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EXPONENTS OF TWO-COLORED DIGRAPHS

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Abstract. We consider the primitive two-colored digraphs whose uncolored digraph has $n+s$ vertices and consists of one n-cycle and one $(n-3)$-cycle. We give bounds on the exponents and characterizations of extremal two-colored digraphs.

Keywords: exponent, digraph, primitivity

MSC 2010: 15A18, 15A48

1. Introduction

A *two-colored digraph* is a digraph whose arcs are colored red or blue. We allow loops and both a red arc and a blue arc from i to j. Let D be a two-colored digraph. D is *strongly connected* if for each pair (i,j) of vertices there is a walk in D from i to j. Given a walk w in D, let $r(w)$ and $b(w)$, denote the number of red and blue arcs, respectively, of w. We call w an $(r(w),b(w))$-*walk*, and define the *composition* of w to be the vector $(r(w),b(w))$ or

$$\begin{bmatrix} r(w) \\ b(w) \end{bmatrix}.$$

A two-colored digraph D is primitive if there exist nonnegative integers h and k with $h+k>0$ such that for each pair (i,j) of vertices there exists an (h,k)-walk in D from i to j. The exponent $\exp(D)$ is the minimum value of $h+k$ taken over all pairs (h,k) such that for each pair (i,j) of vertices there exists an (h,k)-walk from i to j ([2]).

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Let D be a two-colored digraph and let $C = \{\gamma_1, \gamma_2, \ldots, \gamma_l\}$ be the set of cycles of D. Set M to be the $2 \times l$ matrix whose ith column is the composition of γ_i, $i = 1, 2, \ldots, l$. We call M the *cycle matrix* of D. The *content* of M, denoted content(M), is defined to be 0 if the rank of M is less than 2, and the greatest common divisor of the determinants of the 2×2 submatrices of M, otherwise.

There is a natural correspondence between two-colored digraphs and nonnegative matrix pairs ([2]). The concept of the exponent of a nonnegative matrix pair arises naturally in the study of finite Markov chains, and some results have already been obtained ([1], [2], [3], [4], [5]).

Lemma 1.1 ([2]). *Let D be a two-colored digraph. Then D is primitive if and only if D is strongly connected and* content$(M) = 1$.

We consider the two-colored digraphs that have at least one red arc and one blue arc, and whose uncolored digraph is the digraph as given in Fig. 1, where $s \geqslant 0$, $m \geqslant s+1$ and $n \geqslant m+1$.

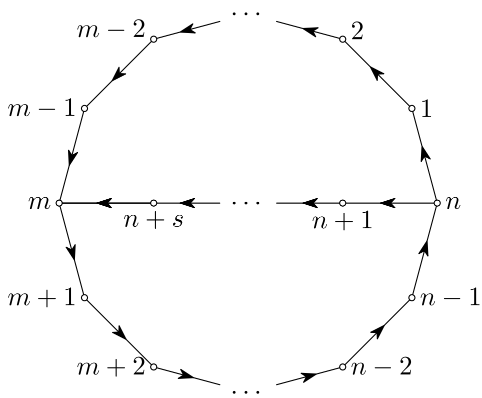

Fig. 1. Digraph D

Clearly, D has only two cycles. One is an n-cycle and the other is an $(n-m+s+1)$-cycle. Without loss of generality we may assume that the cycle matrix of D is

$$M = \begin{bmatrix} a & b \\ n-a & n-m+s+1-b \end{bmatrix}$$

for some integers a and b with $n/2 \leqslant a \leqslant n$.

Theorem 1.2 ([4]). *Let D be a two-colored digraph as given in Fig.* 1 *and let $m = s+1+t$. Then D is primitive if and only if $t \geqslant 1$, $(at+1)/n$ or $(at-1)/n$ is integer, and $b = a - (at+1)/n$ or $b = a - (at-1)/n$.*

Theorem 1.3. *Let D be a two-colored digraph as given in Fig.* 1 *and let $m = s + 1 + t$. If D is primitive, then* $\gcd\{t, n\} = 1$.

P r o o f. Note that

$$|M| = \begin{vmatrix} a & b \\ n-a & n-t-b \end{vmatrix} = \begin{vmatrix} a & b \\ n & n-t \end{vmatrix} = \begin{vmatrix} a & b-a \\ n & -t \end{vmatrix}.$$

Since $|M| = \pm 1$, we have $\gcd\{t, n\} = 1$. □

Theorem 1.4. *Let D be a two-colored digraph as given in Fig.* 1 *and let $m = s + 1 + t$. Then D is primitive if and only if $|a(n-t) - bn| = 1$.*

P r o o f. Since $|M| = a(n-t) - bn$, the theorem follows from Lemma 1.1. □

Theorem 1.5. *Let D be a two-colored digraph as given in Fig.* 1 *and let $m = s+4$. Then D is primitive if and only if*
(1) $n = 3q + 1$, $a = 2q + 1$, *and* $b = 2q - 1$; *or*
(2) $n = 3q + 2$, $a = 2q + 1$, *and* $b = 2q - 1$.

P r o o f. By Theorem 1.3 we have $3 \nmid n$. So let $n = 3q + 1$ or $n = 3q + 2$, where $q \geqslant 2$.

When $n = 3q + 1$, then by Theorem 1.2, $(3a + 1)/(3q + 1)$ or $(3a - 1)/(3q + 1)$ is integer. Noting that $n/2 \leqslant a \leqslant n$, we have $a = 2q + 1$ and $b = 2q - 1$. So the cycle matrix of D is

$$M = \begin{bmatrix} 2q+1 & 2q-1 \\ q & q-1 \end{bmatrix}.$$

When $n = 3q + 2$, then by Theorem 1.2, $(3a + 1)/(3q + 2)$ or $(3a - 1)/(3q + 2)$ is integer. Noting that $n/2 \leqslant a \leqslant n$, we have $a = 2q + 1$ and $b = 2q - 1$. So the cycle matrix of D is

$$M = \begin{bmatrix} 2q+1 & 2q-1 \\ q+1 & q \end{bmatrix}.$$

The theorem follows. □

Let D be the two-colored digraph D as given in Fig. 1. In [4], we considered D with $m = s + 2$ and gave the set of exponents of families of D. In [5], we considered D with $m = s + 3$ and gave the bounds on the exponents and characterizations of extremal two-colored digraphs. In this paper we consider D with $m = s + 4$ (that is $t = 3$), $n \geqslant 9$, give bounds on the exponents and characterizations of extremal two-colored digraphs. Throughout the rest of the paper, we let $D_{n,s}$ denote the collection of primitive two-colored digraphs that have at least one red arc and one blue arc, and whose uncolored digraph is the digraph as given in Fig. 1 with $m = s + 4$.

2. The case $n = 3q + 1$

Let $n = 3q + 1$, and let the cycle matrix of D be

$$M = \begin{bmatrix} 2q+1 & 2q-1 \\ q & q-1 \end{bmatrix},$$

where $q \geqslant 3$. Clearly,

$$M^{-1} = \begin{bmatrix} 1-q & 2q-1 \\ q & -2q-1 \end{bmatrix}.$$

Theorem 2.1. *Let $D \in D_{3q+1,s}$. Then*

$$18q^2 - 12q - 3 \leqslant \exp(D) \leqslant \begin{cases} 12q^3 - 2q^2 - 3q, & \text{if } s \leqslant q-3, \\ 12q^3 - 2q^2 + 1, & \text{if } s = q-2, \\ 6q^3 + 2(3s+7)q^2 - 2(2s+5)q - s - 2, & \text{if } s \geqslant q-1. \end{cases}$$

P r o o f. First, we show that

$$\exp(D) \geqslant 18q^2 - 12q - 3.$$

Suppose that (h,k) is a pair of nonnegative integers such that for all pairs (i,j) of vertices there is an (h,k)-walk from i to j. By considering $i = j = n$, we see that there exist nonnegative integers u and v with

$$\begin{bmatrix} h \\ k \end{bmatrix} = M \begin{bmatrix} u \\ v \end{bmatrix}.$$

Since there are $2q+1$ red arcs and q blue arcs on the n-cycle, there is a red path w of length 3 on the n-cycle. Taking i and j to be the initial vertex and terminal vertex of w, respectively, each walk from i to j can be decomposed into the path w and cycles. Hence,

$$Mz = \begin{bmatrix} h-3 \\ k \end{bmatrix}$$

has a nonnegative integer solution. Then

$$z = M^{-1}\begin{bmatrix} h-3 \\ k \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix} - M^{-1}\begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix} - \begin{bmatrix} 3-3q \\ 3q \end{bmatrix} \geqslant 0.$$

So $v \geqslant 3q$. Finally, take i and j to be the terminal and initial vertices of w, respectively. Then the path from i to j has composition either $(2q-2, q)$ or $(2q-4, q-1)$, so we have that

$$Mz = \begin{bmatrix} h-(2q-2) \\ k-q \end{bmatrix} \quad \text{or} \quad Mz = \begin{bmatrix} h-(2q-4) \\ k-(q-1) \end{bmatrix}$$

has a nonnegative integer solution. Then

$$z = \begin{bmatrix} u \\ v \end{bmatrix} - M^{-1} \begin{bmatrix} 2q-2 \\ q \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix} - \begin{bmatrix} 3q-2 \\ -3q \end{bmatrix} \geqslant 0,$$

or

$$z = \begin{bmatrix} u \\ v \end{bmatrix} - M^{-1} \begin{bmatrix} 2q-4 \\ q-1 \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix} - \begin{bmatrix} 3q-3 \\ -3q+1 \end{bmatrix} \geqslant 0.$$

So $u \geqslant 3q-3$. Thus

$$h+k = [\,1 \quad 1\,]\, M \begin{bmatrix} u \\ v \end{bmatrix} \geqslant [\,3q+1 \quad 3q-2\,] \begin{bmatrix} 3q-3 \\ 3q \end{bmatrix} = 18q^2 - 12q - 3,$$

and $\exp(D) \geqslant 18q^2 - 12q - 3$.

Now, we prove the upper bounds for $\exp(D)$. Let p_{ij} be the shortest path in D from vertex i to vertex j, $r = r(p_{ij})$, and $b = b(p_{ij})$.

First, we show that $\exp(D) \leqslant 12q^3 - 2q^2 - 3q$ when $s \leqslant q-3$.

Note that

$$(2.1) \qquad \begin{bmatrix} r \\ b \end{bmatrix} + ((q-1)r - (2q-1)b + 2q^2 - q) \begin{bmatrix} 2q+1 \\ q \end{bmatrix} + ((2q+1)b - qr + 2q^2 + q) \begin{bmatrix} 2q-1 \\ q-1 \end{bmatrix} = \begin{bmatrix} 8q^3 - 2q \\ 4q^3 - 2q^2 - q \end{bmatrix}.$$

Consider the following three cases.

Case 1. Both the vertices i and j are on the n-cycle.

Clearly, $b \leqslant q$ and $r \leqslant 2q+1$. Thus $(q-1)r - (2q-1)b + 2q^2 - q \geqslant (q-1)r - (2q-1)q + 2q^2 - q = (q-1)r \geqslant 0$ and $(2q+1)b - qr + 2q^2 + q \geqslant (2q+1)b - q(2q+1) + 2q^2 + q = (2q+1)b \geqslant 0$. If $(q-1)r - (2q-1)b + 2q^2 - q = 0$, then $b = q$, $r = 0$. Since $q \geqslant s+3$, so either i or j is on the $(n-3)$-cycle.

Case 2. Both the vertices i and j are on the $(n-3)$-cycle.

Clearly, $b \leqslant q-1$ and $r \leqslant 2q-1$. Thus $(q-1)r - (2q-1)b + 2q^2 - q \geqslant -(2q-1)(q-1) + 2q^2 - q = 2q-1 > 0$ and $(2q+1)b - qr + 2q^2 + q \geqslant -q(2q-1) + 2q^2 + q = 2q > 0$.

Case 3. The vertex i (or j) is on the path $1 \to 2 \to \ldots \to s+3$, and the vertex j (or i) is on the path $n+1 \to \ldots \to n+s$.

Clearly, the path p_{ij} contains the path $s+4 \to s+5 \to \ldots \to n$. Let the number of red arcs and blue arcs in the path $s+4 \to s+5 \to \ldots \to n$ be x and y, respectively. Then $x + y = 3q - s - 3$, and the number of red arcs and blue arcs in D is $4q - x = q + s + y + 3$ and $2q - y - 1$, respectively. Since $s \leqslant q-3$, we see that

$$2q - y \leqslant 3q - s - y - 3 \leqslant r \leqslant q + s + y + 3 \leqslant 2q + y,$$
$$y \leqslant b \leqslant 2q - 1 - y.$$

Thus $(q-1)r-(2q-1)b+2q^2-q \geqslant (q-1)(2q-y)-(2q-1)(2q-1-y)+2q^2-q = yq+q-1 > 0$, $(2q+1)b-qr+2q^2+q \geqslant (2q+1)y-q(2q+y)+2q^2+q = yq+y+q > 0$.

By virtue of (2.1), the walk that starts at vertex i, follows p_{ij} to vertex j, and along the way goes around the n-cycle $(q-1)r-(2q-1)b+2q^2-q$ times and around the $(n-3)$-cycle $(2q+1)b-qr+2q^2+q$ times is an $(8q^3-2q, 4q^3-2q^2-q)$-walk from i to j. So $\exp(D) \leqslant 12q^3-2q^2-3q$ when $s \leqslant q-3$.

Secondly, we show that $\exp(D) \leqslant 12q^3-2q^2+1$ when $s = q-2$.

Note that

$$\begin{bmatrix} r \\ b \end{bmatrix} + ((q-1)r-(2q-1)b+2q^2-q+1)\begin{bmatrix} 2q+1 \\ q \end{bmatrix} + ((2q+1)b-qr+2q^2+q)\begin{bmatrix} 2q-1 \\ q-1 \end{bmatrix} = \begin{bmatrix} 8q^3+1 \\ 4q^3-2q^2 \end{bmatrix}. \tag{2.2}$$

Similarly to the above, we can show that the walk that starts at vertex i, follows p_{ij} to vertex j, and along the way goes around the n-cycle $(q-1)r-(2q-1)b+2q^2-q+1$ times and around the $(n-3)$-cycle $(2q+1)b-qr+2q^2+q$ times is an $(8q^3+1, 4q^3-2q^2)$-walk from i to j. So $\exp(D) \leqslant 12q^3-2q^2+1$ when $s = q-2$.

Finally, we show that $\exp(D) \leqslant 6q^3+2(3s+7)q^2-2(2s+5)q-s-2$ when $s \geqslant q-1$.

Note that

$$\begin{bmatrix} r \\ b \end{bmatrix} + ((q-1)r-(2q-1)b+q^2+2q+sq-s-2)\begin{bmatrix} 2q+1 \\ q \end{bmatrix} + ((2q+1)b-qr+q^2+sq+3q)\begin{bmatrix} 2q-1 \\ q-1 \end{bmatrix} = \begin{bmatrix} 4q^3+2(2s+5)q^2-(2s+5)q-s-2 \\ 2q^3+2(s+2)q^2-(2s+5)q \end{bmatrix}. \tag{2.3}$$

Consider the following three cases.

Case 1. Both the vertices i and j are on the n-cycle.

Clearly, $b \leqslant q$ and $r \leqslant 2q+1$. Thus $(q-1)r-(2q-1)b+q^2+2q+sq-s-2 \geqslant -(2q-1)q+q^2+2q+(q-1)^2-2 = q-1 > 0$ and $(2q+1)b-qr+q^2+sq+3q \geqslant -q(2q+1)+q^2+(q-1)q+3q = q > 0$.

Case 2. Both the vertices i and j are on the $(n-3)$-cycle.

Clearly, $b \leqslant q-1$ and $r \leqslant 2q-1$. Thus $(q-1)r-(2q-1)b+q^2+2q+sq-s-2 \geqslant -(2q-1)(q-1)+q^2+2q+(q-1)^2-2 = 3q-2 > 0$ and $(2q+1)b-qr+q^2+sq+3q \geqslant -q(2q-1)+q^2+(q-1)q+3q = 3q > 0$.

Case 3. The vertex i (or j) is on the path $1 \to 2 \to \ldots \to s+3$, and the vertex j (or i) is on the path $n+1 \to \ldots \to n+s$.

Clearly, the path p_{ij} contains the path $s+4 \to s+5 \to \ldots \to n$. Let the number of red arcs and blue arcs in the path $s+4 \to s+5 \to \ldots \to n$ be x and y, respectively.

Then $x+y=3q-s-3$, and the numbers of red arcs and blue arcs in D are $4q-x=q+s+y+3$ and $2q-y-1$, respectively. We see that

$$3q-s-y-3 \leqslant r \leqslant q+s+y+3,$$
$$y \leqslant b \leqslant 2q-1-y.$$

Thus $(q-1)r-(2q-1)b+q^2+2q+sq-s-2 \geqslant (q-1)(3q-s-y-3)-(2q-1)\times(2q-1-y)+q^2+2q+sq-s-2=yq \geqslant 0$, and $(2q+1)b-qr+q^2+sq+3q \geqslant (2q+1)y-q(q+s+y+3)+q^2+sq+3q=y(q+1) \geqslant 0$.

By virtue of (2.3), the walk that starts at vertex i, follows p_{ij} to vertex j and along the way goes around the n-cycle $(q-1)r-(2q-1)b+q^2+2q+sq-s-2$ times and around the $(n-3)$-cycle $(2q+1)b-qr+q^2+sq+3q$ times is a $(4q^3+2(2s+5)q^2-(2s+5)q-s-2, 2q^3+2(s+2)q^2-(2s+5)q)$-walk from i to j. So $\exp(D) \leqslant 6q^3+2(3s+7)q^2-2(2s+5)q-s-2$ when $s \geqslant q-1$.

The theorem now follows. □

3. Extremal two-colored digraphs for the case $n=3q+1$

In this section we give characterizations of extremal two-colored digraphs for the case $n=3q+1$. The main results are Theorems 3.4, 3.6, 3.7 and 3.11.

If the arcs in a walk w of length t are all red (blue), then we say that these arcs are t *consecutive red (blue) arcs*, or w is t *consecutive red (blue) arcs*. Since there are $2q+1$ red arcs and q blue arcs on the n-cycle, the n-cycle has at least one 3 consecutive red arcs. Similarly, the $(n-3)$-cycle has at least one 3 consecutive red arcs.

Lemma 3.1. *Let $D \in D_{3q+1,s}$. If D has a 3 consecutive red arcs in the path $n-2 \to n-1 \to n \to 1 \to \ldots \to s+6$, then*

$$\exp(D) > 18q^2-12q-3.$$

Proof. Let $a \to a+1, a+1 \to a+2, a+2 \to a+3$ be a 3 consecutive red arcs in the path $n-2 \to n-1 \to n \to 1 \to \ldots \to s+6$. Suppose that (h,k) is a pair of nonnegative integers such that for all pairs (i,j) of vertices there is an (h,k)-walk from i to j. Considering $i=j=n$, we see that there exist nonnegative integers u and v with

$$\begin{bmatrix} h \\ k \end{bmatrix} = M \begin{bmatrix} u \\ v \end{bmatrix}.$$

Taking i and j to be a and $a+3$, respectively, there is a unique path from i to j, and each walk from i to j can be decomposed into the path from i to j and cycles. Hence

$$Mz = \begin{bmatrix} h-3 \\ k \end{bmatrix}$$

has a nonnegative integer solution. Necessarily

$$z = M^{-1}\begin{bmatrix} h-3 \\ k \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix} - M^{-1}\begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix} - \begin{bmatrix} 3-3q \\ 3q \end{bmatrix} \geqslant 0.$$

So $v \geqslant 3q$. Next, take i and j to be $a+3$ and a, respectively. Since there is a unique path from i to j, and this path has composition $(2q-2, q)$, hence

$$Mz = \begin{bmatrix} h-(2q-2) \\ k-q \end{bmatrix}$$

has a nonnegative integer solution. Necessarily

$$z = M^{-1}\begin{bmatrix} h-(2q-2) \\ k-q \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix} - M^{-1}\begin{bmatrix} 2q-2 \\ q \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix} - \begin{bmatrix} 3q-2 \\ -3q \end{bmatrix} \geqslant 0.$$

So $u \geqslant 3q-2$. Thus

$$h+k = [1 \quad 1]\, M \begin{bmatrix} u \\ v \end{bmatrix} \geqslant [3q+1 \quad 3q-2] \begin{bmatrix} 3q-2 \\ 3q \end{bmatrix} = 18q^2 - 9q - 2,$$

and $\exp(D) \geqslant 18q^2 - 9q - 2 > 18q^2 - 12q - 3$. □

Lemma 3.2. *Let $D \in D_{3q+1,s}$. If D has a 2 consecutive blue arcs or has a blue-red-blue path of length 3, then*

$$\exp(D) > 18q^2 - 12q - 3.$$

Proof. If D has a 2 consecutive blue arcs, we can prove that $u \geqslant 4q-2$ and $v \geqslant 4q+2$ similarly to the proof of Lemma 3.1. So

$$\exp(D) \geqslant [3q+1 \quad 3q-2] \begin{bmatrix} 4q-2 \\ 4q+2 \end{bmatrix} = 24q^2 - 4q - 6 > 18q^2 - 12q - 3.$$

If D has a blue-red-blue path of length 3, we can prove that $u \geqslant 3q-1$ and $v \geqslant 3q+2$ similarly to the proof of Lemma 3.1. So

$$\exp(D) \geqslant [3q+1 \quad 3q-2] \begin{bmatrix} 3q-1 \\ 3q+2 \end{bmatrix} = 18q^2 - 5 > 18q^2 - 12q - 3.$$

□

Lemma 3.3. *Let $D \in D_{3q+1,s}$. If D has exactly one 3 consecutive red arcs, and the remaining arcs of D alternate between one blue arc and two red arcs, then*

$$\exp(D) = 18q^2 - 12q - 3.$$

Proof. We only need to show that $\exp(D) \leqslant 18q^2 - 12q - 3$.

Let w be the 3 consecutive red arcs. It is clear that w must be in the path $s+4 \to s+5 \to \ldots \to n$.

Let (i, j) be a pair of vertices and let p_{ij} be the shortest path from i to j. Denote $r = r(p_{ij})$ and $b = b(p_{ij})$. We see that

$$\begin{bmatrix} r \\ b \end{bmatrix} + ((q-1)r - (2q-1)b + 3q - 3) \begin{bmatrix} 2q+1 \\ q \end{bmatrix} + ((2q+1)b - qr + 3q) \begin{bmatrix} 2q-1 \\ q-1 \end{bmatrix} = \begin{bmatrix} 12q^2 - 6q - 3 \\ 6q^2 - 6q \end{bmatrix}. \tag{3.1}$$

Note that $r \leqslant 2(b+1)+1$ and $2(b-1) \leqslant r$ when $b \geqslant 1$. Consider the following three cases.

Case 1. Both the vertices i and j are on the $(n-3)$-cycle.

If $b = 0$, $r = 3$, then $(2q+1)b - qr + 3q = 0$, and both i and j are on the n-cycle. If $b = 0$, $r \leqslant 2$, then $(2q+1)b - qr + 3q > 0$. If $b \geqslant 1$, since $r \leqslant 2(b+1)+1$, we see that $(2q+1)b - qr + 3q \geqslant (2q+1)b - q(2b+3) + 3q = b > 0$.

If $b = 0$, then $(q-1)r - (2q-1)b + 3q - 3 > 0$. If $b \geqslant 1$, noting that $r \geqslant 2(b-1)$, we obtain $(q-1)r - (2q-1)b + 3q - 3 \geqslant 2(q-1)(b-1) - (2q-1)b + 3q - 3 = q - b - 1 \geqslant 0$.

Case 2. Both the vertices i and j are on the n-cycle and either i or j is not on the $(n-3)$-cycle.

Clearly, $r \leqslant 2q+1$ and $b \leqslant q$. If $0 \leqslant b \leqslant q-2$, then $(q-1)r - (2q-1)b + 3q - 3 \geqslant 2(q-1)(b-1) - (2q-1)b + 3q - 3 = q - b - 1 > 0$. If $b = q-1$, $r > 2q-4$, then $(q-1)r - (2q-1)b + 3q - 3 > (q-1)(2q-4) - (2q-1)(q-1) + 3q - 3 = 0$. If $b = q-1$, $r = 2q-4$, then $(q-1)r - (2q-1)b + 3q - 3 = 0$ and p_{ij} must contain a vertex which is on the $(n-3)$-cycle. If $b = q$, and either i or j is not on the $(n-3)$-cycle, then $r \geqslant 2q-1$ and $(q-1)r - (2q-1)b + 3q - 3 > (q-1)(2q-1) - (2q-1)q + 3q - 3 = q - 2 > 0$.

Noticing that $r \leqslant 2(b+1)+1$, we see that $(2q+1)b - qr + 3q \geqslant (2q+1)b - q(2b+3) + 3q = b \geqslant 0$.

Case 3. The vertex i (or j) is on the path $1 \to 2 \to \ldots \to s+3$, and the vertex j (or i) is on the path $n+1 \to \ldots \to n+s$.

Clearly, the path p_{ij} contains the path $s+4 \to s+5 \to \ldots \to n$. Let the number of red arcs and blue arcs in the path $s+4 \to s+5 \to \ldots \to n$ be x and y, respectively. Then $x + y = 3q - s - 3$, and the number of red arcs and blue arcs in D is $4q - x = q + s + y + 3$ and $2q - y - 1$, respectively.

If $b \leqslant 1$, then $(q-1)r-(2q-1)b+3q-3 \geqslant (q-1)r+q-2 \geqslant 0$. If $b \geqslant 2$, noting that $r \geqslant 2(b-1)+1$, we obtain $(q-1)r-(2q-1)b+3q-3 \geqslant (q-1)(2b-1)-(2q-1)b+3q-3 = 2q-b-2$. When $y=0$, since D has exactly one 3 consecutive red arcs, then $n \to 1$, $n \to n+1$, $s+3 \to s+4$ and $n+s \to s+4$ are blue. So $b \leqslant 2q-1-y-2 = 2q-3$ and $(q-1)r-(2q-1)b+3q-3 > 0$. When $y \geqslant 1$, then $b \leqslant 2q-1-y \leqslant 2q-2$ and $(q-1)r-(2q-1)b+3q-3 \geqslant 0$.

Noticing that $r \leqslant 2(b+1)+1$, we see that $(2q+1)b-qr+3q \geqslant (2q+1)b-q(2b+3)+3q = b \geqslant 0$.

By virtue of (3.1), the walk that starts at vertex i, follows p_{ij} to vertex j, and along the way goes around the n-cycle $(q-1)r-(2q-1)b+3q-3$ times and around the $(n-3)$-cycle $(2q+1)b-qr+3q$ times is a $(12q^2-6q-3, 6q^2-6q)$-walk from i to j. So $\exp(D) \leqslant 18q^2-12q-3$. □

Lemmas 3.1, 3.2, 3.3 yield the following theorem.

Theorem 3.4. *Let $D \in D_{3q+1,s}$. Then $\exp(D) = 18q^2-12q-3$ if and only if D has exactly one 3 consecutive red arcs, and the remaining arcs of D alternate between one blue arc and two red arcs.*

Now, we characterize the extremal digraphs in $D_{3q+1,s}$ whose exponents attain the upper bounds.

Lemma 3.5. *Let $D \in D_{3q+1,s}$ with $s \leqslant q-2$. If $2q+1$ red arcs on the n-cycle are not consecutive, then*

$$\exp(D) < 12q^3 - 2q^2 - 3q.$$

Proof. Let (i,j) be a pair of vertices and let p_{ij} be the shortest path from i to j. Denote $r = r(p_{ij})$ and $b = b(p_{ij})$. We see that

$$(3.2)\qquad \begin{bmatrix} r \\ b \end{bmatrix} + ((q-1)r-(2q-1)b+2q^2-q)\begin{bmatrix} 2q+1 \\ q \end{bmatrix} + ((2q+1)b-qr+2q^2+q-1)\begin{bmatrix} 2q-1 \\ q-1 \end{bmatrix} = \begin{bmatrix} 8q^3-4q+1 \\ 4q^3-2q^2-2q+1 \end{bmatrix}.$$

Consider the following three cases.

Case 1. Both the vertices i and j are on the n-cycle.

Clearly, $b \leqslant q$ and $r \leqslant 2q+1$. If $b \leqslant q-1$, then $(q-1)r-(2q-1)b+2q^2-q \geqslant (q-1)r-(2q-1)(q-1)+2q^2-q = (q-1)r+2q-1 > 0$. If $b = q$, since the q blue arcs on the n-cycle are not consecutive, $r \geqslant 1$ and $(q-1)r-(2q-1)b+2q^2-q \geqslant (q-1)-(2q-1)q+2q^2-q = q-1 > 0$.

If $r = 2q+1$, then $b \geqslant 1$ and $(2q+1)b - qr + 2q^2 + q - 1 \geqslant (2q+1) - q(2q+1) + 2q^2 + q - 1 = 2q > 0$. Otherwise $r \leqslant 2q$ and $(2q+1)b - qr + 2q^2 + q - 1 \geqslant (2q+1)b - 2q^2 + 2q^2 + q - 1 = (2q+1)b + q - 1 > 0$.

Case 2. Both the vertices i and j are on the $(n-3)$-cycle.

Clearly, $b \leqslant q-1$ and $r \leqslant 2q-1$. So $(q-1)r-(2q-1)b+2q^2-q \geqslant -(2q-1)(q-1)+2q^2-q = 2q-1 > 0$ and $(2q+1)b-qr+2q^2+q-1 \geqslant (2q+1)b-q(2q-1)+2q^2+q-1 = (2q+1)b+2q-1 > 0$.

Case 3. The vertex i (or j) is on the path $1 \to 2 \to \ldots \to s+3$, and the vertex j (or i) is on the path $n+1 \to \ldots \to n+s$.

Clearly, the path p_{ij} contains the path $s+4 \to s+5 \to \ldots \to n$. Let the number of red arcs and blue arcs in the path $s+4 \to s+5 \to \ldots \to n$ be x and y, respectively. Then $x+y = 3q-s-3$, and

$$\begin{aligned} 2q-y-1 \leqslant 3q-s-y-3 \leqslant r \leqslant q+s+y+3 \leqslant 2q+1+y, \\ y \leqslant b \leqslant 2q-1-y. \end{aligned}$$

Thus $(q-1)r-(2q-1)b+2q^2-q \geqslant (q-1)(2q-y-1)-(2q-1)(2q-1-y)+2q^2-q = qy \geqslant 0$, and $(2q+1)b-qr+2q^2+q-1 \geqslant (2q+1)y-q(2q+1+y)+2q^2+q-1 = qy+y-1$. If $y > 0$, then $(2q+1)b - qr + 2q^2 + q - 1 \geqslant qy + y - 1 > 0$. If $y = 0$, $r \leqslant 2q$, then $(2q+1)b - qr + 2q^2 + q - 1 \geqslant q - 1 > 0$. If $y = 0$, $r = 2q+1$, then $b \geqslant 1$ and $(2q+1)b - qr + 2q^2 + q - 1 \geqslant 2q + 1 - q(2q+1) + 2q^2 + q - 1 = 2q > 0$.

By virtue of (3.2), the walk that starts at vertex i, follows p_{ij} to vertex j, and along the way goes around the n-cycle $(q-1)r-(2q-1)b+2q^2-q$ times and around the $(n-3)$-cycle $(2q+1)b-qr+2q^2+q-1$ times is a $(8q^3-4q+1, 4q^3-2q^2-2q+1)$-walk from i to j. So $\exp(D) \leqslant 12q^3 - 2q^2 - 6q + 2 < 12q^3 - 2q^2 - 3q$. □

Theorem 3.6. *Let $D \in D_{3q+1,s}$ with $s \leqslant q-3$. Then $\exp(D) = 12q^3 - 2q^2 - 3q$ if and only if $2q+1$ red arcs on the n-cycle are consecutive.*

P r o o f. We only need to show that if $2q+1$ red arcs on the n-cycle are consecutive, then $\exp(D) \geqslant 12q^3 - 2q^2 - 3q$.

Suppose that (h,k) is a pair of nonnegative integers such that for all pairs (i,j) of vertices there is an (h,k)-walk from i to j. Considering $i = j = n$, we see that there exist nonnegative integers u and v with

$$\begin{bmatrix} h \\ k \end{bmatrix} = M \begin{bmatrix} u \\ v \end{bmatrix}.$$

Since there are $2q+1$ consecutive red arcs on the n-cycle, the remaining q arcs of the n-cycle are consecutive blue arcs. Taking i and j to be the initial vertex and the terminal vertex of $2q+1$ consecutive red arcs on the n-cycle, respectively, there is a unique path from i to j, and this path has composition $(2q+1,0)$. Hence

$$Mz = \begin{bmatrix} h-(2q+1) \\ k \end{bmatrix}$$

has a nonnegative integer solution. Necessarily

$$z = M^{-1}\begin{bmatrix} h-(2q+1) \\ k \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix} - M^{-1}\begin{bmatrix} 2q+1 \\ 0 \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix} - \begin{bmatrix} q+1-2q^2 \\ 2q^2+q \end{bmatrix} \geqslant 0.$$

So $v \geqslant 2q^2+q$. Next, taking i and j to be the initial vertex and the terminal vertex of q consecutive blue arcs on the n-cycle, respectively, there is a unique path from i to j, and this path has composition $(0,q)$. Hence

$$Mz = \begin{bmatrix} h \\ k-q \end{bmatrix}$$

has a nonnegative integer solution. Necessarily

$$z = \begin{bmatrix} u \\ v \end{bmatrix} - M^{-1}\begin{bmatrix} 0 \\ q \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix} - \begin{bmatrix} 2q^2-q \\ -2q^2-q \end{bmatrix} \geqslant 0.$$

So $u \geqslant 2q^2-q$. Thus

$$h+k = [1 \quad 1]\, M \begin{bmatrix} u \\ v \end{bmatrix} \geqslant [3q+1 \quad 3q-2]\begin{bmatrix} 2q^2-q \\ 2q^2+q \end{bmatrix} = 12q^3-2q^2-3q,$$

and $\exp(D) \geqslant 12q^3-2q^2-3q$. □

Theorem 3.7. *Let $D \in D_{3q+1,s}$ with $s=q-2$. Then $\exp(D) = 12q^3-2q^2+1$ if and only if $s+3 \to s+4 \to s+5 \to \ldots \to n \to 1$ are red, and the other arcs are blue.*

Proof. Necessity. Let $\exp(D) = 12q^3-2q^2+1$. By Lemma 3.5, $2q+1$ red arcs on the n-cycle are consecutive. Assuming that there is at least one blue arc in the path $s+3 \to s+4 \to s+5 \to \ldots \to n \to 1$, we show that $\exp(D) \leqslant 12q^3-2q^2-3q$.

Let (i,j) be a pair of vertices and let p_{ij} be the shortest path in D from i to j. Denote $r = r(p_{ij})$ and $b = b(p_{ij})$. We see that

$$\begin{bmatrix} r \\ b \end{bmatrix} + ((q-1)r-(2q-1)b+2q^2-q)\begin{bmatrix} 2q+1 \\ q \end{bmatrix} + ((2q+1)b-qr+2q^2+q)\begin{bmatrix} 2q-1 \\ q-1 \end{bmatrix} = \begin{bmatrix} 8q^3-2q \\ 4q^3-2q^2-q \end{bmatrix}. \tag{3.3}$$

Consider the following three cases.

Case 1. Both the vertices i and j are on the n-cycle.

Clearly, $b \leqslant q$ and $r \leqslant 2q+1$. Thus $(q-1)r-(2q-1)b+2q^2-q \geqslant (q-1)r-(2q-1)q+2q^2-q = (q-1)r \geqslant 0$ and $(2q+1)b-qr+2q^2+q \geqslant (2q+1)b-q(2q+1)+2q^2+q = (2q+1)b \geqslant 0$. If $(q-1)r-(2q-1)b+2q^2-q = 0$, then $r = 0$, $b = q$ and p_{ij} contains the vertex which is on the $(n-3)$-cycle.

Case 2. Both the vertices i and j are on the $(n-3)$-cycle.

Clearly, $b \leqslant q-1$ and $r \leqslant 2q-1$. Thus $(q-1)r-(2q-1)b+2q^2-q \geqslant -(2q-1)\times(q-1)+2q^2-q = 2q-1 > 0$ and $(2q+1)b-qr+2q^2+q \geqslant -q(2q-1)+2q^2+q = 2q > 0$.

Case 3. The vertex i (or j) is on the path $1 \to 2 \to \ldots \to s+3$ and the vertex j (or i) is on the path $n+1 \to \ldots \to n+s$.

Clearly, the path p_{ij} contains the path $s+4 \to s+5 \to \ldots \to n$. Let the number of red arcs and blue arcs in the path $s+4 \to s+5 \to \ldots \to n$ be x and y, respectively. Then $x+y = 2q-1$, and the number of red arcs and blue arcs in D is $4q-x = 2q+y+1$ and $2q-y-1$, respectively. We see that $2q-y-1 \leqslant r \leqslant 2q+y+1$ and $y \leqslant b \leqslant 2q-y-1$. Thus $(q-1)r-(2q-1)b+2q^2-q \geqslant (q-1)(2q-y-1)-(2q-1)(2q-y-1)+2q^2-q = yq \geqslant 0$, and $(2q+1)b-qr+2q^2+q \geqslant (2q+1)y-q(2q+y+1)+2q^2+q = yq+y \geqslant 0$.

By virtue of (3.3), the walk that starts at vertex i, follows p_{ij} to vertex j, and along the way goes around the n-cycle $(q-1)r-(2q-1)b+2q^2-q$ times and around the $(n-3)$-cycle $(2q+1)b-qr+2q^2+q$ times is a $(8q^3-2q, 4q^3-2q^2-q)$-walk from i to j. So $\exp(D) \leqslant 12q^3-2q^2-3q < 12q^3-2q^2+1$, a contradiction.

Sufficiency. Let $s+3 \to s+4 \to s+5 \to \ldots \to n \to 1$ be red and the other arcs be blue. We only need to show that $\exp(D) \geqslant 12q^3-2q^2+1$.

Suppose that (h,k) is a pair of nonnegative integers such that for all pairs (i,j) of vertices there is an (h,k)-walk from i to j. Considering $i = j = n$, we see that there exist nonnegative integers u and v with

$$\begin{bmatrix} h \\ k \end{bmatrix} = M \begin{bmatrix} u \\ v \end{bmatrix}.$$

Taking $i = s+3$ and $j = 1$, there is a unique path from i to j, and this path has composition $(2q+1, 0)$. Hence

$$Mz = \begin{bmatrix} h-(2q+1) \\ k \end{bmatrix}$$

has a nonnegative integer solution. Necessarily

$$z = M^{-1}\begin{bmatrix} h-(2q+1) \\ k \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix} - M^{-1}\begin{bmatrix} 2q+1 \\ 0 \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix} - \begin{bmatrix} q+1-2q^2 \\ 2q^2+q \end{bmatrix} \geqslant 0.$$

So $v \geqslant 2q^2 + q$. Next, taking $i = 1$ and $j = s + 3$, there is a unique path from i to j, and this path has composition $(0, q)$. Noting that this path does not contain any vertex on the $(n-3)$-cycle, we infer that each walk of length greater than q from i to j can be decomposed into the path from i to j and z_1 n-cycles and z_2 $(n-3)$-cycles, and $z_1 > 0$. This implies that there are integers $z_1 > 0$ and $z_2 \geqslant 0$ such that

$$M \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} h \\ k - q \end{bmatrix}.$$

Necessarily

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix} - M^{-1} \begin{bmatrix} 0 \\ q \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix} - \begin{bmatrix} 2q^2 - q \\ -2q^2 - q \end{bmatrix}.$$

So $u \geqslant 2q^2 - q + 1$. Thus

$$h + k = [1 \quad 1]\, M \begin{bmatrix} u \\ v \end{bmatrix} \geqslant [3q+1 \quad 3q-2] \begin{bmatrix} 2q^2 - q + 1 \\ 2q^2 + q \end{bmatrix} = 12q^3 - 2q^2 + 1,$$

and $\exp(D) \geqslant 12q^3 - 2q^2 + 1$. Sufficiency is proved. □

Let the number of red arcs and blue arcs in the path $s + 4 \to s + 5 \to \ldots \to n$ be x and y, respectively. Note that $x = 3q - y - s - 3 \leqslant 3q - s - 3$. Let r denote the number of red arcs in D. Then $r = 4q - x \geqslant q + s + 3$, and $r = q + s + 3$ if and only if $x = 3q - s - 3$, that is, the arcs $s + 4 \to s + 5$, $s + 5 \to s + 6$, …, $n - 1 \to n$ must be red.

Lemma 3.8. *Let $D \in D_{3q+1,s}$ with $s \geqslant q - 1$, and let D have exactly $q + s + 3$ red arcs. If the $q + s + 3$ red arcs are consecutive, then*

$$\exp(D) = 6q^3 + 2(3s + 7)q^2 - 2(2s + 5)q - s - 2.$$

Proof. We only need to show that $\exp(D) \geqslant 6q^3 + 2(3s+7)q^2 - 2(2s+5)q - s - 2$.

Suppose that (h, k) is a pair of nonnegative integers such that for all pairs (i, j) of vertices there is an (h, k)-walk from i to j. Considering $i = j = n$, we see that there exist nonnegative integers u and v with

$$\begin{bmatrix} h \\ k \end{bmatrix} = M \begin{bmatrix} u \\ v \end{bmatrix}.$$

Since D has exactly $q + s + 3$ red arcs, the arcs $s + 4 \to s + 5$, $s + 5 \to s + 6$, …, $n - 1 \to n$ are red. This implies that there exist $s - q + 4$ red arcs in the path $n \to 1 \to 2 \to \ldots \to s + 4$ and $s - q + 2$ red arcs in the path $n \to n + 1 \to \ldots \to n + s \to s + 4$, respectively.

Taking i and j to be the initial vertex and the terminal vertex of $q+s+3$ consecutive red arcs, respectively, then there is a unique path from i to j, and this path has composition $(q+s+3, 0)$. Hence

$$Mz = \begin{bmatrix} h-(q+s+3) \\ k \end{bmatrix}$$

has a nonnegative integer solution. Necessarily

$$\begin{aligned} z &= M^{-1}\begin{bmatrix} h-(q+s+3) \\ k \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix} - M^{-1}\begin{bmatrix} q+s+3 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} u \\ v \end{bmatrix} - \begin{bmatrix} -q^2-(s+2)q+(s+3) \\ q^2+(s+3)q \end{bmatrix} \geqslant 0. \end{aligned}$$

So $v \geqslant q^2+(s+3)q$. Next, taking i and j to be the terminal vertex and the initial vertex of $q+s+3$ consecutive red arcs, respectively, there is a unique path from i to j, and this path has composition $(3q-s-3, 2q-1)$. Hence

$$Mz = \begin{bmatrix} h-(3q-s-3) \\ k-(2q-1) \end{bmatrix}$$

has a nonnegative integer solution. Necessarily

$$\begin{aligned} z &= M^{-1}\begin{bmatrix} h-(3q-s-3) \\ k-(2q-1) \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix} - M^{-1}\begin{bmatrix} 3q-s-3 \\ 2q-1 \end{bmatrix} \\ &= \begin{bmatrix} u \\ v \end{bmatrix} - \begin{bmatrix} q^2+(s+2)q-(s+2) \\ -q^2-(s+3)q+1 \end{bmatrix} \geqslant 0. \end{aligned}$$

So $u \geqslant q^2+(s+2)q-(s+2)$. Thus

$$\begin{aligned} h+k &= [1 \quad 1]\, M \begin{bmatrix} u \\ v \end{bmatrix} \geqslant [3q+1 \quad 3q-2] \begin{bmatrix} q^2+(s+2)q-(s+2) \\ q^2+(s+3)q \end{bmatrix} \\ &= 6q^3+2(3s+7)q^2-2(2s+5)q-s-2, \end{aligned}$$

and $\exp(D) \geqslant 6q^3+2(3s+7)q^2-2(2s+5)q-s-2$. □

Lemma 3.9. *Let $D \in D_{3q+1,s}$ with $s \geqslant q-1$, and let D have exactly $q+s+3$ red arcs. If the $q+s+3$ red arcs are not consecutive, then*

$$\exp(D) < 6q^3+2(3s+7)q^2-2(2s+5)q-s-2.$$

Proof. Let (i,j) be a pair of vertices and let p_{ij} be the shortest path in D from i to j. Denote $r = r(p_{ij})$ and $b = b(p_{ij})$. We see that

$$\begin{aligned}(3.4)\qquad &\begin{bmatrix} r \\ b \end{bmatrix} + ((q-1)r - (2q-1)b + q^2 + 2q + sq - s - 2)\begin{bmatrix} 2q+1 \\ q \end{bmatrix} \\ &+ ((2q+1)b - qr + q^2 + sq + 2q)\begin{bmatrix} 2q-1 \\ q-1 \end{bmatrix} \\ &= \begin{bmatrix} 4q^3 + 2(2s+4)q^2 - (2s+4)q - s - 2 \\ 2q^3 + (2s+3)q^2 - (2s+4)q \end{bmatrix}.\end{aligned}$$

Consider the following three cases.

Case 1. Both the vertices i and j are on the n-cycle.

Clearly, $b \leqslant q$ and $r \leqslant 2q+1$. Thus $(q-1)r - (2q-1)b + q^2 + 2q + sq - s - 2 \geqslant -(2q-1)q + q^2 + 2q + (q-1)^2 - 2 = q - 1 > 0$ and $(2q+1)b - qr + q^2 + sq + 2q \geqslant -q(2q+1) + q^2 + (q-1)q + 2q = 0$.

Case 2. Both the vertices i and j are on the $(n-3)$-cycle.

Clearly, $b \leqslant q-1$ and $r \leqslant 2q-1$. Thus $(q-1)r - (2q-1)b + q^2 + 2q + sq - s - 2 \geqslant -(2q-1)(q-1) + q^2 + 2q + (q-1)^2 - 2 = 3q - 2 > 0$ and $(2q+1)b - qr + q^2 + sq + 2q \geqslant -q(2q-1) + q^2 + (q-1)q + 2q = 2q > 0$.

Case 3. The vertex i (or j) is on the path $1 \to 2 \to \ldots \to s+3$, and the vertex j (or i) is on the path $n+1 \to \ldots \to n+s$.

Clearly, the path p_{ij} contains the path $s+4 \to s+5 \to \ldots \to n$, and the arcs $s+4 \to s+5$, $s+5 \to s+6$, …, $n-1 \to n$ must be red. So

$$\begin{aligned} 3q - s - 3 \leqslant r \leqslant q + s + 3, \\ 0 \leqslant b \leqslant 2q - 1. \end{aligned}$$

Thus $(q-1)r - (2q-1)b + q^2 + 2q + sq - s - 2 \geqslant (q-1)(3q-s-3) - (2q-1)(2q-1) + q^2 + 2q + sq - s - 2 = 0$. If $r \leqslant q + s + 2$, then $(2q+1)b - qr + q^2 + sq + 2q \geqslant -q(q+s+2) + q^2 + sq + 2q = 0$. If $r = q + s + 3$, then $b \geqslant 1$, and $(2q+1)b - qr + q^2 + sq + 2q \geqslant 2q + 1 - q(q+s+3) + q^2 + sq + 2q = q + 1 > 0$.

By virtue of (3.4), the walk that starts at vertex i, follows p_{ij} to vertex j, and along the way goes around the n-cycle $(q-1)r - (2q-1)b + q^2 + 2q + sq - s - 2$ times and around the $(n-3)$-cycle $(2q+1)b - qr + q^2 + sq + 2q$ times is a $\bigl(4q^3 + 2(2s+4)q^2 - (2s+4)q - s - 2, 2q^3 + (2s+3)q^2 - (2s+4)q\bigr)$-walk from i to j. So $\exp(D) \leqslant 6q^3 + (6s+11)q^2 - 2(2s+4)q - s - 2 < 6q^3 + 2(3s+7)q^2 - 2(2s+5)q - s - 2$. □

Lemma 3.10. *Let $D \in D_{3q+1,s}$ with $s \geqslant q-1$ and let there be at least one blue arc in the path $s+4 \to s+5 \to \ldots \to n$. Then*

$$\exp(D) < 6q^3 + 2(3s+7)q^2 - 2(2s+5)q - s - 2.$$

P r o o f. Let (i,j) be a pair of vertices and let p_{ij} be the shortest path from i to j. Denote $r = r(p_{ij})$ and $b = b(p_{ij})$. Let the number of red arcs and blue arcs in the path $s+4 \to s+5 \to \ldots \to n$ be x and y, respectively. Then $y \geqslant 1$ and $x \leqslant 3q - s - 4$. We see that

$$\begin{aligned} &\begin{bmatrix} r \\ b \end{bmatrix} + ((q-1)r - (2q-1)b + q^2 + 2q + sq - s - 2) \begin{bmatrix} 2q+1 \\ q \end{bmatrix} \\ &\quad + ((2q+1)b - qr + q^2 + sq + 3q - 1) \begin{bmatrix} 2q-1 \\ q-1 \end{bmatrix} \\ &\quad = \begin{bmatrix} 4q^3 + 2(2s+5)q^2 - (2s+7)q - s - 1 \\ 2q^3 + 2(s+2)q^2 - (2s+6)q + 1 \end{bmatrix}. \end{aligned} \tag{3.5}$$

Consider the following three cases.

Case 1. Both the vertices i and j are on the n-cycle.

Clearly, $b \leqslant q$ and $r \leqslant 2q+1$. Thus $(q-1)r - (2q-1)b + q^2 + 2q + sq - s - 2 \geqslant -(2q-1)q + q^2 + 2q + (q-1)^2 - 2 = q - 1 > 0$ and $(2q+1)b - qr + q^2 + sq + 3q - 1 \geqslant -q(2q+1) + q^2 + (q-1)q + 3q - 1 = q - 1 > 0$.

Case 2. Both the vertices i and j are on the $(n-3)$-cycle.

Clearly, $b \leqslant q-1$ and $r \leqslant 2q-1$. Thus $(q-1)r - (2q-1)b + q^2 + 2q + sq - s - 2 \geqslant -(2q-1)(q-1) + q^2 + 2q + (q-1)^2 - 2 = 3q - 2 > 0$ and $(2q+1)b - qr + q^2 + sq + 3q - 1 \geqslant -q(2q-1) + q^2 + (q-1)q + 3q - 1 = 3q - 1 > 0$.

Case 3. The vertex i (or j) is on the path $1 \to 2 \to \ldots \to s+3$, and the vertex j (or i) is on the path $n+1 \to \ldots \to n+s$.

Clearly, the path p_{ij} contains the path $s+4 \to s+5 \to \ldots \to n$. So

$$\begin{aligned} 3q - s - y - 3 \leqslant r &\leqslant q + s + y + 3, \\ y \leqslant b &\leqslant 2q - 1 - y. \end{aligned}$$

Thus $(q-1)r - (2q-1)b + q^2 + 2q + sq - s - 2 \geqslant (q-1)(3q-s-y-3) - (2q-1)(2q-1-y) + q^2 + 2q + sq - s - 2 = yq > 0$ and $(2q+1)b - qr + q^2 + sq + 3q - 1 \geqslant (2q+1)y - q(q+s+y+3) + q^2 + sq + 3q - 1 = y(q+1) - 1 > 0$.

By virtue of (3.5), the walk that starts at vertex i, follows p_{ij} to vertex j, and along the way goes around the n-cycle $(q-1)r - (2q-1)b + q^2 + 2q + sq - s - 2$ times and around the $(n-3)$-cycle $(2q+1)b - qr + q^2 + sq + 3q - 1$ times is a $(4q^3 + 2(2s+5)q^2 - (2s+7)q - s - 1, 2q^3 + 2(s+2)q^2 - (2s+6)q + 1)$-walk from i to j. So $\exp(D) \leqslant 6q^3 + 2(3s+7)q^2 - (4s+13)q - s < 6q^3 + 2(3s+7)q^2 - 2(2s+5)q - s - 2$.

□

Lemmas 3.8, 3.9, and 3.10 yield the following result.

Theorem 3.11. *Let $D \in D_{3q+1,s}$ with $s \geqslant q-1$. Then $\exp(D) = 6q^3 + 2(3s+7)q^2 - 2(2s+5)q - s - 2$ if and only if there are exactly $q+s+3$ red arcs in D, and all the red arcs are consecutive.*

4. The case $n = 3q + 2$

Let $n = 3q+2$ and let the cycle matrix of D be

$$M = \begin{bmatrix} 2q+1 & 2q-1 \\ q+1 & q \end{bmatrix},$$

where $q \geqslant 3$. Clearly,

$$M^{-1} = \begin{bmatrix} q & -2q+1 \\ -q-1 & 2q+1 \end{bmatrix}.$$

Theorem 4.1. *Let $D \in D_{3q+2,s}$. Then*

$$18q^2 - 5 \leqslant \exp(D) \leqslant \begin{cases} 12q^3 + 14q^2 + 2q - 1, & \text{if } s \leqslant q-2, \\ 6q^3 + 2(3s+8)q^2 + 2(2s+5)q - (s+3), & \text{if } s \geqslant q-1. \end{cases}$$

Proof. First, we show that $\exp(D) \geqslant 18q^2 - 5$.

Suppose that (h,k) is a pair of nonnegative integers such that for all pairs (i,j) of vertices there is an (h,k)-walk from i to j. Considering $i = j = n$, we see that there exist nonnegative integers u and v with

$$\begin{bmatrix} h \\ k \end{bmatrix} = M \begin{bmatrix} u \\ v \end{bmatrix}.$$

Let the length of the longest red path in D be l. Since there are $2q+1$ red arcs and $q+1$ blue arcs on the n-cycle, we see that $l \geqslant 2$.

Case 1. $l = 2$.

In this case, there is a blue-red-blue path w of length 3 on the n-cycle. Taking i and j to be the initial vertex and terminal vertex of w, respectively, the path from i to j has composition $(1,2)$. So

$$Mz = \begin{bmatrix} h-1 \\ k-2 \end{bmatrix}$$

has a nonnegative integer solution. Then

$$z = \begin{bmatrix} u \\ v \end{bmatrix} - M^{-1} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix} - \begin{bmatrix} -3q+2 \\ 3q+1 \end{bmatrix} \geqslant 0.$$

So $v \geqslant 3q+1$. Next, let i and j be the terminal and initial vertices of w, respectively. Then the path from i to j has composition either $(2q, q-1)$ or $(2q-2, q-2)$, so we have that

$$Mz = \begin{bmatrix} h-2q \\ k-(q-1) \end{bmatrix} \quad \text{or} \quad Mz = \begin{bmatrix} h-(2q-2) \\ k-(q-2) \end{bmatrix}$$

has a nonnegative integer solution. Then

$$z = \begin{bmatrix} u \\ v \end{bmatrix} - M^{-1} \begin{bmatrix} 2q \\ q-1 \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix} - \begin{bmatrix} 3q-1 \\ -3q-1 \end{bmatrix} \geqslant 0,$$

or

$$z = \begin{bmatrix} u \\ v \end{bmatrix} - M^{-1} \begin{bmatrix} 2q-2 \\ q-2 \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix} - \begin{bmatrix} 3q-2 \\ -3q \end{bmatrix} \geqslant 0.$$

So $u \geqslant 3q-2$. Thus

$$h+k = [1 \quad 1]\, M \begin{bmatrix} u \\ v \end{bmatrix} \geqslant [3q+2 \quad 3q-1] \begin{bmatrix} 3q-2 \\ 3q+1 \end{bmatrix} = 18q^2 - 5.$$

Case 2. $l \geqslant 3$.

In this case, there is a red path w of length 3. Taking i and j as the initial vertex and terminal vertex of w, respectively, the path from i to j has composition $(3,0)$. So

$$Mz = \begin{bmatrix} h-3 \\ k \end{bmatrix}$$

has a nonnegative integer solution. Then

$$z = M^{-1} \begin{bmatrix} h-3 \\ k \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix} - M^{-1} \begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix} - \begin{bmatrix} 3q \\ -3q-3 \end{bmatrix} \geqslant 0.$$

So $u \geqslant 3q$. Next, let i and j be the terminal and initial vertices of w, respectively. Then the path from i to j has composition either $(2q-2, q+1)$, $(2q-4, q)$, or $(4q-3, 2q+1)$ (this case arises only if $s+4 = n-1$, $i = n+1$ and $j = s+3$ or $i = 1$ and $j = n+s$), so we have that

$$Mz = \begin{bmatrix} h-(2q-2) \\ k-(q+1) \end{bmatrix}, \quad Mz = \begin{bmatrix} h-(2q-4) \\ k-q \end{bmatrix}, \quad \text{or} \quad Mz = \begin{bmatrix} h-(4q-3) \\ k-(2q+1) \end{bmatrix}$$

has a nonnegative integer solution. Then

$$z = \begin{bmatrix} u \\ v \end{bmatrix} - M^{-1} \begin{bmatrix} 2q-2 \\ q+1 \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix} - \begin{bmatrix} -3q+1 \\ 3q+3 \end{bmatrix} \geqslant 0,$$

$$z = \begin{bmatrix} u \\ v \end{bmatrix} - M^{-1} \begin{bmatrix} 2q-4 \\ q \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix} - \begin{bmatrix} -3q \\ 3q+4 \end{bmatrix} \geqslant 0,$$

or

$$z = \begin{bmatrix} u \\ v \end{bmatrix} - M^{-1} \begin{bmatrix} 4q-3 \\ 2q+1 \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix} - \begin{bmatrix} -3q+1 \\ 3q+4 \end{bmatrix} \geqslant 0.$$

So $v \geqslant 3q+3$. Thus

$$h+k = [1 \quad 1]\, M \begin{bmatrix} u \\ v \end{bmatrix} \geqslant [3q+2 \quad 3q-1] \begin{bmatrix} 3q \\ 3q+3 \end{bmatrix} = 18q^2+12q-3,$$

and $\exp(D) \geqslant 18q^2-5$.

Next, we show that $\exp(D) \leqslant 12q^3+14q^2+2q-1$ when $s \leqslant q-2$.

Let (i,j) be a pair of vertices and let p_{ij} be the shortest path in D from i to j. Denote $r = r(p_{ij})$ and $b = b(p_{ij})$. We see that

$$\begin{aligned} (4.1) \qquad & \begin{bmatrix} r \\ b \end{bmatrix} + ((2q-1)b - qr + 2q^2 + q) \begin{bmatrix} 2q+1 \\ q+1 \end{bmatrix} \\ & \quad + ((q+1)r - (2q+1)b + 2q^2 + 3q + 1) \begin{bmatrix} 2q-1 \\ q \end{bmatrix} = \begin{bmatrix} 8q^3+8q^2-1 \\ 4q^3+6q^2+2q \end{bmatrix}. \end{aligned}$$

Consider the following three cases.

Case 1. Both the vertices i and j are on the n-cycle.

Clearly, $b \leqslant q+1$ and $r \leqslant 2q+1$. If $b=0$ and $r=2q+1$, then $(2q-1)b-qr+2q^2+q = -q(2q+1)+2q^2+q = 0$ and either i or j is on the $(n-3)$-cycle. Otherwise, $(2q-1)b-qr+2q^2+q > -q(2q+1)+2q^2+q = 0$. For $(q+1)r-(2q+1)b+2q^2+3q+1$, we have $(q+1)r-(2q+1)b+2q^2+3q+1 \geqslant (q+1)r-(2q+1)(q+1)+2q^2+3q+1 = (q+1)r \geqslant 0$.

Case 2. Both the vertices i and j are on the $(n-3)$-cycle.

Clearly, $b \leqslant q$ and $r \leqslant 2q-1$. Thus $(2q-1)b-qr+2q^2+q \geqslant -q(2q-1)+2q^2+q = 2q > 0$ and $(q+1)r-(2q+1)b+2q^2+3q+1 \geqslant -(2q+1)q+2q^2+3q+1 = 2q+1 > 0$.

Case 3. The vertex i (or j) is on the path $1 \to 2 \to \ldots \to s+3$, and the vertex j (or i) is on the path $n+1 \to \ldots \to n+s$.

Clearly, the path p_{ij} contains the path $s+4 \to s+5 \to \ldots \to n$. Let the number of red arcs and blue arcs in the path $s+4 \to s+5 \to \ldots \to n$ be x and y, respectively. Then $x+y = 3q-s-2$, and the number of red arcs and blue arcs in D is $4q-x = q+s+y+2$ and $2q-y+1$, respectively. Since $s \leqslant q-2$, we see that

$$\begin{aligned} 2q-y \leqslant 3q-s-y-2 \leqslant r \leqslant q+s+y+2 \leqslant 2q+y, \\ y \leqslant b \leqslant 2q-y+1. \end{aligned}$$

Thus $(2q-1)b-qr+2q^2+q \geqslant (2q-1)y-q(2q+y)+2q^2+q = q+(q-1)y > 0$, and $(q+1)r-(2q+1)b+2q^2+3q+1 \geqslant (q+1)(2q-y)-(2q+1)(2q-y+1)+2q^2+3q+1 = yq+q > 0$.

By virtue of (4.1), the walk that starts at vertex i, follows p_{ij} to vertex j, and along the way goes around the n-cycle $(2q-1)b-qr+2q^2+q$ times and around the $(n-3)$-cycle $(q+1)r-(2q+1)b+2q^2+3q+1$ times is a $(8q^3+8q^2-1, 4q^3+6q^2+2q)$-walk from i to j. So $\exp(D) \leqslant 12q^3+14q^2+2q-1$ when $s \leqslant q-2$.

Finally, we show that $\exp(D) \leqslant 6q^3+2(3s+8)q^2+2(2s+5)q-(s+3)$ when $s \geqslant q-1$.

Let (i,j) be a pair of vertices and let p_{ij} be the shortest path in D from i to j. Denote $r = r(p_{ij})$ and $b = b(p_{ij})$. We see that

$$\begin{aligned}(4.2)\qquad & \begin{bmatrix} r \\ b \end{bmatrix} + ((2q-1)b-qr+q^2+2q+sq)\begin{bmatrix} 2q+1 \\ q+1 \end{bmatrix} \\ & + ((q+1)r-(2q+1)b+q^2+sq+3q+s+3)\begin{bmatrix} 2q-1 \\ q \end{bmatrix} \\ & = \begin{bmatrix} 4q^3+2(2s+5)q^2+(2s+5)q-s-3 \\ 2q^3+2(s+3)q^2+(2s+5)q \end{bmatrix}.\end{aligned}$$

Consider the following three cases.

Case 1. Both the vertices i and j are on the n-cycle.

Clearly, $b \leqslant q+1$ and $r \leqslant 2q+1$. Thus $(2q-1)b-qr+q^2+2q+sq \geqslant (2q-1)b-q(2q+1)+q^2+2q+(q-1)q = (2q-1)b \geqslant 0$ and $(q+1)r-(2q+1)b+q^2+sq+3q+s+3 \geqslant (q+1)r-(q+1)(2q+1)+q^2+(q-1)(q+1)+3q+3 = (q+1)r+1 > 0$. If $(2q-1)b-qr+q^2+2q+sq = 0$, hence $b=0$, $r=2q+1$, $s=q-1$, and either i or j is on the $(n-3)$-cycle.

Case 2. Both the vertices i and j are on the $(n-3)$-cycle.

Clearly, $b \leqslant q$ and $r \leqslant 2q-1$. Thus $(2q-1)b-qr+q^2+2q+sq \geqslant -q(2q-1)+q^2+2q+(q-1)q = 2q > 0$ and $(q+1)r-(2q+1)b+q^2+sq+3q+s+3 \geqslant -(2q+1)q+q^2+(q-1)(q+1)+3q+3 = 2q+2 > 0$.

Case 3. The vertex i (or j) is on the path $1 \to 2 \to \ldots \to s+3$ and the vertex j (or i) is on the path $n+1 \to \ldots \to n+s$.

Clearly, the path p_{ij} contains the path $s+4 \to s+5 \to \ldots \to n$. Let the number of red arcs and blue arcs in the path $s+4 \to s+5 \to \ldots \to n$ be x and y, respectively. Then $x+y = 3q-s-2$, and the number of red arcs and blue arcs in D is $4q-x = q+s+y+2$ and $2q-y+1$, respectively. We see that

$$\begin{aligned} 3q-s-y-2 \leqslant r &\leqslant q+s+y+2, \\ y \leqslant b &\leqslant 2q-y+1. \end{aligned}$$

Thus $(2q-1)b-qr+q^2+2q+sq \geqslant (2q-1)y-q(q+s+y+2)+q^2+2q+sq = y(q-1) \geqslant 0$, and $(q+1)r-(2q+1)b+q^2+sq+3q+s+3 \geqslant (q+1)(3q-s-y-2)-(2q+1)\times(2q-y+1)+q^2+sq+3q+s+3 = yq \geqslant 0$.

By virtue of (4.2), the walk that starts at vertex i, follows p_{ij} to vertex j, and along the way goes around the n-cycle $(2q-1)b-qr+q^2+2q+sq$ times and around the $(n-3)$-cycle $(q+1)r-(2q+1)b+q^2+sq+3q+s+3$ times is a $(4q^3+2(2s+5)q^2+(2s+5)q-s-3, 2q^3+2(s+3)q^2+(2s+5)q)$-walk from i to j. So $\exp(D) \leqslant 6q^3+2(3s+8)q^2+2(2s+5)q-(s+3)$ when $s \geqslant q-1$.

The theorem follows. □

5. Extremal two-colored digraphs for the case $n = 3q+2$

In this section we give characterizations of extremal two-colored digraphs for the case $n = 3q+2$. The main results are Theorems 5.4, 5.6 and 5.10.

Lemma 5.1. *Let $D \in D_{3q+2,s}$. If the length of the longest red path in D is greater than or equal to 3, then*

$$\exp(D) > 18q^2 - 5.$$

Proof. From the proof of Theorem 4.1, it is clear. □

Lemma 5.2. *Let $D \in D_{3q+2,s}$. If the length of the longest red path in D is 2 and there is a blue-red-blue path w in the path $n-2 \to n-1 \to n \to 1 \to \ldots \to s+6$, then*

$$\exp(D) > 18q^2 - 5.$$

Proof. Suppose that (h,k) is a pair of nonnegative integers such that for all pairs (i,j) of vertices there is an (h,k)-walk from i to j. Considering $i = j = n$, we see that there exist nonnegative integers u and v with

$$\begin{bmatrix} h \\ k \end{bmatrix} = M \begin{bmatrix} u \\ v \end{bmatrix}.$$

Taking i and j to be the initial vertex and terminal vertex of w, respectively, then the path from i to j has composition $(1,2)$. So we have that

$$Mz = \begin{bmatrix} h-1 \\ k-2 \end{bmatrix}$$

has a nonnegative integer solution. Then

$$z = \begin{bmatrix} u \\ v \end{bmatrix} - M^{-1}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix} - \begin{bmatrix} -3q+2 \\ 3q+1 \end{bmatrix} \geqslant 0.$$

So $v \geqslant 3q+1$. Next, let i and j be the terminal and initial vertices of w, respectively. Then the path from i to j has composition $(2q, q-1)$, so we have that

$$Mz = \begin{bmatrix} h-2q \\ k-(q-1) \end{bmatrix}$$

has a nonnegative integer solution. Then

$$z = \begin{bmatrix} u \\ v \end{bmatrix} - M^{-1}\begin{bmatrix} 2q \\ q-1 \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix} - \begin{bmatrix} 3q-1 \\ -3q-1 \end{bmatrix} \geqslant 0.$$

So $u \geqslant 3q-1$. Thus

$$h+k = [1 \quad 1]\, M \begin{bmatrix} u \\ v \end{bmatrix} \geqslant [3q+2 \quad 3q-1]\begin{bmatrix} 3q-1 \\ 3q+1 \end{bmatrix} = 18q^2+3q-3 > 18q^2-5.$$

This implies the lemma. $\square$

Lemma 5.3. *Let $D \in D_{3q+2,s}$. If the length of the longest red path in D is 2, and there is a blue-red-blue path w in the path $s+4 \to s+5 \to \ldots \to n$, then*

$$\exp(D) = 18q^2 - 5.$$

Proof. We only need to show that

$$\exp(D) \leqslant 18q^2 - 5.$$

Let (i,j) be a pair of vertices and let p_{ij} be the shortest path from i to j. Denote $r = r(p_{ij})$, $b = b(p_{ij})$. We see that

$$\begin{aligned}(5.1)\qquad &\begin{bmatrix} r \\ b \end{bmatrix} + ((2q-1)b - qr + 3q - 2)\begin{bmatrix} 2q+1 \\ q+1 \end{bmatrix} \\ &\quad + ((q+1)r - (2q+1)b + 3q + 1)\begin{bmatrix} 2q-1 \\ q \end{bmatrix} = \begin{bmatrix} 12q^2-2q-3 \\ 6q^2+2q-2 \end{bmatrix}.\end{aligned}$$

Noting that $r \leqslant 2(b+1) = 2b+2$ and $r \geqslant 2(b-1)-1 = 2b-3$ when $b \geqslant 2$, we have $b \leqslant \frac{1}{2}(r+3)$. When $r = 0$, then $b \leqslant 1$, and $(q+1)r-(2q+1)b+3q+1 \geqslant q > 0$. When $r \geqslant 1$, then $(q+1)r-(2q+1)b+3q+1 \geqslant (q+1)r-(2q+1)\frac{1}{2}(r+3)+3q+1 = \frac{1}{2}(r-1) \geqslant 0$, and if $(q+1)r-(2q+1)b+3q+1=0$ then $r=1$ and $b=2$. This implies that p_{ij} is the path w, and both i and j are on the n-cycle.

Now we prove that $(2q-1)b-qr+3q-2 \geqslant 0$ and if $(2q-1)b-qr+3q-2=0$ then p_{ij} must contain a vertex which is on the $(n-3)$-cycle.

Case 1. Both the vertices i and j are on the $(n-3)$-cycle.

Clearly, $b \leqslant q$, $r \leqslant 2q-1$, and $r \leqslant 2b+2$. If $r \leqslant 2b+1$, then $(2q-1)b-qr+3q-2 \geqslant (2q-1)b-q(2b+1)+3q-2 = 2q-2-b \geqslant 2q-2-q = q-2 \geqslant 0$. If $r=2b+2$, noticing that $r \leqslant 2q-1$, we infer that $b \leqslant q-2$ and $(2q-1)b-qr+3q-2 = (2q-1)b-q(2b+2)+3q-2 = q-2-b \geqslant 0$.

Case 2. Both the vertices i and j are on the n-cycle, and either i or j is not on the $(n-3)$-cycle.

Clearly $b \leqslant q+1$ and $r \leqslant 2b+2$. If $r \leqslant 2b$, then $(2q-1)b-qr+3q-2 \geqslant (2q-1)b-2qb+3q-2 = 3q-2-b \geqslant 3q-2-(q+1) > 0$. If $r=2b+1$, noticing that $r \leqslant 2q+1$, we infer that $b \leqslant q$ and $(2q-1)b-qr+3q-2 = 2q-b-2 \geqslant q-2 > 0$. If $r=2b+2$, noticing $r \leqslant 2q+1$, then $b \leqslant q-1$. If $b \leqslant q-3$, $r=2b+2$, then $(2q-1)b-qr+3q-2 = (2q-1)b-q(2b+2)+3q-2 = q-b-2 > 0$. If $b=q-2$, $r=2b+2=2q-2$, since the length of the longest red path in D is 2 and there is a blue-red-blue path in $s+4 \to s+5 \to \ldots \to n$, so in this case we have $(2q-1)b-qr+3q-2 = (2q-1)b-q(2b+2)+3q-2 = q-b-2 = 0$ and either i or j is on the $(n-3)$-cycle. If $b=q-1$, $r=2b+2=2q$, then i and j are the terminal and initial vertices of w, respectively, and both i and j are on the $(n-3)$-cycle, so this is not the case.

Case 3. The vertex i (or j) is on the path $1 \to 2 \to \ldots \to s+3$ and the vertex j (or i) is on the path $n+1 \to \ldots \to n+s$.

Clearly, the path p_{ij} contains the path $s+4 \to s+5 \to \ldots \to n$. So $r \leqslant 2(b+1)-1 = 2b+1$. Let the number of blue arcs in the path $s+4 \to s+5 \to \ldots \to n$ be y. Then $2 \leqslant y \leqslant b \leqslant 2q-y+1$. If $b=2q-y+1$, then $n \to 1$, $n \to n+1$, $s+3 \to s+4$ and $n+s \to s+4$ are red. So $r \leqslant 2(b+1)-1-2 = 2b-1$, and $(2q-1)b-qr+3q-2 \geqslant (2q-1)b-q(2b-1)+3q-2 = 4q-2-b = 4q-2-2q+y-1 = 2q-3+y > 0$. If $b \leqslant 2q-y \leqslant 2q-2$, then $(2q-1)b-qr+3q-2 \geqslant (2q-1)b-q(2b+1)+3q-2 = 2q-2-b \geqslant 0$.

By virtue of (5.1), the walk that starts at vertex i, follows p_{ij} to vertex j, and goes $(2q-1)b-qr+3q-2$ times around the n-cycle and $(q+1)r-(2q+1)b+3q+1$ times around the $(n-3)$-cycle is a $(12q^2-2q-3, 6q^2+2q-2)$-walk from i to j. So $\exp(D) \leqslant 18q^2-5$. □

Lemmas 5.1, 5.2, 5.3 yield the following theorem.

Theorem 5.4. *Let $D \in D_{3q+2,s}$. Then $\exp(D) = 18q^2 - 5$ if and only if the length of the longest red path in D is 2, and there is a blue-red-blue path in the path $s+4 \to s+5 \to \ldots \to n$.*

Now, we characterize the extremal digraphs in $D_{3q+2,s}$ whose exponents attain the upper bounds.

Lemma 5.5. *Let $D \in D_{3q+2,s}$ with $s \leqslant q-2$. If $2q+1$ red arcs on the n-cycle are not consecutive, then*

$$\exp(D) < 12q^3 + 14q^2 + 2q - 1.$$

P r o o f. Let (i,j) be a pair of vertices and let p_{ij} be the shortest path in D from i to j. Denote $r = r(p_{ij})$ and $b = b(p_{ij})$. We see that

$$(5.2)\quad \begin{bmatrix} r \\ b \end{bmatrix} + ((2q-1)b - qr + 2q^2 + q) \begin{bmatrix} 2q+1 \\ q+1 \end{bmatrix} + \bigl((q+1)r - (2q+1)b + 2q^2 + 3q\bigr) \begin{bmatrix} 2q-1 \\ q \end{bmatrix} = \begin{bmatrix} 8q^3 + 8q^2 - 2q \\ 4q^3 + 6q^2 + q \end{bmatrix}.$$

Consider the following three cases.

Case 1. Both the vertices i and j are on the n-cycle.

Clearly, $b \leqslant q+1$ and $r \leqslant 2q+1$. Thus $(2q-1)b - qr + 2q^2 + q \geqslant (2q-1)b - q(2q+1) + 2q^2 + q = (2q-1)b \geqslant 0$. If $(2q-1)b - qr + 2q^2 + q = 0$, then $b = 0$ and $r = 2q+1$. Noting that $s+4 \leqslant q+2 < 2q+3$, we infer that either i or j is on the $(n-3)$-cycle. For $(q+1)r - (2q+1)b + 2q^2 + 3q$, if $b \leqslant q$, then $(q+1)r - (2q+1)b + 2q^2 + 3q \geqslant (q+1)r - (2q+1)q + 2q^2 + 3q = (q+1)r + 2q > 0$. If $b = q+1$, noting that the $q+1$ blue arcs on the n-cycle are not consecutive, then $r \geqslant 1$ and $(q+1)r - (2q+1)b + 2q^2 + 3q \geqslant (q+1) - (2q+1)(q+1) + 2q^2 + 3q = q > 0$.

Case 2. Both the vertices i and j are on the $(n-3)$-cycle.

Clearly, $b \leqslant q$ and $r \leqslant 2q-1$. Thus $(2q-1)b - qr + 2q^2 + q \geqslant -q(2q-1) + 2q^2 + q = 2q > 0$ and $(q+1)r - (2q+1)b + 2q^2 + 3q \geqslant -(2q+1)q + 2q^2 + 3q = 2q > 0$.

Case 3. The vertex i (or j) is on the path $1 \to 2 \to \ldots \to s+3$, and the vertex j (or i) is on the path $n+1 \to \ldots \to n+s$.

Clearly, the path p_{ij} contains the path $s+4 \to s+5 \to \ldots \to n$. Let the number of red arcs and blue arcs in the path $s+4 \to s+5 \to \ldots \to n$ be x and y, respectively. Then $x + y = 3q - s - 2$, and

$$2q - y \leqslant 3q - s - y - 2 \leqslant r \leqslant q + s + y + 2 \leqslant 2q + y,$$
$$y \leqslant b \leqslant 2q - y + 1.$$

Thus $(2q-1)b-qr+2q^2+q \geqslant (2q-1)y-q(2q+y)+2q^2+q = q+(q-1)y > 0$ and $(q+1)r-(2q+1)b+2q^2+3q \geqslant (q+1)(2q-y)-(2q+1)(2q-y+1)+2q^2+3q = yq+q-1 > 0$.

By virtue of (5.2), the walk that starts at vertex i, follows p_{ij} to vertex j, and along the way goes around the n-cycle $(2q-1)b-qr+2q^2+q$ times and around the $(n-3)$-cycle $(q+1)r-(2q+1)b+2q^2+3q$ times is an $(8q^3+8q^2-2q, 4q^3+6q^2+q)$-walk from i to j. So $\exp(D) \leqslant 12q^3+14q^2-q < 12q^3+14q^2+2q-1$. □

Theorem 5.6. *Let $D \in D_{3q+2,s}$ with $s \leqslant q-2$. Then $\exp(D) = 12q^3+14q^2+2q-1$ if and only if $2q+1$ red arcs on the n-cycle are consecutive.*

Proof. By Lemma 5.5 and Theorem 4.1, we only need to show that if $2q+1$ red arcs on the n-cycle are consecutive, then $\exp(D) \geqslant 12q^3+14q^2+2q-1$.

Suppose that (h,k) is a pair of nonnegative integers such that for all pairs (i,j) of vertices there is an (h,k)-walk from i to j. Considering $i=j=n$, we see that there exist nonnegative integers u and v with

$$\begin{bmatrix} h \\ k \end{bmatrix} = M \begin{bmatrix} u \\ v \end{bmatrix}.$$

Since there are $2q+1$ consecutive red arcs on the n-cycle, the remaining $q+1$ arcs of the n-cycle are consecutive blue arcs. Taking i and j to be the initial vertex and the terminal vertex of $2q+1$ consecutive red arcs on the n-cycle, respectively, there is a unique path from i to j, and this path has composition $(2q+1,0)$. Hence

$$Mz = \begin{bmatrix} h-(2q+1) \\ k \end{bmatrix}$$

has a nonnegative integer solution. Necessarily

$$z = M^{-1}\begin{bmatrix} h-(2q+1) \\ k \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix} - M^{-1}\begin{bmatrix} 2q+1 \\ 0 \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix} - \begin{bmatrix} 2q^2+q \\ -2q^2-3q-1 \end{bmatrix} \geqslant 0.$$

So $u \geqslant 2q^2+q$. Next, taking i and j to be the initial vertex and the terminal vertex of q consecutive blue arcs on the n-cycle, respectively, there is a unique path from i to j, and this path has composition $(0,q+1)$. Hence

$$Mz = \begin{bmatrix} h \\ k-(q+1) \end{bmatrix}$$

has a nonnegative integer solution. Necessarily

$$z = \begin{bmatrix} u \\ v \end{bmatrix} - M^{-1}\begin{bmatrix} 0 \\ q+1 \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix} - \begin{bmatrix} -2q^2-q+1 \\ 2q^2+3q+1 \end{bmatrix} \geqslant 0.$$

So $v \geqslant 2q^2 + 3q + 1$. Thus

$$h + k = [\,1 \quad 1\,]\, M \begin{bmatrix} u \\ v \end{bmatrix} \geqslant [\,3q+2 \quad 3q-1\,] \begin{bmatrix} 2q^2 + q \\ 2q^2 + 3q + 1 \end{bmatrix} = 12q^3 + 14q^2 + 2q - 1,$$

and $\exp(D) \geqslant 12q^3 + 14q^2 + 2q - 1$. □

Let the number of red arcs and blue arcs in the path $s+4 \to s+5 \to \ldots \to n$ be x and y, respectively. Note that $x = 3q - y - s - 2 \leqslant 3q - s - 2$. Let r denote the number of red arcs in D. Then $r = 4q - x \geqslant q + s + 2$, and $r = q + s + 2$ if and only if $x = 3q - s - 2$, that is, the arcs $s+4 \to s+5$, $s+5 \to s+6$, …, $n-1 \to n$ must be red.

Lemma 5.7. *Let $D \in D_{3q+2,s}$ with $s \geqslant q-1$, and let D have exactly $q+s+2$ red arcs. If the $q+s+2$ red arcs are consecutive, then*

$$\exp(D) = 6q^3 + 2(3s+8)q^2 + 2(2s+5)q - (s+3).$$

Proof. We only need to show that $\exp(D) \geqslant 6q^3 + 2(3s+8)q^2 + 2(2s+5)q - (s+3)$.

Suppose that (h,k) is a pair of nonnegative integers such that for all pairs (i,j) of vertices there is an (h,k)-walk from i to j. Considering $i = j = n$, we see that there exist nonnegative integers u and v with

$$\begin{bmatrix} h \\ k \end{bmatrix} = M \begin{bmatrix} u \\ v \end{bmatrix}.$$

Since D has exactly $q+s+2$ red arcs, the arcs $s+4 \to s+5$, $s+5 \to s+6$, …, $n-1 \to n$ are red. This implies that there exist $s-q+3$ red arcs in the path $n \to 1 \to 2 \to \ldots \to s+4$ and $s-q+1$ red arcs in the path $n \to n+1 \to \ldots \to n+s \to s+4$, respectively.

Taking i and j to be the initial vertex and the terminal vertex of $q+s+2$ consecutive red arcs, respectively, then there is a unique path from i to j, and this path has composition $(q+s+2, 0)$. Hence

$$Mz = \begin{bmatrix} h - (q+s+2) \\ k \end{bmatrix}$$

has a nonnegative integer solution. Necessarily

$$\begin{aligned} z &= M^{-1} \begin{bmatrix} h - (q+s+2) \\ k \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix} - M^{-1} \begin{bmatrix} q+s+2 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} u \\ v \end{bmatrix} - \begin{bmatrix} q^2 + (s+2)q \\ -q^2 - (s+3)q - (s+2) \end{bmatrix} \geqslant 0. \end{aligned}$$

So $u \geqslant q^2 + (s+2)q$. Next, taking i and j to be the terminal vertex and the initial vertex of $q+s+2$ consecutive red arcs, respectively, there is a unique path from i to j, and this path has composition $(3q-s-2, 2q+1)$. Hence

$$Mz = \begin{bmatrix} h-(3q-s-2) \\ k-(2q+1) \end{bmatrix}$$

has a nonnegative integer solution. Necessarily

$$\begin{aligned} z &= M^{-1}\begin{bmatrix} h-(3q-s-2) \\ k-(2q+1) \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix} - M^{-1}\begin{bmatrix} 3q-s-2 \\ 2q+1 \end{bmatrix} \\ &= \begin{bmatrix} u \\ v \end{bmatrix} - \begin{bmatrix} -q^2-(s+2)q+1 \\ q^2+(s+3)q+(s+3) \end{bmatrix} \geqslant 0. \end{aligned}$$

So $v \geqslant q^2 + (s+3)q + (s+3)$. Thus

$$\begin{aligned} h+k &= [1 \quad 1]\, M \begin{bmatrix} u \\ v \end{bmatrix} \geqslant [3q+2 \quad 3q-1] \begin{bmatrix} q^2+(s+2)q \\ q^2+(s+3)q+(s+3) \end{bmatrix} \\ &= 6q^3 + 2(3s+8)q^2 + 2(2s+5)q - (s+3), \end{aligned}$$

and $\exp(D) \geqslant 6q^3 + 2(3s+8)q^2 + 2(2s+5)q - (s+3)$. □

Lemma 5.8. *Let $D \in \mathcal{D}_{3q+2,s}$ with $s \geqslant q-1$, and let D have exactly $q+s+2$ red arcs. If the $q+s+2$ red arcs are not consecutive, then*

$$\exp(D) < 6q^3 + 2(3s+8)q^2 + 2(2s+5)q - (s+3).$$

P r o o f. Let (i,j) be a pair of vertices and let p_{ij} be the shortest path in D from i to j. Denote $r = r(p_{ij})$ and $b = b(p_{ij})$. We see that

$$\begin{aligned} \begin{bmatrix} r \\ b \end{bmatrix} &+ ((2q-1)b - qr + q^2 + 2q + sq)\begin{bmatrix} 2q+1 \\ q+1 \end{bmatrix} \\ &+ \big((q+1)r - (2q+1)b + q^2 + sq + 3q + s + 2\big)\begin{bmatrix} 2q-1 \\ q \end{bmatrix} \\ &= \begin{bmatrix} 4q^3 + 2(2s+5)q^2 + (2s+3)q - s - 2 \\ 2q^3 + 2(s+3)q^2 + (2s+4)q \end{bmatrix}. \end{aligned} \tag{5.3}$$

Consider the following three cases.

Case 1. Both the vertices i and j are on the n-cycle.

Clearly, $b \leqslant q+1$ and $r \leqslant 2q+1$. Thus $(2q-1)b - qr + q^2 + 2q + sq \geqslant (2q-1)b - q(2q+1) + q^2 + 2q + (q-1)q = (2q-1)b \geqslant 0$ and $(q+1)r - (2q+1)b + q^2 + sq + 3q + s + 2 \geqslant (q+1)r - (q+1)(2q+1) + q^2 + (q-1)(q+1) + 3q + 2 = (q+1)r \geqslant 0$.

If $(2q-1)b-qr+q^2+2q+sq=0$, then $b=0$, $r=2q+1$, $s=q-1$, and either i or j is on the $(n-3)$-cycle.

Case 2. Both the vertices i and j are on the $(n-3)$-cycle.

Clearly, $b\leqslant q$ and $r\leqslant 2q-1$. Thus $(2q-1)b-qr+q^2+2q+sq\geqslant -q(2q-1)+q^2+2q+(q-1)q=2q>0$ and $(q+1)r-(2q+1)b+q^2+sq+3q+s+2\geqslant -(2q+1)q+q^2+(q-1)(q+1)+3q+2=2q+1>0$.

Case 3. The vertex i (or j) is on the path $1\to 2\to\ldots\to s+3$, and the vertex j (or i) is on the path $n+1\to\ldots\to n+s$.

Clearly, the path p_{ij} contains the path $s+4\to s+5\to\ldots\to n$. So

$$3q-s-2\leqslant r\leqslant q+s+2,$$
$$0\leqslant b\leqslant 2q+1.$$

Thus $(2q-1)b-qr+q^2+2q+sq\geqslant -q(q+s+2)+q^2+2q+sq=0$. If $b\leqslant 2q$, then $(q+1)r-(2q+1)b+q^2+sq+3q+s+2\geqslant (q+1)(3q-s-2)-2q(2q+1)+q^2+sq+3q+s+2=2q>0$. Let $b=2q+1$. Since the $q+s+2$ red arcs are not consecutive, we have $r\geqslant 3q-s-1$ and $(q+1)r-(2q+1)b+q^2+sq+3q+s+2\geqslant (q+1)(3q-s-1)-(2q+1)(2q+1)+q^2+sq+3q+s+2=q>0$.

By virtue of (5.3), the walk that starts at vertex i, follows p_{ij} to vertex j, and along the way goes around the n-cycle $(2q-1)b-qr+q^2+2q+sq$ times and around the $(n-3)$-cycle $(q+1)r-(2q+1)b+q^2+sq+3q+s+2$ times is a $(4q^3+2(2s+5)q^2+(2s+3)q-s-2,\,2q^3+2(s+3)q^2+(2s+4)q)$-walk from i to j. So $\exp(D)\leqslant 6q^3+2(3s+8)q^2+(4s+7)q-(s+2)<6q^3+2(3s+8)q^2+2(2s+5)q-(s+3)$. □

Lemma 5.9. *Let $D\in D_{3q+2,s}$ with $s\geqslant q-1$ and let there be at least one blue arc in the path $s+4\to s+5\to\ldots\to n$. Then*

$$\exp(D)<6q^3+2(3s+8)q^2+2(2s+5)q-(s+3).$$

P r o o f. Let (i,j) be a pair of vertices and let p_{ij} be the shortest path in D from i to j. Denote $r=r(p_{ij})$ and $b=b(p_{ij})$. Let the number of red arcs and blue arcs in the path $s+4\to s+5\to\ldots\to n$ be x and y, respectively. Then $y\geqslant 1$ and $x\leqslant 3q-s-3$. We see that

$$(5.4)\qquad \begin{bmatrix} r\\ b\end{bmatrix}+((2q-1)b-qr+q^2+2q+sq)\begin{bmatrix}2q+1\\ q+1\end{bmatrix}$$
$$+((q+1)r-(2q+1)b+q^2+sq+3q+s+2)\begin{bmatrix}2q-1\\ q\end{bmatrix}$$
$$=\begin{bmatrix}4q^3+2(2s+5)q^2+(2s+3)q-s-2\\ 2q^3+2(s+3)q^2+(2s+4)q\end{bmatrix}.$$

Consider the following three cases.

Case 1. Both the vertices i and j are on the n-cycle.

Clearly, $b \leqslant q+1$ and $r \leqslant 2q+1$. Thus $(2q-1)b - qr + q^2 + 2q + sq \geqslant (2q-1)b - q(2q+1) + q^2 + 2q + (q-1)q = (2q-1)b \geqslant 0$ and $(q+1)r - (2q+1)b + q^2 + sq + 3q + s + 2 \geqslant (q+1)r - (q+1)(2q+1) + q^2 + (q-1)(q+1) + 3q + 2 = (q+1)r \geqslant 0$. If $(2q-1)b - qr + q^2 + 2q + sq = 0$, then $b = 0$, $r = 2q+1$, $s = q-1$, and either i or j is on the $(n-3)$-cycle.

Case 2. Both the vertices i and j are on the $(n-3)$-cycle.

Clearly, $b \leqslant q$ and $r \leqslant 2q-1$. Thus $(2q-1)b - qr + q^2 + 2q + sq \geqslant -q(2q-1) + q^2 + 2q + (q-1)q = 2q > 0$ and $(q+1)r - (2q+1)b + q^2 + sq + 3q + s + 2 \geqslant -(2q+1)q + q^2 + (q-1)(q+1) + 3q + 2 = 2q+1 > 0$.

Case 3. The vertex i (or j) is on the path $1 \to 2 \to \ldots \to s+3$ and the vertex j (or i) is on the path $n+1 \to \ldots \to n+s$.

Clearly, the path p_{ij} contains the path $s+4 \to s+5 \to \ldots \to n$. So

$$3q - s - y - 2 \leqslant r \leqslant q + s + y + 2,$$
$$y \leqslant b \leqslant 2q - y + 1.$$

Thus $(2q-1)b - qr + q^2 + 2q + sq \geqslant (2q-1)y - q(q+s+y+2) + q^2 + 2q + sq = y(q-1) \geqslant 0$, and $(q+1)r - (2q+1)b + q^2 + sq + 3q + s + 2 \geqslant (q+1)(3q-s-y-2) - (2q+1) \times (2q-y+1) + q^2 + sq + 3q + s + 2 = yq - 1 > 0$.

By virtue of (5.4), the walk that starts at vertex i, follows p_{ij} to vertex j and along the way goes around the n-cycle $(2q-1)b - qr + q^2 + 2q + sq$ times and around the $(n-3)$-cycle $(q+1)r - (2q+1)b + q^2 + sq + 3q + s + 2$ times is a $(4q^3 + 2(2s+5)q^2 + (2s+3)q - s - 2, 2q^3 + 2(s+3)q^2 + (2s+4)q)$-walk from i to j. So $\exp(D) \leqslant 6q^3 + 2(3s+8)q^2 + (4s+7)q - (s+2) < 6q^3 + 2(3s+8)q^2 + 2(2s+5)q - (s+3)$. □

Lemmas 5.7, 5.8, and 5.9 yield the following result.

Theorem 5.10. *Let $D \in D_{3q+2,s}$ with $s \geqslant q-1$. Then $\exp(D) = 6q^3 + 2(3s+7)q^2 + (2s+5)q - 2(s+3)$ if and only if there are exactly $q+s+2$ red arcs in D, and all red arcs are consecutive.*

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