

I. Monterde; Vicente Montesinos
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CONVEX-COMPACT SETS AND BANACH DISCS

I. MONTERDE, and V. MONTESINOS

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Abstract. Every relatively convex-compact convex subset of a locally convex space is contained in a Banach disc. Moreover, an upper bound for the class of sets which are contained in a Banach disc is presented. If the topological dual E' of a locally convex space E is the $\sigma(E', E)$ -closure of the union of countably many $\sigma(E', E)$ -relatively countably compact sets, then every weakly (relatively) convex-compact set is weakly (relatively) compact.

Keywords: weakly compact sets, convex-compact sets, Banach discs

1. INTRODUCTION

The possibility to include a certain bounded subset A of a locally convex space (E, \mathcal{T}) in a *Banach disc* (i.e., a bounded absolutely convex set in (E, \mathcal{T}) such that $E_A := \bigcup_n nA$, endowed with the norm $\|\cdot\|_A$ given by the Minkowski gauge of A , is a Banach space) has a big impact on its structure (in particular, the set A becomes *strongly bounded*, i.e., bounded on bounded subsets of the space $(E', \sigma(E', E))$ —the topological dual E' of (E, \mathcal{T}) endowed with the topology $\sigma(E', E)$ of the pointwise convergence on all points in E), and is the basic fact in the proof of the important Banach-Mackey theorem (see, for example, [4, §20.11(3)]). It is then convenient to be able to check if this happens with a minimum of requirements. This is so for sequentially complete absolutely convex bounded subsets of a locally convex space ([4, §20.11(2)]) and for convex relatively countably compact subsets ([2, p.17]).

Let (E, τ) be a locally convex space. An *adherent point of a filter* $(F_i)_I$ in E is an element in $\bigcap_I \overline{F_i}$. An *adherent point of a net* (x_i) in E is an adherent point (in the former sense) of the filter $(F_i := \{x_j : j \geq i\})$.

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We collect in the following definition several of the most useful concepts when dealing with compactness in a general locally convex space.

Definition 1. A subset A of a locally convex space (E, τ) is said to be

- *(relatively) countably compact* ((R)NK) if every sequence of points in A has an adherent point in A (in \bar{A});
- *(relatively) sequentially compact* ((R)SK) if every sequence in A has a subsequence which converges to a point in A (in \bar{A});
- *(relatively) compact* ((R)K) if every net in A has an adherent point in A (in \bar{A});
- *(relatively) convex-compact* ((R)CK) if the following holds: suppose that $K_1 \supset K_2 \supset \dots$ is a decreasing sequence of closed convex subsets of E for which all the intersections $K_n \cap A$ are non-empty; then the sequence $(K_n \cap A)$ has an adherent point in A (in \bar{A}).

Obviously, (R)K sets are (R)NK and (R)SK sets are (R)NK, too. It is easy to prove (see, for example, [4, §24.3(3)]) that every (R)NK set is (R)CK. The converse does not hold. A RCK set is always bounded ([4, §24.3(3)]). The closure of a RCK set does not need to be CK (see Example 9 below). The concept of (R)CK is due to Šmulian (see references in [1, Ch. III, §2]).

As we mentioned before, the following result holds:

Theorem 2 ([2], p. 17). *Every convex RNK subset A of a locally convex space E is contained in a Banach disk $U \subset E$.*

In this paper, we extend this result to the class of RCK sets. We provide also an upper bound for classes of sets which are contained in a Banach disc together with some other results about CK sets; in particular, we prove that $\sigma(E, E')$ -(R)CK implies $\sigma(E, E')$ -(R)K when there is a sequence of $\sigma(E', E)$ -RNK subsets of E' whose union is $\sigma(E', E)$ -dense in E' (in particular, if (E, τ) has a coarser metrizable locally convex topology).

2. BANACH DISCS

The following result is well known, so its proof will be omitted.

Theorem 3. *Let A be a bounded subset of a complete locally convex space E . Then, the map*

$$T: \ell_1(A) \longrightarrow E$$

given by

$$(\alpha_a)_{a \in A} \xrightarrow{T} \sum_{a \in A} \alpha_a a$$

is well defined and continuous and $D := T(B_{\ell_1(A)})$ is a Banach disc.

The following result extends [4, §20.11(2)] and [2, p. 17, Lemma] to the class of convex and CK subsets of an arbitrary locally convex space.

Theorem 4. *Every convex, RCK subset A of a locally convex space (E, \mathcal{T}) is contained in a Banach disc $D \subset E$.*

Proof. Let \tilde{E} be the completion of E and D the Banach disc in \tilde{E} constructed in Theorem 3. We shall prove that, in fact, $D \subset E$. This will conclude the proof.

To that end, let us denote by \tilde{E}_D the Banach space generated by D in \tilde{E} and let $\|\cdot\|_D$ be its norm. Given $a \in D$, it can be written as $a = \sum_{i=1}^{\infty} \alpha_i a_i$ (the sum converges in $\|\cdot\|_D$ and, in particular, also in \mathcal{T}), where $a_i \in A$, $\alpha_i \neq 0$ for every i and $\sum_{i=1}^{\infty} |\alpha_i| \leq 1$. We can split this sum as

$$a = \underbrace{\sum_{i=1}^{\infty} \beta_i b_i}_b - \underbrace{\sum_{i=1}^{\infty} \gamma_i c_i}_c,$$

where $\beta_i > 0$, $\gamma_i > 0$, $b_i \in A$ and $c_i \in A$.

Let $s_n = \sum_1^n \beta_i$, $s = \sum_1^{\infty} \beta_i$ and $x_n = (1/s_n) \sum_1^n \beta_i b_i$. Then $x_n \in A$ and $(x_n)_{\|\cdot\|_D}$ -tends to $(1/s)b \in \tilde{E}$.

Let K_n be the sequence of closed convex sets in (E, \mathcal{T}) defined as

$$K_n = \overline{\text{conv}} \left[\{x_i : i \in \mathbb{N}\} \cap \left(\frac{1}{s} b + \frac{1}{n} D \right) \right]$$

Thus, $K_1 \supset K_2 \supset \dots$ and $K_n \cap A \neq \emptyset$ (observe that D contains the open unit ball in the norm $\|\cdot\|_D$). Therefore, there exists $x \in E$ such that $x \in \bigcap_1^{\infty} \overline{(K_n \cap A)}$. Let $U(0)$ be any closed neighborhood of 0 in (E, \mathcal{T}) . By the fact that D is bounded, there exists $n \in \mathbb{N}$ such that $(1/n)D \subset U(0)$. Then (all closures taken in (E, \mathcal{T})),

$$\begin{aligned} x \in \overline{K_n \cap A} &\subset K_n \subset \overline{\text{conv}}[(1/s)b + (1/n)D] \\ &= \overline{[(1/s)b + (1/n)D]} \subset \overline{[(1/s)b + U(0)]} = (1/s)b + U(0). \end{aligned}$$

Therefore, $x = (1/s)b$, so $b \in E$. Analogously, $c \in E$. This implies, finally, that $a \in E$. \square

If A is absolutely convex, we can be a little bit more precise, since we have $D = A$ if A is CK. In case that A is just RCK, we can only say that $A \subseteq D \subseteq \bar{A}$.

Corollary 5. *Let A be an absolutely convex, (R)CK subset of a locally convex space. Then A is a Banach disc (A is contained in a Banach disc D such that $D \subseteq \bar{A}$).*

Since convex RCK sets are contained in a Banach disc, we can use, for example, [4, §20.11(3)] to conclude the following result.

Corollary 6. *Every convex, RCK subset A of a locally convex space (E, τ) is strongly bounded, i.e., $\sup_{u \in B, x \in A} |u(x)| < \infty$, for each $\sigma(E', E)$ -bounded set $B \in E'$.*

Further criteria for weak compactness use, for example, the interchangeable limit condition, as in [5] and [3]. Given a locally convex space (E, τ) , we say that two sets, $A \subset E$ and $B \subset E'$, *interchange limits* (and we write $A \sim B$) if $\lim_n \lim_m \langle x_n, x'_m \rangle = \lim_m \lim_n \langle x_n, x'_m \rangle$ whenever (x_n) (resp. (x'_m)) is a sequence in A (resp. in B) such that both iterated limits exist. Let $\mu(E, E')$ be the *Mackey topology* on E , i.e., the topology on E of the uniform convergence on the family of all absolutely convex and $\sigma(E', E)$ compact subsets of E' . A central result in [3] is that *a bounded subset of a $\mu(E, E')$ -quasicomplete locally convex space E is $\sigma(E, E')$ -RK if and only if it interchanges limits with every absolutely convex $\sigma(E', E)$ -K subset of E'* . If A is RCK then $A \sim B$ for every absolutely convex and $\sigma(E', E)$ -K subset of E' . This can be easily deduced from the following fact. Here, E'^* denotes the algebraic dual of the topological dual E' of E .

Lemma 7. *Every (R)CK set A in a locally convex space (E, τ) satisfies the following property: for every sequence (f_n) in E' and for every element $z \in \bar{A}^{(E'^*, \sigma(E'^*, E'))}$, there exists $a \in A$ ($\in \bar{A}^{(E, \sigma(E, E'))}$) such that $\langle z - a, f_n \rangle = 0$ for all $n \in \mathbb{N}$.*

This can be proved just by considering the decreasing sequence of closed convex sets $K_n := \{x \in E : \sup\{\langle z - x, f_i \rangle : i = 1, 2, \dots, n\} \leq 1/n\}$.

With the following example we bound the class of sets in (E, τ) which are included in a Banach disc.

Example 8. There exists a locally convex space (E, τ) and a bounded subset of E interchanging limits with every absolutely convex $\sigma(E', E)$ -K subset of E' and yet not included in a Banach disc.

Proof. Let $(E, \tau) := (\ell_1, \sigma(\ell_1, \varphi))$, where $\varphi \subset \ell_\infty$ is the linear space of all eventually zero sequences (so $\sigma(\ell_1, \varphi)$ is the topology on ℓ_1 of the pointwise convergence) and let $A := \prod_{n=1}^{\infty} [-n, n] \cap \ell_1$, a convex and bounded subset of E .

Observe that A is not $\beta(\ell_1, \varphi)$ -bounded, where $\beta(\ell_1, \varphi)$ denotes the *strong* topology on ℓ_1 for the dual pair $\langle \ell_1, \varphi \rangle$, i.e., the topology of the uniform convergence on all the $\sigma(\varphi, \ell_1)$ -bounded subsets of φ . In order to see this, notice that the set $M :=$

$[-1, 1]^{\mathbb{N}} \cap \varphi$ is $\sigma(\varphi, \ell_1)$ -bounded and yet $\sup\{\langle ne_n, e_n \rangle : n \in \mathbb{N}\} = +\infty$, where e_n is the n -th vector of the canonical basis of ℓ_1 .

We shall prove that $A \sim U$ for every absolutely convex and $\sigma(\varphi, \ell_1)$ -compact subset of φ . The set U is $\beta(\varphi, \ell_1)$ -bounded by the Banach-Mackey theorem (see, for example, [4, §20.11(3)]). The topology $\beta(\varphi, \ell_1)$ is compatible with the dual pair $\langle \mathbb{R}^{\mathbb{N}}, \varphi \rangle$ (this can be seen as follows: given $x := (x_n) \in \mathbb{R}^{\mathbb{N}}$, the sequence $(\sum_{k=1}^n x_k e_k)_n$ is in ℓ_1 and $\sigma(\mathbb{R}^{\mathbb{N}}, \varphi)$ -converges to x , so x is in the $\sigma(\mathbb{R}^{\mathbb{N}}, \varphi)$ -closure of a $\sigma(\ell_1, \varphi)$ -bounded subset of ℓ_1). It follows then that U lies in a finite-dimensional subspace of φ , say $\text{span}\{w_i : i = 1, 2, \dots, k\}$. Assume now that for two sequences (a_m) in A and (u_n) in U the iterated limits

$$\lim_n \lim_m \langle a_m, u_n \rangle, \quad \lim_m \lim_n \langle a_m, u_n \rangle$$

exists. Put $u_n := \sum_{i=1}^k \lambda_i^n w_i$, $n \in \mathbb{N}$, where λ_i^n are real numbers. Let $u_0 := \sum_{i=1}^k \lambda_i^0 w_i$ be a $\sigma(\varphi, \ell_1)$ -adherent point of the sequence (u_n) and $a_0 \in \mathbb{R}^{\mathbb{N}}$ a $\sigma(\mathbb{R}^{\mathbb{N}}, \varphi)$ -adherent point of the sequence (a_n) . It follows that

$$\lim_n \lim_m \langle a_m, u_n \rangle = \lim_n \langle a_0, u_n \rangle, \quad \lim_m \lim_n \langle a_m, u_n \rangle = \lim_m \langle a_m, u_0 \rangle = \langle a_0, u_0 \rangle.$$

The element u_0 is also $\sigma(\varphi, \varphi)$ -adherent to the sequence (u_n) , so, in particular, λ_i^0 is adherent to the sequence $(\lambda_i^n)_n$ for $i = 1, 2, \dots, k$. It follows that

$$\langle a_0, u_n \rangle = \sum_{i=1}^k \lambda_i^n \langle a_0, w_i \rangle \xrightarrow{n} \sum_{i=1}^k \lambda_i^0 \langle a_0, w_i \rangle = \langle a_0, u_0 \rangle$$

and this proves the assertion. Again by the Banach-Mackey theorem, A is not contained in a Banach disc as it is not $\beta(\ell_1, \varphi)$ -bounded. \square

3. SOMETIMES CONVEX-COMPACTNESS IMPLIES COMPACTNESS

In ([2, p. 9]), an example of an absolutely convex sequentially compact subset A in a locally convex space (E, τ) such that \bar{A} is not countably compact is given. We can prove that, in fact, \bar{A} is not even convex-compact. This provides an example of a relatively convex-compact set whose closure is not convex-compact.

Example 9. There exists a locally convex space with an absolutely convex, sequentially compact (and then countably compact and so convex-compact) subset whose closure is not convex-compact.

To present the example, take a X_n be a disjoint sequence of uncountable sets and define $X := \bigcup_{n=1}^{\infty} X_n$. For $f: X \rightarrow \mathbb{R}$, the support of f is defined as $\text{supp } f := \{x \in X \mid f(x) \neq 0\}$. Let the vector space

$$E := \left\{ f: X \rightarrow \mathbb{R} \mid \exists n \in \mathbb{N}: \text{supp } f \cap \bigcup_{m=n}^{\infty} X_m \text{ is countable} \right\}$$

be endowed with the restriction of the topology \mathcal{T}_p in \mathbb{R}^X of pointwise convergence on X , denoted again \mathcal{T}_p . Clearly, (E, \mathcal{T}_p) turns out to be a locally convex space. By using a diagonal procedure, it is easy to see that the set

$$A := \{f \in E \mid \text{supp } f \text{ iscountable, } \|f\|_{\infty} \leq 1\}$$

is sequentially compact. However, the closure

$$\bar{A}^{(E, \mathcal{T}_p)} \quad (= \{f \in E \mid \|f\|_{\infty} \leq 1\})$$

is not convex-compact. To see this, let f_n be the characteristic function of $\bigcup_{i=1}^n X_i$, $n \in \mathbb{N}$. The sequence (f_n) is in $\bar{A}^{(E, \mathcal{T}_p)}$ and \mathcal{T}_p -converges to $f \in \mathbb{R}^X$, the characteristic function of X , which is not in E . Consider now the sets

$$K_n = \overline{\text{conv}}\{f_i\}_n^{\infty}, \quad n \in \mathbb{N}.$$

They form a decreasing sequence of closed convex sets in E such that $K_n \cap \bar{A}^{(E, \mathcal{T}_p)} \neq \emptyset$. If $g \in K_n$ then $g(x) = 1$ for all $x \in \bigcup_{k=1}^n X_k$. Thus the sequence $K_n \cap \bar{A}^{(E, \mathcal{T}_p)}$ has no adherent point in E . □

In Fréchet spaces or in locally convex spaces E with $\sigma(E', E)$ -separable dual E' , several concepts of weak compactness coincide (theorems of Eberlein and Eberlein-Šmulian, see for example [4, §24]). A criterium for weak compactness in the spirit of the Eberlein-Šmulian theorem is given in [2, 3.10]:

Theorem 10. *A locally convex space E which admits $\sigma(E', E)$ -relatively countably compact sets $M_n \subset E'$, $n \in \mathbb{N}$, such that*

$$E' = \bigcup_{n=1}^{\infty} \overline{M_n}^{\sigma(E', E)}$$

is $\sigma(E, E')$ -angelic (i.e., every $\sigma(E, E')$ -relatively countably compact subset of E is $\sigma(E, E')$ -relatively compact, and its $\sigma(E, E')$ -closure coincides with its $\sigma(E, E')$ -sequential closure). In particular, the following classes of subsets coincide:

- (i) $\sigma(E, E')$ -RNK, $\sigma(E, E')$ -RSK, $\sigma(E, E')$ -RK,
- (ii) $\sigma(E, E')$ -NK, $\sigma(E, E')$ -SK, $\sigma(E, E')$ -K.

We shall prove that there is a similar Eberlein-Šmulian theorem for the class of (R)CK sets. In fact, it can be stated for a more general class of sets (see the following definition) including the CK ones.

Definition 11. A subset A of a locally convex space (E, \cdot) is said to be $\sigma(E, E')$ -*(relatively) numerably compact* (briefly, $\sigma(E, E')$ -(R) Ξ K) if it is bounded and, given a sequence (a_n) in A and a $\sigma(E'^*, E')$ -adherent point $a'^* \in E'^*$ of (a_n) then, for any sequence (x'_n) in E' , there exists a point $a \in A \cap \overline{\text{span}}\{a_n; n \in \mathbb{N}\}$ ($a \in \overline{\text{span}}\{a_n; n \in \mathbb{N}\}$) such that $\langle a'^* - a, x'_n \rangle = 0$ for all $n \in \mathbb{N}$.

It is easy to see that $\sigma(E, E')$ -(R)CK sets are $\sigma(E, E')$ -(R) Ξ K. Indeed, it is easy to check that they are bounded; the second condition can be proved just by considering the decreasing sequence of closed convex sets $K_n := \{x \in \overline{\text{span}}\{a_n; n \in \mathbb{N}\}; \sup\{|\langle a'^* - x, x'_i \rangle|; i = 1, 2, \dots, n\} \leq 1/n\}$.

Theorem 12. Let (E, \cdot) be a locally convex space such that in E' there is a sequence (M_n) of $\sigma(E', E)$ -RNK subsets such that $\bigcup_{n \in \mathbb{N}} M_n$ is $\sigma(E', E)$ -dense in E' (in particular, this is the case if $E(T)$ has a locally convex topology coarser than T and metrizable, or more particularly, if E' is $\sigma(E', E)$ -separable). Then every $\sigma(E, E')$ -(R) Ξ K set is $\sigma(E, E')$ -(R)K.

Proof. Let us assume first that $(E', \sigma(E', E))$ is separable. Let A be a $\sigma(E, E')$ -(R) Ξ K subset of E and $a'^* \in \bar{A}^{\sigma(E'^*, E')}$ a $\sigma(E'^*, E')$ -adherent point of a sequence (a_n) in A . By definition, given a countable subset $N \subset E'$, there exists $a_N \in A \cap \overline{\text{span}}\{a_n; n \in \mathbb{N}\}$ ($a_N \in \overline{\text{span}}\{a_n; n \in \mathbb{N}\}$), such that $a_N|_N = a'^*|_N$. Let D be a countable and $\sigma(E', E)$ -dense subset of E' and let $x' \in E'$ be an arbitrary point. Let us consider the points $a_{D \cup x'}$ and a_D in E . They coincide on D , so $a_{D \cup x'} = a_D$. Moreover, $\langle a'^*, x' \rangle = \langle a_{D \cup x'}, x' \rangle (= \langle a_D, x' \rangle)$. Therefore $a'^*|_{E'} = a_D|_{E'}$, and so $a'^* \in E$ and A is $\sigma(E, E')$ -(R)K since it is bounded. Assume now that (E, \cdot) satisfies the requirement and let (a_n) be any sequence in A . Let us consider the separable locally convex space $F = \overline{\text{span}}\{a_n\}_{n \in \mathbb{N}}$. Its dual is $F' = q(E') = E'/F^\perp$, where $q: E' \rightarrow E'/F^\perp$ is the canonical mapping. It is easy to see that $q(M_n)$ is $\sigma(F', F)$ -RNK and that $\bigcup_{n \in \mathbb{N}} q(M_n)$ is dense in $(F', \sigma(F', F))$. Furthermore, the dual of $(F', \sigma(F', F))$ is F , which is separable. Therefore we can apply Theorem 10 to conclude that $q(M_n)$ is $\sigma(F', F)$ -RK and so, metrizable in $(F', \sigma(F', F))$. Thus, $q(M_n)$ is separable, and so it is $(F', \sigma(F', F))$, too.

We claim now that $A \cap F$ is $\sigma(E, E')$ -(R) Ξ K. Indeed, let f'^* be a $\sigma(F'^*, F')$ -adherent point of a given sequence (x_n) in F , and let (f'_n) be a sequence in F' . The

element $f'^* \circ q$ ($\in E'^*$) is a $\sigma(E'^*, E')$ -adherent point to (x_n) in E , and there exists a sequence (e'_n) in E' such that $q(e'_n) = f'_n$ for all $n \in \mathbb{N}$. By the assumption, we can find $a \in A \cap \overline{\text{span}}\{x_n; n \in \mathbb{N}\} \subset A \cap F$ ($a \in \overline{\text{span}}\{x_n; n \in \mathbb{N}\} (\subset F)$) such that $\langle e'^* - a, e'_n \rangle = 0$, i.e., $\langle f'^* - a, f'_n \rangle = 0$, for all $n \in \mathbb{N}$. This proves the claim.

We can then apply the first part of the proof to the set $A \cap F$ to obtain that the set $\{a_n : n \in \mathbb{N}\}$ is $\sigma(F, F')$ -RNK (with an adherent point in A (in $\overline{A}^{\sigma(F, F')}$)). This implies that A is $\sigma(E, E')$ -(R)NK. By Theorem 10, A is $\sigma(E, E')$ -(R)K. \square

We can extend now Theorem 10 to include the class of $\sigma(E, E')$ -(R) Ξ K sets (and so $\sigma(E, E')$ -(R)CK sets).

Theorem 13. *Let $(E,)$ be a locally convex space which admits $\sigma(E', E)$ -relatively countably compact sets $M_n \subset E'$, $n \in \mathbb{N}$, such that*

$$E' = \bigcup_{n=1}^{\infty} \overline{M_n}^{\sigma(E', E)} .$$

Then, the following classes of sets (in the topology $\sigma(E, E')$) coincide:

- (i) $\sigma(E, E')$ -K, $\sigma(E, E')$ -SK, $\sigma(E, E')$ -NK, $\sigma(E, E')$ -CK, $\sigma(E, E') - \Xi$ K.
- (ii) $\sigma(E, E')$ -RK, $\sigma(E, E')$ -RSK, $\sigma(E, E')$ -RNK, $\sigma(E, E')$ -RCK, $\sigma(E, E')$ -R Ξ K.

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Authors' addresses: I. Monterde, Universidad Politécnica de Valencia, e-mail: nachomonterde@gmail.com; V. Montesinos, Instituto de Matemática Pura y Aplicada, Universidad Politécnica de Valencia, e-mail: vmontesinos@mat.upv.es.