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ON THE DISTRIBUTIVE RADICAL OF AN ARCHIMEDEAN
LATTICE-ORDERED GROUP

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Abstract. Let G be an Archimedean ℓ -group. We denote by G^d and $R_D(G)$ the divisible hull of G and the distributive radical of G , respectively. In the present note we prove the relation $(R_D(G))^d = R_D(G^d)$. As an application, we show that if G is Archimedean, then it is completely distributive if and only if it can be regularly embedded into a completely distributive vector lattice.

Keywords: Archimedean ℓ -group, divisible hull, distributive radical, complete distributivity

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Throughout the paper, ℓ -group will be used as a shorthand for lattice-ordered group.

The distributive radical $R_D(G)$ of an ℓ -group G was investigated by Byrd and Lloyd [2]; cf. also Darnel [3], Section 2.2.

Assume that G is an Archimedean ℓ -group. The symbol G^d denotes the divisible hull of G . In this paper we prove that the relation

$$(R_D(G))^d = R_D(G^d)$$

is valid.

In other words, we prove that the operators d and R_D commute on the class \mathcal{A} of all Archimedean ℓ -groups defined by

$$\begin{aligned} d: G &\rightarrow G^d, \\ R_D: G &\rightarrow R_D(G). \end{aligned}$$

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As an application, we show that if G is Archimedean, then it is completely distributive if and only if it can be regularly embedded into a completely distributive vector lattice.

1. ℓ -IDEALS IN G

The group operation in an ℓ -group will be written additively; for the terminology, cf. [1] and [2].

It is well-known that each Abelian ℓ -group can be embedded into a divisible ℓ -group. Each Archimedean ℓ -group is Abelian. If G is an Archimedean ℓ -group, then according to the results of [4], the divisible hull G^d is an Archimedean ℓ -group and is characterized by the following properties:

- (i) for each element y of G^d there exist a positive integer n and an element $x \in G$ such that $ny = x$; in such case we write

$$(1) \quad y = \frac{x}{n};$$

- (ii) for each $z \in G^d$ and each positive integer m there is $t \in G^d$ with $mt = z$;
- (iii) G is regularly embedded into G^\wedge (that is, all suprema and infima that exist in G are preserved by the embedding into G^\wedge).

This terminology is in accordance with that applied by Sikorski [8] for Boolean algebras.

In what follows we assume that G is an Archimedean ℓ -group.

If H is an ℓ -group, then we denote by $J(H)$ the system of all ℓ -ideals of H ; this system is partially ordered by the set-theoretic inclusion. In fact, $J(H)$ is a complete lattice. For an Abelian ℓ -group H , the notion of ℓ -ideal coincides with the notion of convex ℓ -subgroup of H .

The following three lemmas are easy to verify; the proofs will be omitted.

Lemma 1.1. *Let H_1 be a subset of an ℓ -group H such that*

- (i) H_1 is a subgroup of the group H ;
- (ii) if $0 < x \in H_1$, $0 < y \in H$ and $y \leq x$, then $y \in H_1$.

Then H_1 is a convex ℓ -subgroup of H .

Let $A \in J(G)$. We denote by $f(A)$ the set of all elements y of G^d which can be expressed in the form (1), where $x \in A$ and $n \in \mathbb{N}$.

Lemma 1.2. *Let $A \in J(G)$. Then $f(A) \in J(G^d)$.*

Let $B \in J(G^d)$. We put $g(B) = B \cap G$.

Lemma 1.3. *Under the notation as above, $g(B)$ belongs to $J(G)$.*

Lemma 1.4. *Let $A \in J(G)$ and $B = f(A)$. Then $g(B) = A$.*

Proof. a) Let $x \in A$. Then $x \in B$, whence $x \in g(B)$. Thus $A \subseteq g(B)$.

b) Let $z \in (g(B))^+$. Hence there are $n \in \mathbb{N}$ and $x \in A$ such that $nz = x$. In view of $z \geq 0$ we have $x \geq 0$. Moreover, $0 \leq z \leq x$. Since $z \in G$, we infer that $z \in A$. Thus $(g(B))^+ \subseteq A$.

c) If $x \in A$, then $-x$ also belongs to A . Hence b) yields $(g(B))^- \subseteq A$. The group $g(B)$ is generated by its subset

$$(g(B))^+ \cup (g(B))^-$$

and thus $g(B) \subseteq A$. □

As a consequence of 1.3 and 1.4 we get that f is a one-to-one mapping of the set $J(G)$ onto the set $J(G^d)$.

From the definitions of the mappings f and g we obtain that if $A_1, A_2 \in J(G)$, then

$$A_1 \subseteq A_2 \Rightarrow f(A_1) \subseteq f(A_2).$$

Similarly, if $B_1, B_2 \in J(G^d)$, then

$$B_1 \subseteq B_2 \Rightarrow g(B_1) \subseteq g(B_2).$$

Hence we conclude

Lemma 1.5. *The mapping f is an isomorphism of the lattice $J(G)$ onto the lattice $J(G^d)$.*

For each ℓ -group H and each subset X of H the polar $X^{\delta(H)}$ is defined by

$$X^{\delta(H)} = \{y \in H: |y| \wedge |x| = 0 \text{ for all } x \in X\}.$$

It is well-known that $X^{\delta(H)}$ is a convex ℓ -subgroup of H .

From the definition of polar we easily obtain

Lemma 1.6. *Let H be an Abelian ℓ -group and let A be an ℓ -ideal of H . Then $A^{\delta(H)}$ is the largest element of the set*

$$\{A_1 \in J(H): A_1 \wedge A = \{0\}\},$$

where $A_1 \wedge A$ denotes the infimum of $\{A_1, A\}$ in the lattice $J(H)$.

Lemma 1.7. *Let $A \in J(G)$. Then*

$$f(A^{\delta(G)}) = f(A)^{\delta(G^d)}.$$

Proof. This is a consequence of 1.5 and 1.6. □

Lemma 1.8. *Let $A \in J(G)$. The following conditions are equivalent:*

- (i) *A is a prime ideal in G .*
- (ii) *$f(A)$ is a prime ideal in G^d .*

Proof. a) Let (i) be valid and let $y_1, y_2 \in G^d$ be such that $y_1 \wedge y_2 = 0$. Hence there are $x_1, x_2 \in G$ and $n_1, n_2 \in \mathbb{N}$ such that

$$y_i = \frac{x_i}{n_i} \quad (i = 1, 2).$$

Then we have $x_i \geq 0$ ($i = 1, 2$). Moreover, from the relation $y_1 \wedge y_2 = 0$ we obtain

$$(n_1 y_1) \wedge (n_2 y_2) = 0,$$

whence $x_1 \wedge x_2 = 0$. The condition (i) yields that either $x_1 \in A$ or $x_2 \in A$. Thus either $y_1 \in f(A)$ or $y_2 \in f(A)$. Therefore (ii) is valid.

b) Conversely, assume that (ii) holds. Let $x_1, x_2 \in G$, $x_1 \wedge x_2 = 0$. Then $x_1, x_2 \in G^d$ and the relation $x_1 \wedge x_2 = 0$ is valid in G^d . In view of (ii), either $x_1 \in f(A)$ or $x_2 \in f(A)$. Hence according to 1.4, either $x_1 \in A$ or $x_2 \in A$. Therefore (i) holds. □

2. DISTRIBUTIVE RADICAL AND COMPLETE DISTRIBUTIVITY

We recall the notions concerning higher degrees of distributivity.

Let α and β be nonzero cardinals. Further, let T and S be nonempty sets with $\text{card } T \leq \alpha$ and $\text{card } S \leq \beta$. A lattice L is (α, β) -*distributive* if the following identities hold in L

$$(d_1) \quad \bigwedge_{t \in T} \bigvee_{s \in S} x_{t,s} = \bigvee_{\varphi \in S^T} \bigwedge_{t \in T} x_{t,\varphi(t)},$$

$$(d_2) \quad \bigvee_{t \in T} \bigwedge_{s \in S} x_{t,s} = \bigwedge_{\varphi \in S^T} \bigvee_{t \in T} x_{t,\varphi(t)}$$

whenever all joins and meets appearing in (d_1) or (d_2) exist in L .

If L is (α, β) -distributive for any nonzero cardinals α and β , then L is said to be completely distributive.

Definition 2.1 (Cf. [2]). Let H be an ℓ -group. The distributive radical $R_D(H)$ of H is defined to be the set

$$\bigcap A_i^{\delta(H)} \quad (i \in I),$$

where $\{A_i\}_{i \in I}$ is the system of all minimal prime ideals of H .

Lemma 2.2. $R_D(G^d) = f(R_D(G))$.

Proof. Let $A \in J(G)$. In view of 1.7, 1.8 and 1.5 the following conditions are equivalent:

- (i) A is a minimal prime ideal of G .
- (ii) $f(A)$ is a minimal prime ideal of G^d .

By applying 1.5 again we conclude that the assertion of the lemma is valid. □

Corollary 2.2.1. $R_D(G) = \{0\}$ if and only if $R_D(G^d) = \{0\}$.

Proof. This is a consequence of 2.2 and 1.5. □

Theorem 2.3. Let G be an Archimedean ℓ -group. Then the relation

$$(R_D(G))^d = R_D(G^d)$$

is valid.

Proof. Denote

$$P = R_D(G), \quad Q = R_D(G^d).$$

We have to verify that $Q = P^d$.

It is obvious that P is an ℓ -subgroup of Q and that Q is Archimedean. Consider the conditions (i), (ii) and (iii) from the definition of the divisible hull of an ℓ -group (cf. Section 1). Let us apply these conditions for P and Q .

Let $q \in Q$. In view of 2.2, there exist $p \in P$ and $n \in \mathbb{N}$ such that $nq = p$. Hence (i) is valid.

In view of the definition of the divisible hull, Q is divisible (since it is an ℓ -ideal of a divisible ℓ -group); thus the condition (ii) is satisfied.

Suppose that $X \subseteq P$, $x \in P$ and that the relation $x = \sup X$ is valid in P . Since P is an ℓ -ideal in G , the relation $x = \sup X$ holds in G . Then, according to the definition of the divisible hull, $x = \sup X$ is valid in G^d as well. Since Q is an ℓ -ideal of G^d and $X \subseteq Q$, $x \in Q$, we conclude that the relation $x = \sup X$ holds in Q . Analogously we verify the corresponding dual condition. Therefore the condition (iii) is satisfied. \square

The following theorem is due to Byrd and Lloyd [2].

Theorem 2.4. *Let H be an ℓ -group. The following conditions are equivalent:*

- (i) $R_D(H) = \{0\}$.
- (ii) H is completely distributive.

Theorem 2.5. *The following conditions are equivalent:*

- (i) G is completely distributive.
- (ii) G^d is completely distributive.

Proof. This is a consequence of 2.2.1 and 2.4. \square

The Dedekind completion of an Archimedean ℓ -group H will be denoted by H^\wedge .

We remark that the implication (i) \Rightarrow (ii) is a consequence of the following more general result which is due to Darnel (private communication):

- (*) Let H be an Abelian ℓ -group. Suppose that H is completely distributive. Then the divisible hull H^d of H is completely distributive as well.

Let α and β be cardinals. The question whether the complete distributivity in (*) can be replaced by (α, β) -distributivity remains open.

Lemma 2.6. (Cf. [4], [6].) G^{d^\wedge} is a complete vector lattice.

The following result was proved independently in [6] and [7].

Lemma 2.7. (Cf. [6], [7].) G is regularly embedded into G^{d^\wedge} .

Lemma 2.8. *The following conditions are equivalent:*

- (i) G^d is completely distributive.
- (ii) $G^{d\wedge}$ is completely distributive.

Proof. This follows from Theorem 2.2 in [5]. □

Theorem 2.9. *Let G be an Archimedean ℓ -group. The following conditions are equivalent:*

- (i) G is completely distributive.
- (ii) G can be regularly embedded into a completely distributive complete vector lattice.

Proof. The implication (ii) \Rightarrow (i) is obvious. The converse implication is a consequence of 2.5, 2.6, 2.7, 2.3 and 2.8. (Thus the only new ingredient to prove Theorem 2.9 is Theorem 2.3.) □

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