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ON THE $2k$-TH POWER MEAN OF $\frac{L'}{L}(1,\chi)$ WITH THE WEIGHT OF GAUSS SUMS

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Cordially dedicated to editor V. Dlab

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Abstract. The main purpose of this paper is to study the hybrid mean value of $\frac{L'}{L}(1,\chi)$ and Gauss sums by using the estimates for trigonometric sums as well as the analytic method. An asymptotic formula for the hybrid mean value $\sum_{\chi \neq \chi_0} |\tau(\chi)||\frac{L'}{L}(1,\chi)|^{2k}$ of $\frac{L'}{L}$ and Gauss sums will be proved using analytic methods and estimates for trigonometric sums.

Keywords: Dirichlet L-function, Gauss sums, asymptotic formula

MSC 2010: 11M20

§1. Introduction

Let $\chi$ be the Dirichlet character modulo $q \geq 3$. For any integer $m$, the classical Gauss sum $G(m, \chi)$ is defined as

$$G(m, \chi) = \sum_{a=1}^{q} \chi(a)e\left(\frac{ma}{q}\right),$$

where $e(y) = e^{2\pi iy}$. In particular, when $m = 1$, we denote by $\tau(\chi) = G(1, \chi) = \sum_{a=1}^{q} \chi(a)e(a/q)$.

Perhaps the most important property of $G(m, \chi)$ is: when $(m, q) = 1$ and $\chi$ is a primitive character modulo $q$, we have $|G(m, \chi)| = \sqrt{q}$. For a nonprimitive

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character, the value of $|G(m, \chi)|$ varies, i.e. the value of $|G(m, \chi)|$ is irregular as $\chi$ varies. However, $G(m, \chi)$ enjoys many good value distribution properties in some problems of the weighted mean value.

Similarly, many books about the analytic number theory include a discussion on the properties of $\tau(\chi)$ (see Ref. [1]–[3]). Maybe the most important property of $\tau(\chi)$ is: when $\chi$ is a primitive character mod $q$, then $|\tau(\chi)| = \sqrt{q}$. For a nonprimitive character, the value of $\tau(\chi)$ is irregular as $\chi$ varies and sometimes it may be zero. But $\tau(\chi)$ surprisingly enjoys many good value distribution properties in some problems of the weighted mean value.

Yi Yuan and Zhang Wenpeng [4] studied the first power mean of the Dirichlet L-function with the weight of Gauss sums and obtained the asymptotic formula

$$
\sum_{\chi \neq \chi_0} |G(m, \chi)|^2 |L(1, \chi)| = \varphi^2(q) \cdot \sum_{n=1}^{\infty} \frac{r^2(n)}{n^2} + O(q^{3+\varepsilon}),
$$

where $q$ is an integer with $q > 2$, $m$ is an integer satisfying $(m, q) = 1$, $\chi_0$ is the principal character modulo $q$, $\sum'$ denotes the sum over all $n$ which are coprime with $q$, $\varphi(q)$ is the Euler function, $\varepsilon$ is any given positive number, and $r(n)$ is defined as follows: for any prime $p$ and positive integer $\alpha$, $r(1) = 1$, $r(p^{\alpha}) = 4^{-\alpha}C_{2\alpha}^\alpha$, $C_{2\alpha}^\alpha = (2\alpha)!/(\alpha!)^2$. For any positive integer $n$, when its standard factorization is $p_1^{\alpha_1}p_2^{\alpha_2}\ldots p_k^{\alpha_k}$, we can easily get

$$
r(n) = \frac{1}{4^{\alpha_1 + \alpha_2 + \ldots + \alpha_k}} C_{2\alpha_1}^{\alpha_1} C_{2\alpha_2}^{\alpha_2} \ldots C_{2\alpha_k}^{\alpha_k}.
$$

Yi Yuan and Zhang Wenpeng [5] studied the $2k$-th power mean of the Dirichlet L-function with the weight of Gauss sums and obtained

$$
\sum_{\chi \neq \chi_0} |\tau(\chi)|^m |L(1, \chi)|^{2k} = N^{2k-1} \varphi^2(N) \zeta^{2k-1}(2) \prod_{p|q} \left(1 - \frac{1}{p^{2k-1}}\right) \prod_{p|\alpha} \left(1 - \frac{1 - C_{2k-2}^{k-1}}{p^{2k-1}}\right) \prod_{p|M} \left(p^{2k-1} - 2p^{k-1} + 1\right) + O(p^{2k+\varepsilon}),
$$

where $q \geq 3$ is an integer and $q = MN$, $M = \prod_{p|q} p$, $(M, N) = 1$, $m$ is any positive number, $k$ is any positive integer, $\prod$ denotes all prime factors of $q$ such that $p \mid q$ and $p^2 \not\mid q$, $\varphi(q)$ is the Euler function, $\zeta(s)$ is the Riemann Zeta function and $\varepsilon$ is any given positive number.

Let $\chi$ be the Dirichlet character modulo $q$ and let $L(s, \chi)$ denote the corresponding Dirichlet L-function. $\frac{L'}{L}(1, \chi)$ has long history and plays an important role in number
theory [6], but one can hardly estimate $L'(1, \chi)$. In fact, it enjoys good mean value properties. Zhang Wenpeng [7] studied the asymptotic properties of the sums

$$\sum_{q \leq Q} \frac{q}{\varphi(q)} \sum_{\chi \not= \chi_0} \left| \frac{L'}{L}(1, \chi) \right|^4, \quad \sum_{q \leq Q} \frac{1}{\varphi(q)} \sum_{\chi \not= \chi_0} \left| \frac{L'}{L}(1, \chi) \right|^4,$$

where $Q > 3$ is a real number and $\varphi(q)$ is the Euler function. Liu Huaning and Zhang Xiaobeng [8] studied the mean value of $|L'(1, \chi)|^{2k}$ and obtained

$$\sum_{\chi \mod q} \left| \frac{L'}{L}(1, \chi) \right|^{2k} = A(k, q) \varphi(q) + O(q^\varepsilon),$$

where

$$A(k, q) = \sum_{n=1}^{\infty} \frac{\tau^2_k(n)}{n^2}$$

is a constant depending on $k$ and $q$, with

$$\tau_k(n) = \sum_{m_1 m_2 \ldots m_k = n} \Lambda(m_1) \Lambda(m_2) \ldots \Lambda(m_k),$$

$\Lambda(n)$ the Mangoldt function, $\varphi(q)$ the Euler function and $\varepsilon$ any given positive number.

In what follows we shall consider the hybrid mean value of $L'/L$ with Gauss sums whose asymptotic behavior has not been studied hitherto. We will use the estimates for trigonometric sums and the analytic method to study the hybrid mean value

$$\sum_{\chi \not= \chi_0} \left| \tau(\chi) \right|^m \left| \frac{L'}{L}(1, \chi) \right|^{2k},$$

and obtain a sharper asymptotic formula for it. That is, we shall prove the following theorem.

**Theorem.** Let $q = MN$, $M = \prod_{p \mid q} p$, $(M, N) = 1$. Then for any positive number $m$ and positive integer $k$ we have the asymptotic formula

$$\sum_{\chi \not= \chi_0} \left| \tau(\chi) \right|^m \left| \frac{L'}{L}(1, \chi) \right|^{2k} = A(k, q) N^{\frac{m}{2} - 1} \varphi^2(N) \prod_{p \mid M} \left( p^{\frac{m}{2} + 1} - 2p^{\frac{m}{2}} + 1 \right) + O\left(q^{\frac{m}{2} + \varepsilon}\right),$$

where $A(k, q)$ is defined as in (1), $\prod_{p \mid q}$ denotes all the prime factors of $q$ such that $p \mid q$ and $p^2 \nmid q$, $\varepsilon$ is any given positive number.

Throughout the paper, we denote by $\mu(n)$ the Möbius function, and $\varepsilon$ always denotes a sufficiently small positive real number which may be different at various occurrence.
§2. Some lemmas

To complete the proof of the theorem, we need the following lemmas.

**Lemma 1.** Let \( q = uv, u \geq 2, v \geq 2 \) and \((u,v) = 1\). Then for any \( \chi \mod q \), there exist one and only one character \( \chi_u \mod u \) one and only one character \( \chi_v \mod v \) such that \( \chi = \chi_u \chi_v \) and

\[
|\tau(\chi)| = |\tau(\chi_u)| \times |\tau(\chi_v)|.
\]

**Proof.** See Theorem 13.3.1. of [2]. \( \square \)

**Lemma 2.** Let \( q \) and \( r \) be integers with \( q \geq 2 \) and \((r,q) = 1\). Then we have the identities

\[
\sum_{\chi \mod q}^* \chi(r) = \sum_{d \mid (q,r-1)} \mu\left(\frac{q}{d}\right) \varphi(d), \quad J(q) = \sum_{d \mid q} \mu(d) \varphi\left(\frac{q}{d}\right),
\]

where \( \sum^* \) denotes the summation over all primitive characters, \( \varphi(q) \) is the Euler function and \( J(q) \) denotes the number of primitive characters \( \mod q \).

**Proof.** From the properties of characters, we know that for any character \( \chi \mod q \) there exists one and only one \( d \mid q \) with a primitive character \( \chi_d^* \mod d \) such that \( \chi = \chi_d^* \chi_0^q \), where \( \chi_0^q \) denotes the principal character \( \mod q \). So we have

\[
\sum_{\chi \mod q} \chi(r) = \sum_{d \mid q} \chi_d^* \chi_0^q \chi(r) = \sum_{d \mid q} \sum_{\chi \mod d} \chi_d^* \chi(r).
\]

Combining this formula with the Möbius transformation and noting the identity

\[
\sum_{\chi \mod q} \chi(r) = \begin{cases} \varphi(q), & \text{if } r \equiv 1 \mod q, \\ 0, & \text{otherwise}, \end{cases}
\]

we have

\[
\sum_{\chi \mod q}^* \chi(r) = \sum_{d \mid q} \mu(d) \sum_{\chi \mod d} \chi(r) = \sum_{d \mid (q,r-1)} \mu \left(\frac{q}{d}\right) \varphi(d).
\]

Taking \( r = 1 \), we immediately get

\[
J(q) = \sum_{d \mid q} \mu(d) \varphi\left(\frac{q}{d}\right).
\]

This proves Lemma 2. \( \square \)
Lemma 3. Let $p$ be a prime, $\alpha$ a positive integer and $\alpha \geq 2$, $n = p^\alpha$. Then for any nonprimitive character $\chi_1 \mod n$ we have the identity
\[
\sum_{a=1}^{p^\alpha} \chi_1(a)e\left(\frac{a}{p^\alpha}\right) = 0.
\]

Proof. See [5]. □

Lemma 4. Let $N = p_1^{\alpha_1}p_2^{\alpha_2}\ldots p_s^{\alpha_s}$, $\alpha_i \geq 2$, $1 \leq i \leq s$, be a positive integer, let a positive integer $M$ have no square factor and let $(M, N) = 1$, $q = MN$. Then for any given positive integer $k$ and $d | M$ we have
\[
\sum_{\chi \mod N}^* \left| \frac{L'(1, \chi_0^0_M)}{L(1, \chi_0^0_M)} \right|^{2k} = A(k, q)\frac{\varphi^2(N)}{N}J(d) + O((MN)^\varepsilon),
\]
where $\sum^*$ denotes the summation over all primitive characters.

Proof. Let $\chi'\chi_0^0_M$ be the nonprincipal real character mod $MN$, then from the properties of the L-function and from C.L. Siegel’s theorem [9] we get
\[
\frac{L'(1, \chi'\chi_0^0_M)}{L(1, \chi_0^0_M)} \ll \frac{(MN)\varepsilon \cdot \log^2(MN)}{C(\varepsilon)},
\]
where $C(\varepsilon)$ is a constant depending on $\varepsilon$.

For any complex character mod $MN$ with $MN \leq \exp(C_1\sqrt{\log x})$, where $C_1$ is any positive constant and $\exp(y) = e^y$, we get from [6]
\[
\psi(x, \chi_0^0_M) = \sum_{n \leq x} \chi(n)\chi_0^0_M(n)\Lambda(n) \ll x \cdot \exp(-C_2\sqrt{\log x})
\]
for some positive $C_2$ depending only on $C_1$.

Let $T \geq \exp(\log^2(MN)/C_1^2)$, then by Abel’s identity we have
\[
\frac{L'(1, \chi_0^0_M)}{L(1, \chi_0^0_M)} = \sum_{n=1}^{\infty} \chi(n)\chi_0^0_M(n)\Lambda(n) \\
= \sum_{1 \leq n \leq T} \chi(n)\chi_0^0_M(n)\Lambda(n) \frac{1}{n} + \int_T^{\infty} \sum_{T \leq n \leq y} \chi(n)\chi_0^0_M(n)\Lambda(n) dy \\
= \sum_{1 \leq n \leq T} \chi(n)\chi_0^0_M(n)\Lambda(n) \frac{1}{n} + O\left(\frac{\log T}{\exp(C_2\sqrt{\log T})}\right).
\]
So if we write $\tau_k(n)$ as in (2), we have

$$\sum_{\chi \mod Nd}^* \left| \frac{L'}{L}(1, \chi \chi_M)^{2k} \right|$$

$$= \sum_{\chi \mod Nd}^* \left| \sum_{1 \leq n \leq T} \frac{\chi(n) \chi_M(n) \Lambda(n)}{n} \right|^{2k}$$

$$+ O\left( \frac{J(Nd) \cdot \log^{2k} T}{\exp(C_2 \sqrt{\log T})} \right) + O\left( \frac{(MN)^{2k \varepsilon} \cdot \log^{4k}(MN)}{C^{2k} \varepsilon} \right)$$

$$= \sum_{\chi \mod Nd}^* \left| \sum_{1 \leq n \leq T} \frac{\chi(n) \chi_M(n) \tau_k(n)}{n} \right|^2$$

$$+ O\left( \frac{J(Nd) \cdot \log^{2k} T}{\exp(C_2 \sqrt{\log T})} \right) + O\left( \frac{(MN)^{2k \varepsilon} \cdot \log^{4k}(MN)}{C^{2k} \varepsilon} \right)$$

$$= \sum_{1 \leq n_1 \leq T^k} \sum_{1 \leq n_2 \leq T^k} \frac{\chi_M(n_1 \overline{n_2}) \tau_k(n_1) \tau_k(n_2)}{n_1 n_2} \sum_{\chi \mod Nd}^* \chi(n_1) \bar{\chi}(n_2)$$

$$+ O\left( \frac{J(Nd) \cdot \log^{2k} T}{\exp(C_2 \sqrt{\log T})} \right) + O\left( \frac{(MN)^{2k \varepsilon} \cdot \log^{4k}(MN)}{C^{2k} \varepsilon} \right)$$

$$= \sum_{l \mid Nd} \mu\left( \frac{N}{l} \right) \varphi(l) \sum_{1 \leq n_1 \leq T^k} \sum_{1 \leq n_2 \leq T^k} \frac{\chi_M(n_1 \overline{n_2}) \tau_k(n_1) \tau_k(n_2)}{n_1 n_2}$$

$$+ O\left( \frac{J(Nd) \cdot \log^{2k} T}{\exp(C_2 \sqrt{\log T})} \right) + O\left( \frac{(MN)^{2k \varepsilon} \cdot \log^{4k}(MN)}{C^{2k} \varepsilon} \right)$$

$$= \sum_{l \mid Nd} \mu\left( \frac{N}{l} \right) \varphi(l) \sum_{1 \leq n \leq T^k} \frac{\tau_k^2(n)}{n^2}$$

$$+ \sum_{l \mid Nd} \mu\left( \frac{N}{l} \right) \varphi(l) \sum_{1 \leq n_1 \leq T^k} \sum_{1 \leq n_2 \leq T^k} \frac{\chi_M(n_1 \overline{n_2}) \tau_k(n_1) \tau_k(n_2)}{n_1 n_2}$$

$$+ O\left( \frac{J(Nd) \cdot \log^{2k} T}{\exp(C_2 \sqrt{\log T})} \right) + O\left( \frac{(MN)^{2k \varepsilon} \cdot \log^{4k}(MN)}{C^{2k} \varepsilon} \right) \right.$$
Noting that $J(N) = \varphi^2(N)/N$ and $(N, d) = 1$, we get

$$
\sum_{\chi \mod Nd}^{*} \left| \frac{L'}{L}(1, \chi \chi_M^0) \right|^{2k} = J(Nd) \sum_{n=1}^{\infty} \frac{\tau^2(n)}{n^2}
$$

$$
+ O\left( \sum_{l|Nd} \varphi(l) \sum_{1 \leq n_1 \leq T_k} \sum_{1 \leq n_2 \leq T_k} \sum_{n_1 \equiv n_2 \pmod{l}, n_1 \neq n_2} \tau_k(n_1) \tau_k(n_2) \right)
$$

$$
+ O\left( J(Nd) \cdot \log^{2k} T \cdot \exp(C_2 \sqrt{\log T}) \right) + O\left( (MN)^{2k \epsilon} \cdot \log^{4k}(MN) \right)
$$

$$
= A(k, q) \frac{\varphi^2(N)}{N} J(d) + O((Nd)^\epsilon \log^{2k+2} T)
$$

$$
+ O\left( J(Nd) \cdot \log^{2k} T \cdot \exp(C_2 \sqrt{\log T}) \right) + O\left( (MN)^{2k \epsilon} \cdot \log^{4k}(MN) \right).
$$

Taking

$$
T = \max \left\{ \exp\left( \frac{\log^2(MN)}{C_2^2} \right), \exp\left( \frac{\log^2(MN)}{C_2^2} \right) \right\},
$$

we get immediately

$$
\sum_{\chi \mod Nd}^{*} \left| \frac{L'}{L}(1, \chi \chi_M^0) \right|^{2k} = A(k, q) \frac{\varphi^2(N)}{N} J(d) + O((MN)^\epsilon).
$$

This proves Lemma 4.

§3. Proof of theorem

In this section we present the proof of the theorem.

Let $q = p_1 p_2 \ldots p_k p_{k+1}^{\alpha_{k+1}} \ldots p_r^{\alpha_r}$ be the standard factorization of $q$, where $\alpha_i > 1$, $k + 1 \leq i \leq r$. Let $M = p_1 p_2 \ldots p_k$, $N = p_{k+1}^{\alpha_{k+1}} \ldots p_r^{\alpha_r}$, so $(M, N) = 1$. For any positive number $m$ and positive integer $k$, Lemma 1 yields

$$
\sum_{\chi \neq \chi_0} |\tau(\chi)|^m \left| \frac{L'}{L}(1, \chi) \right|^{2k} = \sum_{\chi_1 \mod M} \sum_{\chi_2 \mod N} \frac{|\tau(\chi_1 \chi_2)|^m}{\chi_1 \chi_2 \neq \chi_0^0} \left| \frac{L'}{L}(1, \chi_1 \chi_2) \right|^{2k}
$$

$$
= \sum_{\chi_1 \mod M} \sum_{\chi_2 \mod N} \frac{|\tau(\chi_1)|^m |\tau(\chi_2)|^m}{\chi_1 \chi_2 \neq \chi_0^0} \left| \frac{L'}{L}(1, \chi_1 \chi_2) \right|^{2k}.
$$
From Lemma 2, Lemma 3, Lemma 4 and the above expression, we immediately get

$$
\sum_{\chi \neq \chi_0} |\tau(\chi)| m \left| \frac{L'}{L}(1, \chi) \right|^{2k} = \sum_{\chi_1 \mod M} \sum_{\chi_2 \mod N}^* |\tau(\chi_1)| m N \left| \frac{L'}{L}(1, \chi_1 \chi_2) \right|^{2k}
$$

$$
= \sum_{d \mid M} \sum_{\chi_1 \mod M}^* \sum_{\chi_2 \mod N}^* d \frac{N}{d} \left| \frac{L'}{L}(1, \chi_1 \chi_2 \chi_0) \right|^{2k}
$$

$$
= \sum_{d \mid M} d \frac{N}{d} \left( A(k, q) \frac{\varphi^2(N)}{N} J(d) + O((MN)^\varepsilon) \right)
$$

$$
= A(k, q) N \frac{\varphi^2(N)}{N} \sum d \frac{N}{d} J(d) + O \left( N \frac{\varphi^2(N)}{N} \sum d \frac{N}{d} \right)
$$

$$
= A(k, q) N \frac{\varphi^2(N)}{N} \prod_{p \mid M} \left( \sum_{j=0}^{1} (p^j) \frac{\varphi}{p} J(p^j) \right) + O \left( q \frac{\varphi^2(N)}{N} \right)
$$

$$
= A(k, q) N \frac{\varphi^2(N)}{N} \prod_{p \mid M} (1 + p \frac{\varphi}{p} (p - 2)) + O \left( q \frac{\varphi^2(N)}{N} \right)
$$

$$
= A(k, q) N \frac{\varphi^2(N)}{N} \prod_{p \mid M} (p \frac{\varphi}{p} + 1) + O \left( q \frac{\varphi^2(N)}{N} \right).
$$

This completes the proof of Theorem.

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**References**


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