Czechoslovak Mathematical Journal

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Czechoslovak Mathematical Journal, Vol. 59 (2009), No. 3, 781–789

Persistent URL: `http://dml.cz/dmlcz/140516`

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ON THE $2k$-TH POWER MEAN OF $\frac{L'}{L}(1,\chi)$ WITH THE WEIGHT OF GAUSS SUMS

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Cordially dedicated to editor V. Dlab

(Received March 5, 2008)

Abstract. The main purpose of this paper is to study the hybrid mean value of $\frac{L'}{L}(1,\chi)$ and Gauss sums by using the estimates for trigonometric sums as well as the analytic method. An asymptotic formula for the hybrid mean value $\sum\limits_{\chi\neq\chi_0}|\tau(\chi)|\,|\frac{L'}{L}(1,\chi)|^{2k}$ of $\frac{L'}{L}$ and Gauss sums will be proved using analytic methods and estimates for trigonometric sums.

Keywords: Dirichlet L-function, Gauss sums, asymptotic formula

MSC 2010: 11M20

§1. Introduction

Let χ be the Dirichlet character modulo $q \geqslant 3$. For any integer m, the classical Gauss sum $G(m,\chi)$ is defined as

$$G(m,\chi)=\sum_{a=1}^{q}\chi(a)e\Big(\frac{ma}{q}\Big),$$

where $e(y)=\mathrm{e}^{2\pi\mathrm{i}y}$. In particular, when $m=1$, we denote by $\tau(\chi)=G(1,\chi)=\sum\limits_{a=1}^{q}\chi(a)e(a/q)$.

Perhaps the most important property of $G(m,\chi)$ is: when $(m,q)=1$ and χ is a primitive character modulo q, we have $|G(m,\chi)|=\sqrt{q}$. For a nonprimitive

This work is supported by N.S.F. (10601039) of P.R. China.

character, the value of $|G(m,\chi)|$ varies, i.e. the value of $|G(m,\chi)|$ is irregular as χ varies. However, $G(m,\chi)$ enjoys many good value distribution properties in some problems of the weighted mean value.

Similarly, many books about the analytic number theory include a discussion on the properties of $\tau(\chi)$ (see Ref. [1]–[3]). Maybe the most important property of $\tau(\chi)$ is: when χ is a primitive character mod q, then $|\tau(\chi)| = \sqrt{q}$. For a nonprimitive character, the value of $\tau(\chi)$ is irregular as χ varies and sometimes it may be zero. But $\tau(\chi)$ surprisingly enjoys many good value distribution properties in some problems of the weighted mean value.

Yi Yuan and Zhang Wenpeng [4] studied the first power mean of the Dirichlet L-function with the weight of Gauss sums and obtained the asymptotic formula

$$\sum_{\chi\neq\chi_0} |G(m,\chi)|^2 |L(1,\chi)| = \varphi^2(q)\cdot {\sum_{n=1}^{\infty}}' \frac{r^2(n)}{n^2} + O(q^{\frac{3}{2}+\varepsilon}),$$

where q is an integer with $q > 2$, m is an integer satisfying $(m,q) = 1$, χ_0 is the principal character modulo q, $\sum\limits_{n}{}'$ denotes the sum over all n which are coprime with q, $\varphi(q)$ is the Euler function, ε is any given positive number, and $r(n)$ is defined as follows: for any prime p and positive integer α, $r(1) = 1$, $r(p^\alpha) = 4^{-\alpha}C_{2\alpha}^{\alpha}$, $C_{2\alpha}^{\alpha} = (2\alpha)!/(\alpha!)^2$. For any positive integer n, when its standard factorization is $p_1^{\alpha_1}p_2^{\alpha_2}\dots p_k^{\alpha_k}$, we can easily get

$$r(n) = \frac{1}{4^{\alpha_1+\alpha_2+\dots+\alpha_k}} C_{2\alpha_1}^{\alpha_1} C_{2\alpha_2}^{\alpha_2}\dots C_{2\alpha_k}^{\alpha_k}.$$

Yi Yuan and Zhang Wenpeng [5] studied the $2k$-th power mean of the Dirichlet L-function with the weight of Gauss sums and obtained

$$\begin{aligned}\sum_{\chi\neq\chi_0} |\tau(\chi)|^m |L(1,\chi)|^{2k} = {} & N^{\frac{m}{2}-1}\varphi^2(N)\zeta^{2k-1}(2)\prod_{p|q}\left(1-\frac{1}{p^2}\right)^{2k-1} \\ & \times \prod_{p\nmid q}\left(1-\frac{1-C_{2k-2}^{k-1}}{p^2}\right)\prod_{p|M}\left(p^{\frac{m}{2}+1}-2p^{\frac{m}{2}}+1\right) + O\left(p^{\frac{m}{2}+\varepsilon}\right),\end{aligned}$$

where $q \geqslant 3$ is an integer and $q = MN$, $M = \prod\limits_{p\|q} p$, $(M,N) = 1$, m is any positive number, k is any positive integer, $\prod\limits_{p\|q}$ denotes all prime factors of q such that $p \mid q$ and $p^2 \nmid q$, $\varphi(q)$ is the Euler function, $\zeta(s)$ is the Riemann Zeta function and ε is any given positive number.

Let χ be the Dirichlet character modulo q and let $L(s,\chi)$ denote the corresponding Dirichlet L-function. $\frac{L'}{L}(1,\chi)$ has long history and plays an important role in number theory [6], but one can hardly estimate $\frac{L'}{L}(1,\chi)$. In fact, it enjoys good mean value properties. Zhang Wenpeng [7] studied the asymptotic properties of the sums

$$\sum_{q\leqslant Q}\frac{q}{\varphi(q)}\sum_{\chi\neq\chi_0}\left|\frac{L'}{L}(1,\chi)\right|^4,\quad \sum_{q\leqslant Q}\frac{1}{\varphi(q)}\sum_{\chi\neq\chi_0}\left|\frac{L'}{L}(1,\chi)\right|^4,$$

where $Q>3$ is a real number and $\varphi(q)$ is the Euler function. Liu Huaning and Zhang Xiaobeng [8] studied the mean value of $|\frac{L'}{L}(1,\chi)|^{2k}$ and obtained

$$\sum_{\substack{\chi \bmod q\\ \chi\neq\chi_0}}\left|\frac{L'}{L}(1,\chi)\right|^{2k}=A(k,q)\varphi(q)+O(q^{\varepsilon}),$$

where

$$A(k,q)=\sum_{\substack{n=1\\(n,q)=1}}^{\infty}\frac{\tau_k^2(n)}{n^2} \tag{1}$$

is a constant depending on k and q, with

$$\tau_k(n)=\sum_{m_1m_2\ldots m_k=n}\Lambda(m_1)\Lambda(m_2)\ldots\Lambda(m_k), \tag{2}$$

$\Lambda(n)$ the Mangoldt function, $\varphi(q)$ the Euler function and ε any given positive number.

In what follows we shall consider the hybrid mean value of $\frac{L'}{L}$ with Gauss sums whose asymptotic behavior has not been studied hitherto. We will use the estimates for trigonometric sums and the analytic method to study the hybrid mean value $\sum\limits_{\chi\neq\chi_0}|\tau(\chi)|^m|\frac{L'}{L}(1,\chi)|^{2k}$, and obtain a sharper asymptotic formula for it. That is, we shall prove the following theorem.

Theorem. *Let $q=MN$, $M=\prod\limits_{p\|q}p$, $(M,N)=1$. Then for any positive number m and positive integer k we have the asymptotic formula*

$$\sum_{\chi\neq\chi_0}|\tau(\chi)|^m\left|\frac{L'}{L}(1,\chi)\right|^{2k}=A(k,q)N^{\frac{m}{2}-1}\varphi^2(N)\prod_{p|M}\left(p^{\frac{m}{2}+1}-2p^{\frac{m}{2}}+1\right)+O\left(q^{\frac{m}{2}+\varepsilon}\right),$$

where $A(k,q)$ is defined as in (1), $\prod\limits_{p\|q}$ denotes all the prime factors of q such that $p\mid q$ and $p^2\nmid q$, ε is any given positive number.

Throughout the paper, we denote by $\mu(n)$ the Möbius function, and ε always denotes a sufficiently small positive real number which may be different at various occurrence.

§2. Some lemmas

To complete the proof of the theorem, we need the following lemmas.

Lemma 1. *Let $q = uv$, $u \geqslant 2$, $v \geqslant 2$ and $(u, v) = 1$. Then for any χ mod q, there exist one and only one character χ_u mod u one and only one character χ_v mod v such that $\chi = \chi_u \chi_v$ and*

$$|\tau(\chi)| = |\tau(\chi_u)| \times |\tau(\chi_v)|.$$

Proof. See Theorem 13.3.1. of [2]. □

Lemma 2. *Let q and r be integers with $q \geqslant 2$ and $(r, q) = 1$. Then we have the identities*

$$\sum_{\chi \bmod q}^{*} \chi(r) = \sum_{d|(q,r-1)} \mu\Big(\frac{q}{d}\Big)\varphi(d), \quad J(q) = \sum_{d|q} \mu(d)\varphi\Big(\frac{q}{d}\Big),$$

where $\sum^*$ denotes the summation over all primitive characters, $\varphi(q)$ is the Euler function and $J(q)$ denotes the number of primitive characters mod q.

Proof. From the properties of characters, we know that for any character χ mod q there exists one and only one $d \mid q$ with a primitive character ${\chi_d}^*$ mod d such that $\chi = {\chi_d}^* \chi_q^0$, where χ_q^0 denotes the principal character mod q. So we have

$$\sum_{\chi \bmod q} \chi(r) = \sum_{d|q} \sum_{\chi \bmod d}^{*} \chi(r)\chi_q^0(r) = \sum_{d|q} \sum_{\chi \bmod d}^{*} \chi(r).$$

Combining this formula with the Möbius transformation and noting the identity

$$\sum_{\chi \bmod q} \chi(r) = \begin{cases} \varphi(q), & \text{if} \quad r \equiv 1 \pmod q, \\ 0, & \text{otherwise,} \end{cases}$$

we have

$$\sum_{\chi \bmod q}^{*} \chi(r) = \sum_{d|q} \mu(d) \sum_{\chi \bmod q/d} \chi(r) = \sum_{d|(q,r-1)} \mu\Big(\frac{q}{d}\Big)\varphi(d).$$

Taking $r = 1$, we immediately get

$$J(q) = \sum_{d|q} \mu(d)\varphi\Big(\frac{q}{d}\Big).$$

This proves Lemma 2. □

Lemma 3. *Let p be a prime, α a positive integer and $\alpha \geqslant 2$, $n = p^{\alpha}$. Then for any nonprimitive character χ_1 mod n we have the identity*

$$\sum_{a=1}^{p^{\alpha}} \chi_1(a) e\Big(\frac{a}{p^{\alpha}}\Big) = 0.$$

P r o o f. See [5]. □

Lemma 4. *Let $N = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_s^{\alpha_s}$, $\alpha_i \geqslant 2$, $1 \leqslant i \leqslant s$, be a positive integer, let a positive integer M have no square factor and let $(M, N) = 1$, $q = MN$. Then for any given positive integer k and $d \mid M$ we have*

$$\sideset{}{^*}\sum_{\chi \bmod Nd} \Big|\frac{L'}{L}(1, \chi\chi_M^0)\Big|^{2k} = A(k, q) \frac{\varphi^2(N)}{N} J(d) + O((MN)^{\varepsilon}),$$

where $\sum^*$ denotes the summation over all primitive characters.

P r o o f. Let $\chi'\chi_M^0$ be the nonprincipal real character mod MN, then from the properties of the L-function and from C. L. Siegel's theorem [9] we get

$$\frac{L'}{L}(1, \chi'\chi_M^0) \ll \frac{(MN)^{\varepsilon} \cdot \log^2(MN)}{C(\varepsilon)},$$

where $C(\varepsilon)$ is a constant depending on ε.

For any complex character mod MN with $MN \leqslant \exp\big(C_1\sqrt{\log x}\big)$, where C_1 is any positive constant and $\exp(y) = \mathrm{e}^y$, we get from [6]

$$\psi(x, \chi\chi_M^0) = \sum_{n \leqslant x} \chi(n)\chi_M^0(n)\Lambda(n) \ll x \cdot \exp(-C_2\sqrt{\log x})$$

for some positive C_2 depending only on C_1.

Let $T \geqslant \exp(\log^2(MN)/C_1^2)$, then by Abel's identity we have

$$\begin{aligned}\frac{L'}{L}(1, \chi\chi_M^0) &= \sum_{n=1}^{\infty} \frac{\chi(n)\chi_M^0(n)\Lambda(n)}{n} \\ &= \sum_{1 \leqslant n \leqslant T} \frac{\chi(n)\chi_M^0(n)\Lambda(n)}{n} + \int_T^{\infty} \frac{\sum\limits_{T < n \leqslant y} \chi(n)\chi_M^0(n)\Lambda(n)}{y^2}\,\mathrm{d}y \\ &= \sum_{1 \leqslant n \leqslant T} \frac{\chi(n)\chi_M^0(n)\Lambda(n)}{n} + O\Big(\frac{\log T}{\exp(C_2\sqrt{\log T})}\Big).\end{aligned}$$

So if we write $\tau_k(n)$ as in (2), we have

$$
\begin{aligned}
&\sideset{}{^*}\sum_{\chi \bmod Nd} \Big|\frac{L'}{L}(1,\chi\chi_M^0)\Big|^{2k} \\
&\quad = \sideset{}{^*}\sum_{\chi \bmod Nd} \Big|\sum_{1\leqslant n\leqslant T} \frac{\chi(n)\chi_M^0(n)\Lambda(n)}{n}\Big|^{2k} \\
&\qquad + O\Big(\frac{J(Nd)\cdot\log^{2k}T}{\exp(C_2\sqrt{\log T})}\Big) + O\Big(\frac{(MN)^{2k\varepsilon}\cdot\log^{4k}(MN)}{C^{2k}(\varepsilon)}\Big) \\
&\quad = \sideset{}{^*}\sum_{\chi \bmod Nd} \Big|\sum_{1\leqslant n\leqslant T^k} \frac{\chi(n)\chi_M^0(n)\tau_k(n)}{n}\Big|^{2} \\
&\qquad + O\Big(\frac{J(Nd)\cdot\log^{2k}T}{\exp(C_2\sqrt{\log T})}\Big) + O\Big(\frac{(MN)^{2k\varepsilon}\cdot\log^{4k}(MN)}{C^{2k}(\varepsilon)}\Big) \\
&\quad = \sum_{1\leqslant n_1\leqslant T^k}\ \sum_{1\leqslant n_2\leqslant T^k} \frac{\chi_M^0(n_1\overline{n_2})\tau_k(n_1)\tau_k(n_2)}{n_1n_2} \sideset{}{^*}\sum_{\chi \bmod Nd} \chi(n_1)\bar{\chi}(n_2) \\
&\qquad + O\Big(\frac{J(Nd)\cdot\log^{2k}T}{\exp(C_2\sqrt{\log T})}\Big) + O\Big(\frac{(MN)^{2k\varepsilon}\cdot\log^{4k}(MN)}{C^{2k}(\varepsilon)}\Big) \\
&\quad = \sideset{}{'}\sum_{1\leqslant n_1\leqslant T^k}\ \sideset{}{'}\sum_{1\leqslant n_2\leqslant T^k} \frac{\chi_M^0(n_1\overline{n_2})\tau_k(n_1)\tau_k(n_2)}{n_1n_2} \sum_{l\mid(Nd,\,n_1\overline{n_2}-1)} \mu\Big(\frac{Nd}{l}\Big)\varphi(l) \\
&\qquad + O\Big(\frac{J(Nd)\cdot\log^{2k}T}{\exp(C_2\sqrt{\log T})}\Big) + O\Big(\frac{(MN)^{2k\varepsilon}\cdot\log^{4k}(MN)}{C^{2k}(\varepsilon)}\Big) \\
&\quad = \sum_{l\mid Nd} \mu\left(\frac{Nd}{l}\right)\varphi(l) \sideset{}{'}\sum_{1\leqslant n_1\leqslant T^k}\ \sideset{}{'}\sum_{\substack{1\leqslant n_2\leqslant T^k\\ n_1\equiv n_2 \ (\mathrm{mod}\ l)}} \frac{\chi_M^0(n_1\overline{n_2})\tau_k(n_1)\tau_k(n_2)}{n_1n_2} \\
&\qquad + O\Big(\frac{J(Nd)\cdot\log^{2k}T}{\exp(C_2\sqrt{\log T})}\Big) + O\Big(\frac{(MN)^{2k\varepsilon}\cdot\log^{4k}(MN)}{C^{2k}(\varepsilon)}\Big) \\
&\quad = \sum_{l\mid Nd} \mu\left(\frac{Nd}{l}\right)\varphi(l) \sum_{\substack{1\leqslant n\leqslant T^k\\ (n,NM)=1}} \frac{\tau_k^2(n)}{n^2} \\
&\qquad + \sum_{l\mid Nd} \mu\Big(\frac{Nd}{l}\Big)\varphi(l) \sideset{}{'}\sum_{1\leqslant n_1\leqslant T^k}\ \sideset{}{'}\sum_{\substack{1\leqslant n_2\leqslant T^k\\ n_1\equiv n_2 \ (\mathrm{mod}\ l),\, n_1\neq n_2}} \frac{\chi_M^0(n_1\overline{n_2})\tau_k(n_1)\tau_k(n_2)}{n_1n_2} \\
&\qquad + O\Big(\frac{J(Nd)\cdot\log^{2k}T}{\exp(C_2\sqrt{\log T})}\Big) + O\Big(\frac{(MN)^{2k\varepsilon}\cdot\log^{4k}(MN)}{C^{2k}(\varepsilon)}\Big).
\end{aligned}
$$

Noting that $J(N) = \varphi^2(N)/N$ and $(N,d) = 1$, we get

$$\begin{aligned}
\sum_{\chi \bmod Nd}{}^{*} \Big|\frac{L'}{L}(1, \chi\chi_M^0)\Big|^{2k} &= J(Nd) \sum_{\substack{n=1 \\ (n, NM=1)}}^{\infty} \frac{\tau_k^2(n)}{n^2} \\
&\quad + O\Big(\sum_{l|Nd} \varphi(l) \sum_{1\leqslant n_1 \leqslant T^k}{}' \sum_{\substack{1\leqslant n_2\leqslant T^k \\ n_1\equiv n_2 \pmod{l}, n_1\neq n_2}}{}' \frac{\tau_k(n_1)}{n_1}\frac{\tau_k(n_2)}{n_2}\Big) \\
&\quad + O\Big(\frac{J(Nd)\cdot \log^{2k} T}{\exp(C_2\sqrt{\log T})}\Big) + O\Big(\frac{(MN)^{2k\varepsilon}\cdot \log^{4k}(MN)}{C^{2k}(\varepsilon)}\Big) \\
&= A(k,q)\frac{\varphi^2(N)}{N} J(d) + O((Nd)^{\varepsilon} \log^{2k+2} T) \\
&\quad + O\Big(\frac{J(Nd)\cdot \log^{2k} T}{\exp(C_2\sqrt{\log T})}\Big) + O\Big(\frac{(MN)^{2k\varepsilon}\cdot \log^{4k}(MN)}{C^{2k}(\varepsilon)}\Big).
\end{aligned}$$

Taking

$$T = \max\Big\{\exp\Big(\frac{\log^2(MN)}{C_1^2}\Big), \exp\Big(\frac{\log^2(MN)}{C_2^2}\Big)\Big\},$$

we get immediately

$$\sum_{\chi \bmod Nd}{}^{*} \Big|\frac{L'}{L}(1, \chi\chi_M^0)\Big|^{2k} = A(k,q)\frac{\varphi^2(N)}{N} J(d) + O((MN)^{\varepsilon}).$$

This proves Lemma 4. □

§3. Proof of theorem

In this section we present the proof of the theorem.

Let $q = p_1 p_2 \dots p_k p_{k+1}^{\alpha_{k+1}} \dots p_r^{\alpha_r}$ be the standard factorization of q, where $\alpha_i > 1$, $k+1 \leqslant i \leqslant r$. Let $M = p_1 p_2 \dots p_k$, $N = p_{k+1}^{\alpha_{k+1}} \dots p_r^{\alpha_r}$, so $(M,N) = 1$. For any positive number m and positive integer k, Lemma 1 yields

$$\begin{aligned}
\sum_{\chi\neq\chi_0} |\tau(\chi)|^m \Big|\frac{L'}{L}(1,\chi)\Big|^{2k} &= \sum_{\chi_1 \bmod M} \sum_{\substack{\chi_2 \bmod N \\ \chi_1\chi_2\neq\chi_q^0}} |\tau(\chi_1\chi_2)|^m \Big|\frac{L'}{L}(1,\chi_1\chi_2)\Big|^{2k} \\
&= \sum_{\chi_1 \bmod M} \sum_{\substack{\chi_2 \bmod N \\ \chi_1\chi_2\neq\chi_q^0}} |\tau(\chi_1)|^m |\tau(\chi_2)|^m \Big|\frac{L'}{L}(1,\chi_1\chi_2)\Big|^{2k}.
\end{aligned}$$

From Lemma 2, Lemma 3, Lemma 4 and the above expression, we immediately get

$$\begin{aligned}
\sum_{\chi\neq\chi_0}|\tau(\chi)|^m\left|\frac{L'}{L}(1,\chi)\right|^{2k} &= \sum_{\chi_1 \bmod M}\ {\sum_{\chi_2 \bmod N}}^{*} |\tau(\chi_1)|^m N^{\frac{m}{2}}\left|\frac{L'}{L}(1,\chi_1\chi_2)\right|^{2k} \\
&= \sum_{d|M}\ {\sum_{\chi_1 \bmod d}}^{*}\ {\sum_{\chi_2 \bmod N}}^{*} d^{\frac{m}{2}}N^{\frac{m}{2}}\left|\frac{L'}{L}(1,\chi_1\chi_2\chi_M^0)\right|^{2k} \\
&= \sum_{d|M} d^{\frac{m}{2}}N^{\frac{m}{2}}\left(A(k,q)\frac{\varphi^2(N)}{N}J(d)+O\left((MN)^{\varepsilon}\right)\right) \\
&= A(k,q)N^{\frac{m}{2}-1}\varphi^2(N)\sum_{d|M}d^{\frac{m}{2}}J(d)+O\Bigg(N^{\frac{m}{2}+\varepsilon}M^{\varepsilon}\sum_{d|M}d^{\frac{m}{2}}\Bigg) \\
&= A(k,q)N^{\frac{m}{2}-1}\varphi^2(N)\prod_{p|M}\left(\sum_{j=0}^{1}(p^j)^{\frac{m}{2}}J(p^j)\right)+O\left(q^{\frac{m}{2}+\varepsilon}\right) \\
&= A(k,q)N^{\frac{m}{2}-1}\varphi^2(N)\prod_{p|M}\left(1+p^{\frac{m}{2}}(p-2)\right)+O\left(q^{\frac{m}{2}+\varepsilon}\right) \\
&= A(k,q)N^{\frac{m}{2}-1}\varphi^2(N)\prod_{p|M}\left(p^{\frac{m}{2}+1}-2p^{\frac{m}{2}}+1\right)+O\left(q^{\frac{m}{2}+\varepsilon}\right).
\end{aligned}$$

This completes the proof of Theorem.

Acknowledgments. The authors greatly appreciate Prof. Wenpeng Zhang for his kind comments and suggestions, and also the referee for his very helpful and detailed comments.

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