^{2(1+\varepsilon)/(L+\delta)\varepsilon}}{a_N} \right)^{(1-\delta)\varepsilon/(1+\varepsilon)} < 1.$$

Hence

$$0 < \sum_{n=1}^{\infty} \frac{1}{a_n c_n} - \frac{p_N}{q_N} = \sum_{n=N}^{\infty} \frac{1}{a_n c_n} < \frac{1}{q_N^{2(1-\delta)/(1+\delta)(L+\delta)}}$$

for infinitely many N .

Set

$$Y_K := \left\{ x \in \mathbb{R} : \exists \{c_n\}_{n=1}^{\infty} \subseteq \{1, \dots, K\} \text{ s.t. } x = \sum_{n=1}^{\infty} \frac{1}{a_n c_n} \right\}.$$

Then

$$X_B\{a_n\}_{n=1}^{\infty} = \bigcup_{K=1}^{\infty} Y_K \subseteq \bigcap_{N=1}^{\infty} \bigcup_{q=N}^{\infty} \bigcup_{p=1}^{\infty} \left(\frac{p}{q}, \frac{p}{q} + \frac{1}{q_N^{2(1-\delta)/(1+\delta)(L+\delta)}} \right).$$

The Jarník-Besicovitch Theorem implies that

$$\dim X_B\{a_n\}_{n=1}^{\infty} \leq \frac{(1+\delta)(L+\delta)}{1-\delta}.$$

This holds for every small $\delta > 0$. So the result follows. \square

P r o o f of Theorem 4. Use Theorem 3 with $L = 0$ and the fact that $T_n \leq a_{n-1}^{-1}$. \square

P r o o f of Theorem 5. For sequences of the Cantor type we have $T_n = a_{n-1}$. Again use Theorem 3. \square

P r o o f of Theorem 1. Set $n = 2^m$ where $m \in \mathbb{N}$ and m is sufficiently large. Then $a_{n-1} = O(2^{2^{n/2}})$ and $a_n = 2^{2^n}$. So (3) follows. Condition (4) is clear. Now we can apply Theorem 4. \square

P r o o f of Corollary 1. For every sufficiently large $\varepsilon > 0$ we have $(2/S)(A/B) ((1+\varepsilon)/\varepsilon) < 1$. The inequality (5) is obviously fulfilled. Theorem 5 implies that

$$\dim X_B\{a_n\}_{n=1}^{\infty} \leq \lim_{\varepsilon \rightarrow \infty} \frac{2}{S} \frac{A}{B} \frac{1+\varepsilon}{\varepsilon} = \frac{2}{S} \frac{A}{B}.$$

\square

Proof of Corollary 2. Set $B := A$ and use Corollary 1. □

Proof of Corollary 3. This is an immediate consequence of Corollary 1. □

Acknowledgement. The authors thank the referee for his valuable suggestions.

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