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NOTES ON MONOTONE LINDELÖF PROPERTY

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Abstract. We provide a necessary and sufficient condition under which a generalized ordered topological product (GOTP) of two GO-spaces is monotonically Lindelöf.

Keywords: monotone Lindelöf property, generalized ordered topological product, generalized ordered spaces

MSC 2010: 54F05, 54D20

1. Introduction

The monotone Lindelöf property was studied in generalized ordered spaces by H. Bennett, D. Lutzer and M. Matveev in [2]. They presented some classical examples which are not monotonically Lindelöf and proved that any separable GO-space is hereditarily monotonically Lindelöf, $[0, \omega_1]$ is a compact LOTS which is not monotonically Lindelöf, and the lexicographic product of two unit intervals is monotonically Lindelöf. In particular, the double arrow space $[0,1] \times \{0,1\}$ with lexicographic order is monotonically Lindelöf. In [7], [8] we introduced a new topology on the lexicographic product set $X \times Y$, where X, Y are generalized ordered (GO) spaces. This new topology contains the usual open-interval topology of the lexicographic order and also reflects in a natural way the fact that X and Y carry a GO-topology, rather than just the open interval topology of their linear orderings, which is called a *generalized ordered topological product* (GOTP) of the GO-spaces X and Y and is denoted by $\mathrm{GOTP}(X * Y)$.

Recall that a topological space X is *monotonically Lindelöf* if for each open cover $\mathscr{U}$ of X there is a countable open cover $r\mathscr{U}$ of X that refines $\mathscr{U}$ and has the property that if an open cover $\mathscr{U}$ refines an open cover $\mathscr{V}$, then $r\mathscr{U}$ refines $r\mathscr{V}$. In this case, r is

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called a *monotone Lindelöf operator* for the space X. *A linearly ordered topological space* or *a LOTS* is a triple $(X, \lambda, \leqslant)$, where $(X, \leqslant)$ is a linearly ordered set and λ is the interval topology on $(X, \leqslant)$. *A generalized ordered space* or *a GO-space* is a triple $(X, \tau, \leqslant)$, where $(X, \leqslant)$ is a linearly ordered set and τ is a topology on $(X, \leqslant)$ such that $\lambda \subseteq \tau$ and τ has a base consisting of order convex sets, where a set A is called *order convex* if $x \in A$ for every x lying between two points of A.

In this paper we characterize the monotone Lindelöf property of a generalized ordered topological product (GOTP) of two GO-spaces. Let X, Y be GO-spaces. Suppose that Y has both the left endpoint and the right endpoint. We prove that GOTP$(X * Y)$ is monotonically Lindelöf if and only if GOTP$(X * \{0,1\})$ and Y are monotonically Lindelöf. In addition, we show that if Y is monotonically Lindelöf and has either the maximal or the minimal point but not both of them, then GOTP$(X*Y)$ is monotonically Lindelöf if and only if the GO-space $X' = (X, \tau', <)$ is monotonically Lindelöf, where τ' is the topology on X with the subbase $\tau \cup \{[x, \rightarrow) \colon x \in X\}$ $(\tau \cup \{(\leftarrow, x] \colon x \in X\})$.

Throughout this paper, for a set V and a collection $\mathscr{U}$ of sets we will write $V \prec \mathscr{U}$ to mean that V is a subset of some member of $\mathscr{U}$. For a GO-space $X = (X, \tau, <)$ we can define $L_X = \{x \in X \colon (\leftarrow, x] \in \tau - \lambda\}$, $R_X = \{x \in X \colon [x, \rightarrow) \in \tau - \lambda\}$, and $I_X = \{x \in X \colon x \text{ is an isolated point of } X\}$. Conversely, the generalized ordered topology τ on X is determined by its subsets L_X, R_X and I_X. For every set A, *the cardinality* of X is denoted by $|A|$.

For the undefined terminology, the reader may refer to [3] and [6].

2. Results

First we introduce the definition of the generalized ordered topological product of two GO-spaces.

Definition 2.1 ([4]). Let $(X, <_X)$, $(Y, <_Y)$ be linearly ordered sets. Then the lexicographic product $X * Y$ of $(X, <_X)$ and $(Y, <_Y)$ is defined as the ordered set $(X \times Y, \lessdot)$ where $\lessdot$ is the lexicographic ordering, i.e., if $a = \langle x_1, y_1 \rangle$ and $b = \langle x_2, y_2 \rangle \in X \times Y$ then

$$a \lessdot b \text{ if and only if } x_1 <_X x_2 \text{ or } x_1 = x_2 \text{ and } y_1 <_Y y_2.$$

Definition 2.2 ([8]). Let $(X, \tau_X, <_X)$, $(Y, \tau_Y, <_Y)$ be GO-spaces. Let λ_X, λ_Y be the usual interval topologies on X, Y, respectively, and let λ_{X*Y} be the usual interval topology on the linearly ordered set $X * Y$.

By the generalized ordered topology (abbreviated GOT) τ_{X*Y} we mean a topology on $X*Y$ which has a subbase

$$\begin{aligned}\mathscr{B} = \lambda_{X*Y} &\cup \tau_R \cup \tau_L \\ &\cup \{[\langle x,y\rangle,\rightarrow) \subseteq X*Y\colon\ x\in X,\ y\in Y \text{ and } [y,\rightarrow)\in\tau_Y-\lambda_Y\} \\ &\cup \{(\leftarrow,\langle x,y\rangle] \subseteq X*Y\colon\ x\in X,\ y\in Y \text{ and } (\leftarrow,y]\in\tau_Y-\lambda_Y\},\end{aligned}$$

where either

$$\tau_R=\emptyset \text{ and } \tau_L=\emptyset, \text{ if } Y \text{ does not have endpoints,}$$

or

$$\begin{aligned}&\tau_R=\{[\langle x,y_0\rangle,\rightarrow)\colon\ x\in X \text{ and } [x,\rightarrow)\in\tau_X-\lambda_X\} \text{ and } \tau_L=\emptyset,\\ &\qquad \text{if } Y \text{ has a left endpoint } y_0, \text{ but no right one,}\end{aligned}$$

or

$$\begin{aligned}&\tau_R=\emptyset \text{ and } \tau_L=\{(\leftarrow,\langle x,y_1\rangle]\colon\ x\in X \text{ and} (\leftarrow,x]\in\tau_X-\lambda_X\},\\ &\qquad \text{if } Y \text{ has a right endpoint } y_1, \text{ but no left one,}\end{aligned}$$

or

$$\begin{aligned}&\tau_R=\{[\langle x,y_0\rangle,\rightarrow)\colon\ x\in X \text{ and } [x,\rightarrow)\in\tau_X-\lambda_X\} \text{ and}\\ &\tau_L=\{(\leftarrow,\langle x,y_1\rangle]\colon\ x\in X \text{ and} (\leftarrow,x]\in\tau_X-\lambda_X\},\\ &\qquad \text{if } Y \text{ has both a left endpoint } y_0 \text{ and a right endpoint } y_1.\end{aligned}$$

We say that the space $(X*Y,\tau_{X*Y})$ is the generalized ordered topological product (abbreviated as GOTP) of GO-spaces $(X,\tau_X,<_X)$ and $(Y,\tau_Y,<_Y)$, and denote it by GOTP$(X*Y)$. Similarly we denote $(X*Y,\lambda_{X*Y})$ by LOTP$(X*Y)$.

In Definition 2.2, if X, Y are LOTS, then $\tau_{X*Y}=\lambda_{X*Y}$ and if Y has two endpoints, X is a quotient space of GOTP$(X*Y)$ by Lemma 3.5 in [8]. For each $x\in X$, the subspace $\{x\}*Y$ of GOTP$(X*Y)$ is homeomorphic to Y. Moreover, the GOT on $X*Y$ is determined by the topologies on X and Y. So the GOTP is a natural generalization of the lexicographic product with the usual interval topology.

In this paper, we often deal with more than one ordered sets. For different ordered sets, the orderings may be different. But in most cases, we can distinguish them from the context. So we will use the symbol $<$ for all the orderings unless it is necessary to avoid confusions.

For GO-spaces X and Y, let $p\colon X * Y \to X$ be the projection. For a linearly ordered set X and $a, b \in X$ with $a < b$, the symbols (a, b), $[a, b)$, $(a, b]$, $[a, b]$ denote the open interval, left closed and right open interval, left open and right closed, closed interval respectively, as usual. To distinguish in which linearly ordered set the interval is taken, we adopt, for example, $(a, b)_X$ to convey the interval is taken in X.

Lemma 2.3. *Let X, Y be GO-spaces, let Y have both endpoints and let $X' \subset X$, $Y' \subset Y$ with Y' containing the two endpoints of Y. Then* $\mathrm{GOTP}(X' * Y')$ *is a subspace of* $\mathrm{GOTP}(X * Y)$.

Proof. We only need to prove that a convex subset U of $\mathrm{GOTP}(X' * Y')$ is open if and only if U is the intersection of $X' * Y'$ and an open convex subset of $\mathrm{GOTP}(X * Y)$. For convenience, let y_0 be the left endpoint of Y, y_1 the right endpoint of Y. Now let U be an open convex subset of $\mathrm{GOTP}(X' * Y')$. Then for U one of the following cases may occur:

(i) $p(U)$ is a singleton;

(ii) $|p(U)| > 1$, and $p(U)$ has neither the maximum nor the minimum element in X';

(iii) $|p(U)| > 1$, and $p(U)$ contains only one of the maximum and minimum elements in X';

(iv) $|p(U)| > 1$, and $p(U)$ contains both the maximum and minimum elements in X'.

For Case (i), let $p(U) = \{x(U)\}$. Then $U \subset \{x(U)\} * Y'$ and U is open convex in $\{x(U)\} * Y'$. Hence there is an open convex subset V of Y such that $U = \{x(U)\} * (V \cap Y')$. There are three subcases we must consider:

Subcase (i-1). V contains no endpoint of Y. Then $\{x(U)\} * V$ is open in $\mathrm{GOTP}(X * Y)$ since $(\langle x(U), y_0\rangle, \langle x(U), y_1\rangle)$ is open in $\mathrm{GOTP}(X * Y)$.

Subcase (i-2). V contains one of the endpoints of Y but not the other one. For example, V contains the left endpoint but not the right one. Then $\langle x(U), y_0\rangle \in U$ is the minimum element of U. By the definition of GOT, if $[x(U), \to)$ is open in X, then $[\langle x(U), y_0\rangle, \langle x(U), y_1\rangle)$ is open in $\mathrm{GOTP}(X * Y)$ and contains $\{x(U)\} * V$ as its open subset. Hence $\{x(U)\} * V$ is open in $\mathrm{GOTP}(X * Y)$. If $[x(U), \to)$ is not open in X, then $x(U)$ must have no immediate predecessor in X and there is a point $x'(U) \in X$ such that $x'(U) < x(U)$ and $(x'(U), x(U)) \cap X' = \emptyset$. Thus $(\langle x'(U), y_1\rangle, \langle x(U), y_0\rangle] \cup \{x(U)\} * V$ is an open convex subset of $\mathrm{GOTP}(X * Y)$ and $\big((\langle x'(U), y_1\rangle, \langle x(U), y_0\rangle] \cup \{x(U)\} * V\big) \cap (X' * Y') = U$.

Subcase (i-3). V contains both the left and right endpoints. Like in subcase (i-2), we may deal with the right endpoint $\langle x(U), y_1\rangle$ and find an open convex subset W of $\mathrm{GOTP}(X * Y)$ such that $W \cap (X' * Y') = U$.

For Case (ii), $p(U)$ is convex in X' since U is convex in $X' * Y'$ and $p(U)$ is open in X'. Let $V = \{x \in X\colon \exists\, x', x'' \in p(U) \text{ such that } x' < x < x''\}$. Then V is an open convex subset of X and $V \cap X' = p(U)$. Put $W = V * Y$. Then W is open in $\mathrm{GOTP}(X * Y)$ and $W \cap (X' * Y') = U$.

For Case (iii), suppose that $p(U)$ has the minimum element x_0. Then $V = U \cap (\{x_0\} * Y') \neq \emptyset$ and V must contain $\langle x_0, y_1 \rangle$ since Y' contains y_1 and U is convex. Moreover, V is open in $\{x_0\} * Y'$. If V does not contain $\langle x_0, y_0 \rangle$, then $U' = V \cup (\{x \in X \colon \exists\, x', x'' \in p(U) \text{ such that } x' < x < x''\} * Y)$ is open in $\mathrm{GOTP}(X * Y)$ and $U' \cap (X' * Y') = U$. If V also contains $\langle x_0, y_0 \rangle$ and $[x_0, \rightarrow)$ is open in X, then $U' = (\{x_0\} * Y) \cup (\{x \in X \colon \exists\, x', x'' \in p(U) \text{ such that } x' < x < x''\} * Y)$ is open in $\mathrm{GOTP}(X * Y)$ and $U' \cap (X' * Y') = U$. If V contains $\langle x_0, y_0 \rangle$ but $[x_0, \rightarrow)$ is not open in X, then x_0 has no immediate predecessor in X and there is $x'_0 < x_0$ such that $(x'_0, x_0) \cap X' = \emptyset$. Put $U' = \big((x'_0, x_0] \cup \{x \in X \colon \exists\, x', x'' \in p(U) \text{ such that } x' < x < x''\}\big) * Y$. Then U' is open in $\mathrm{GOTP}(X * Y)$ and $U' \cap (X' * Y') = U$. We can similarly discuss the case that $p(U)$ has the maximum element.

For Case (iv), proceed similarly to Case (iii).

Next suppose that U' is a convex subset of $\mathrm{GOTP}(X' * Y')$ which is the intersection of $X' * Y'$ and some open convex subset U of $\mathrm{GOTP}(X * Y)$. Now we prove that U' is open in $\mathrm{GOTP}(X' * Y')$. Notice that for each $x \in X'$, $(\langle x, y_0 \rangle, \langle x, y_1 \rangle)_{X' * Y'} = \{\langle x, y \rangle \colon y \in Y' \text{ and } y_0 < y < y_1\}$ is an open interval in $X' * Y'$ since Y' contains the endpoints of Y. Hence $U \cap (\langle x, y_0 \rangle, \langle x, y_1 \rangle)_{X' * Y'} \subset U'$ is open in $\mathrm{GOTP}(X' * Y')$. So to prove that U' is open in $\mathrm{GOTP}(X' * Y')$, we only need to prove that for each $x \in X'$ if $\langle x, y_0 \rangle \in U'$ (or $\langle x, y_1 \rangle \in U'$), then $\langle x, y_0 \rangle$ (or $\langle x, y_0 \rangle$) is an interior point of U' with respect to the GOT on $X' * Y'$. Suppose that $\langle x, y_0 \rangle \in U'$. If x is not the minimum point of $p(U')$, we may take a point $x' \in p(U')$ such that $x' < x$ and some $y' > y_0$ such that $[\langle x, y_0 \rangle, \langle x, y' \rangle)_{X' * Y'} \subset U'$ since $U \cap (\{x\} * Y)$ is open in $\{x\} * Y$. Therefore $(\langle x', y_1 \rangle, \langle x, y' \rangle) \subset U'$, i.e., $\langle x, y_0 \rangle$ is an interior point of U' with respect to the GOT on $X' * Y'$. If x is the minimum point of $p(U')$, then there are two cases: (a) $[x, \rightarrow)$ is open in X. Then $[x, \rightarrow)_{X'}$ is also open in X, by the definition of GOT, we know that $\langle x, y_0 \rangle$ is an interior point of U'; (b) $[x, \rightarrow)$ is not open in X. Then there must be a point $x' \in X$ such that $x' < x$, satisfying $(x', x) \cap X' = \emptyset$ since $U' = U \cap (X' * Y')$ and U is open in $\mathrm{GOTP}(X * Y)$. Thus $(x', \rightarrow)$ is open in X and $(x', \rightarrow) \cap X' = [x, \rightarrow)_{X'}$. Therefore $[x, \rightarrow)_{X'}$ is open in X'. Hence $\langle x, y_0 \rangle$ is an interior point of U' with respect to the GOT on $X' * Y'$. The argument is similar for the case that $\langle x, y_1 \rangle \in U'$. □

Remark. In Lemma 2.3, the condition that Y' contains the endpoints of Y cannot be removed. For example, let $X = Y = [0, 1]$, $X' = X$ and $Y' = [\frac{1}{3}, \frac{2}{3}]$. Then $\mathrm{GOTP}(X * Y) = \mathrm{LOTP}(X * Y)$ and $\mathrm{GOTP}(X' * Y') = \mathrm{LOTP}(X' * Y')$. In $X' * Y'$ as a subspace of $\mathrm{GOTP}(X * Y)$, $\{\frac{1}{2}\} * Y'$ is open since $\{\frac{1}{2}\} * Y' = (\langle \frac{1}{2}, 0 \rangle, \langle \frac{1}{2}, 1 \rangle) \cap (X' * Y')$.

But $\{\frac{1}{2}\} * Y'$ is not open in $\mathrm{GOTP}(X' * Y')$ since $\langle\frac{1}{2}, 0\rangle$ is not an interior point of $\{\frac{1}{2}\} * Y'$ with respect to the topology on $\mathrm{GOTP}(X' * Y')$.

It is well known that a GO-space X can be embedded as a dense subspace into the compact LOTS $l(X)$ that is called the minimal linearly ordered compactification of X (see [5]). Let X, Y be GO-spaces. By Lemma 2.3, we have the following conclusion.

Theorem 2.4. *Suppose that Y has two endpoints. Then* $\mathrm{GOTP}(X * Y)$ *is a subspace of* $\mathrm{LOTP}(l(X) * l(Y))$.

Lemma 2.5. *Let X, Y be GO-spaces. Suppose that Y has both the left endpoint y_0 and the right endpoint y_1. For every convex open subset U of* $\mathrm{GOTP}(X * \{0,1\})$, *the set*

$$U^{\triangledown} = \bigcup\{\{x\} * Y\colon \text{ there exist } x',\ x'' \in p(U) \text{ such that } x' < x < x''\}$$

is an open convex set in $\mathrm{GOTP}(X*Y)$, *and if V is also a convex subset of* $\mathrm{GOTP}(X*\{0,1\})$ *with $V \subseteq U$, then $V^{\triangledown} \subseteq U^{\triangledown}$.*

Proof. We only need to prove that $U^{\triangledown}$ is convex and open in $\mathrm{GOTP}(X * Y)$. Assume that $z', z'' \in U^{\triangledown}$ with $z' < z''$, $z' = \langle x', y'\rangle$ and $z'' = \langle x'', y''\rangle$. Let $z' < z < z''$ with $z = \langle x, y\rangle$. If $x' = x''$, then $z \in \{x'\} * Y \subseteq U^{\triangledown}$. If $x' \neq x''$, then $x' \leqslant x''$ and for all $x' \leqslant x \leqslant x''$, $\{x\} * Y \subseteq U^{\triangledown}$. Hence $z \in \{x\} * Y \subseteq U^{\triangledown}$.

Next we prove that $U^{\triangledown}$ is open in $\mathrm{GOTP}(X * Y)$. Let $z \in U^{\triangledown}$ with $z = \langle x, y\rangle$. If y is not an endpoint of Y, then $z \in (\langle x, y_0\rangle, \langle x, y_1\rangle) \subseteq U^{\triangledown}$. If $y = y_0$, there are three cases: (i) x is neither the minimum nor the maximum point of $p(U^{\triangledown})$. Then there exist $x', x'' \in p(U^{\triangledown})$ such that $x' < x < x''$. Hence z is an interior point of $U^{\triangledown}$ since $z \in (\langle x', y_0\rangle, \langle x'', y_1\rangle) \subseteq U^{\triangledown}$. (ii) x is the minimum point of $p(U^{\triangledown})$. By the definition of $U^{\triangledown}$, there exists $x' \in p(U)$ with $x' < x$. Moreover, x' is the immediate predecessor of x. Otherwise, we would have a contradiction with the minimality of x. Hence $[\langle x, y_0\rangle, \rightarrow)$ is open in $\mathrm{GOTP}(X * Y)$ by Definition 2.2. Therefore $z = \langle x, y\rangle$ is an interior point of $U^{\triangledown}$ because $z \in [\langle x, y_0\rangle, \langle x, y_1\rangle) \subseteq U^{\triangledown}$. (iii) x is the maximum point of $\pi_1(U^{\triangledown})$. The proof is similar to (ii). Analogously, if $y = y_1$, then we can prove that $z = \langle x, y\rangle$ is an interior point of $U^{\triangledown}$. $\square$

Definition 2.6. Let L be a compact LOTS. For $x \in L$, put

$$0\text{-cf}(x) = \min\{|C|\colon C \text{ is a cofinal subset of } (\leftarrow, x)\}$$

and

$$1\text{-cf}(x) = \min\{|C|\colon C \text{ is a coinitial subset of } (x, \rightarrow)\}.$$

We call 0-cf(x) (1-cf(x)) the left (right) cofinality of x.

Observe that 0-cf$(x) = 0$ (1-cf$(x) = 0$) if x is the left (right) endpoint of L; 0-cf$(x) = 1$ (1-cf$(x) = 1$) if x has an immediate predecessor (successor); 0-cf(x) (1-cf(x)) is a regular cardinal if x is not the left (right) endpoint of L and has no immediate predecessor (successor). For a GO-space X and $x \in X$, the left (right) cofinality 0-cf(x) (1-cf(x)) means the cofinality defined in its minimal linearly ordered compactification $l(X)$. By [5, Lemma 3.5], if 0-cf$(x) \geqslant \omega$ (1-cf$(x) \geqslant \omega$), then there exists a cofinal increasing sequence $\{x_0(\alpha) \in X\colon \alpha < 0\text{-cf}(x)\}$ (a coinitial decreasing sequence $\{x_1(\alpha) \in X\colon \alpha < 1\text{-cf}(x)\}$).

The next definition was introduced by M. Matveev.

Definition 2.7. Let X be a space and x a point of X. X is said to be monotonically Lindelöf at x, if there exists an operator r_x that assigns to every non-empty family $\mathscr{F}$ of neighborhoods of x a non-empty countable family $r_x\mathscr{F}$ of neighborhoods of x so that $r_x\mathscr{F}$ refines $\mathscr{F}$ and $r_x\mathscr{F}$ refines $r_x\mathscr{G}$ provided $\mathscr{F}$ refines $\mathscr{G}$.

To verify Theorem 2.11, we need the following lemmas.

Lemma 2.8 ([8])**.** *For a GO-space X, the following conditions are equivalent.*

(1) *X is monotonically Lindelöf.*

(2) *For any cover $\mathscr{U}$ of X consisting of open convex subsets, there exists a countable open cover $r\mathscr{U}$ refining $\mathscr{U}$ such that if $\mathscr{V}$ is also such an open convex cover of X that refines $\mathscr{U}$, then $r\mathscr{V}$ refines $r\mathscr{U}$.*

(3) *For any cover $\mathscr{U}$ of X consisting of open convex subsets, there exists a countable open cover $r\mathscr{U}$ which also consists of convex subsets refining $\mathscr{U}$ such that if $\mathscr{V}$ is also such an open convex cover of X that refines $\mathscr{U}$, then $r\mathscr{V}$ refines $r\mathscr{U}$.*

Lemma 2.9 ([8])**.** *Let X be a GO-space. If X is monotonically Lindelöf, then both the left and right cofinalities at each point of X are not larger than ω_1.*

Lemma 2.10 ([8])**.** *Suppose that X is a GO-space and $x \in X$. If both the left and right cofinalities of x are not larger than ω_1, then X is monotonically Lindelöf at x.*

Theorem 2.11. *Let X, Y be GO-spaces. Suppose that $|Y| > 1$ and Y has both the left endpoint and the right endpoint. Then* $\mathrm{GOTP}(X * Y)$ *is monotonically Lindelöf if and only if* $\mathrm{GOTP}(X * \{0,1\})$ *and Y are monotonically Lindelöf.*

P r o o f. Necessity. Assume $\mathrm{GOTP}(X * Y)$ is monotonically Lindelöf. Let y_0 be the left endpoint of Y, y_1 the right endpoint. Obviously, $\mathrm{GOTP}(X * \{y_0, y_1\})$ is monotonically Lindelöf since $\mathrm{GOTP}(X * \{y_0, y_1\})$ is a closed subset of $\mathrm{GOTP}(X * Y)$. Thus $\mathrm{GOTP}(X * \{0,1\})$ is monotonically Lindelöf since $\mathrm{GOTP}(X * \{0,1\})$

is homeomorphic to $\mathrm{GOTP}(X * \{y_0, y_1\})$. Y is also monotonically Lindelöf since $\{x\} * Y$ is a closed subspace of $\mathrm{GOTP}(X * Y)$ and homeomorphic to Y for every $x \in X$.

Sufficiency. Assume $\mathrm{GOTP}(X * \{0,1\})$ and Y are monotonically Lindelöf. Then $\mathrm{GOTP}(X * \{y_0, y_1\})$ is monotonically Lindelöf. Suppose that $\mathscr{U}$ is an open cover of $\mathrm{GOTP}(X * Y)$ consisting of convex open subsets. Set $\mathscr{W}_{\mathscr{U}} = \{W = U \cap (X * \{y_0, y_1\})\colon U \in \mathscr{U}\}$. Then $\mathscr{W}_{\mathscr{U}}$ is an open cover of $\mathrm{GOTP}(X * \{y_0, y_1\})$. Let $r_{X*\{y_0,y_1\}}$ be a monotone Lindelöf operator for $\mathrm{GOTP}(X * \{y_0, y_1\})$. Define

$$r_1(\mathscr{U}) = \{U = V^{\nabla}\colon\ V \in r_{X*\{y_0,y_1\}}(\mathscr{W}_{\mathscr{U}}) \text{ and } |V| > 1\}.$$

By Lemma 2.8, we may assume that every member of $r_{X*\{y_0,y_1\}}(\mathscr{W}_{\mathscr{U}})$ is convex in $\mathrm{GOTP}(X * \{y_0, y_1\})$. Therefore, $r_1(\mathscr{U})$ is a countable collection of open subsets of $\mathrm{GOTP}(X * Y)$ by Lemma 2.5. Put $E(\mathscr{U}) = \{x \in X\colon \langle x, y_0\rangle$ or $\langle x, y_1\rangle$ is an endpoint of some member of $r_{X*\{y_0,y_1\}}(\mathscr{W}_{\mathscr{U}})\}$. Clearly, $E(\mathscr{U})$ is a countable subset of X.

Take $x \in E(\mathscr{U})$.

(a) Consider the point $\langle x, y_0\rangle$. It is easy to check that $0\text{-cf}(\langle x, y_0\rangle) = 0\text{-cf}(x) = 0\text{-cf}_{\vartriangle}(\langle x, y_0\rangle)$ and $1\text{-cf}(\langle x, y_0\rangle) = 1\text{-cf}(y_0)$, where $i\text{-cf}(\langle x, y_0\rangle)$, $i = 0, 1$ are the left and right cofinalities of $\langle x, y_0\rangle$ in $\mathrm{GOTP}(X * Y)$, $0\text{-cf}_{\vartriangle}(\langle x, y_0\rangle)$ is the left cofinality of $\langle x, y_0\rangle$ in $\mathrm{GOTP}(X * \{y_0, y_1\})$, $0\text{-cf}(x)$ is the left cofinality of x in X and $1\text{-cf}(y_0)$ is the right cofinality of y_0 in Y. By Lemma 2.9, both the left and right cofinalities of $\langle x, y_0\rangle$ are not larger than ω_1 since $\mathrm{GOTP}(X * \{y_0, y_1\})$ and Y are monotonically Lindelöf. Therefore $\mathrm{GOTP}(X * Y)$ is monotonically Lindelöf at $\langle x, y_0\rangle$ by Lemma 2.10. Let $r_{\langle x,y_0\rangle}$ be a monotone Lindelöf operator at $\langle x, y_0\rangle$. Put

$$\mathscr{O}(x, y_0, \mathscr{U}) = \{O \cap (V^{\nabla} \cup [\langle x, y_0\rangle, \langle x, y_1\rangle))\colon\ \langle x, y_0\rangle \in O \in \mathscr{U} \\ \text{and } \langle x, y_0\rangle \in V \in r_{X*\{y_0,y_1\}}(\mathscr{W}_{\mathscr{U}})\}.$$

Claim 1. For every $V \in r_{X*\{y_0,y_1\}}(\mathscr{W}_{\mathscr{U}})$ with $\langle x, y_0\rangle \in V$, the set $V^{\nabla} \cup [\langle x, y_0\rangle, \langle x, y_1\rangle)$ is open in $\mathrm{GOTP}(X * Y)$. Consider three cases:

(a-1) $\langle x, y_0\rangle$ is the left endpoint of V. By Definition 2.2, $[x, \rightarrow)$ is open in X since V is open in $\mathrm{GOTP}(X * \{y_0, y_1\})$. So $[\langle x, y_0\rangle, \langle x, y_1\rangle)$ is open in $\mathrm{GOTP}(X * Y)$ by Definition 2.2. Therefore $V^{\nabla} \cup [\langle x, y_0\rangle, \langle x, y_1\rangle)$ is open in $\mathrm{GOTP}(X*Y)$ by Lemma 2.5.

(a-2) $\langle x, y_0\rangle$ is the right endpoint of V. If $|V| = 1$, then $V^{\nabla} = \emptyset$ and $x \in R_X$. So $V^{\nabla} \cup [\langle x, y_0\rangle, \langle x, y_1\rangle) = [\langle x, y_0\rangle, \langle x, y_1\rangle)$ is open in $\mathrm{GOTP}(X * Y)$. If $|V| > 1$, then there is a $z \in X * Y$ such that $(z, \langle x, y_0\rangle) \subset V^{\nabla}$ since V is open and convex. So $V^{\nabla} \cup [\langle x, y_0\rangle, \langle x, y_1\rangle) = [\langle x, y_0\rangle, \langle x, y_1\rangle)$ is open in $\mathrm{GOTP}(X * Y)$.

(a-3) $\langle x, y_0\rangle$ is neither the right nor the left endpoint of V. In this case, $[\langle x, y_0\rangle, \langle x, y_1\rangle) \subseteq V^{\nabla}$. Hence $V^{\nabla} \cup [\langle x, y_0\rangle, \langle x, y_1\rangle) = V^{\nabla}$. The proof of Claim 1 is complete.

By Claim 1, $r_{\langle x,y_0\rangle}(\mathscr{O}(x,y_0,\mathscr{U}))$ is a countable open family of neighborhoods of $\langle x,y_0\rangle$ in GOTP$(X*Y)$. Define

$$r_2(\mathscr{U})=\bigcup\{r_{\langle x,y_0\rangle}(\mathscr{O}(x,y_0,\mathscr{U}))\colon\ x\in E(\mathscr{U})\}.$$

(b) Consider the point $\langle x,y_1\rangle$. Similarly to (a), GOTP$(X*Y)$ is monotonically Lindelöf at $\langle x,y_1\rangle$. Define

$$\begin{aligned}\mathscr{O}(x,y_1,\mathscr{U})=\{O\cap(V^{\nabla}\cup(\langle x,y_0\rangle,\langle x,y_1\rangle])\colon\ \langle x,y_1\rangle\in O\in\mathscr{U}\\ \text{and }\langle x,y_1\rangle\in V\in r_{X*\{y_0,y_1\}}(\mathscr{W}_{\mathscr{U}})\},\end{aligned}$$

and

$$r_3(\mathscr{U})=\bigcup\{r_{\langle x,y_1\rangle}(\mathscr{O}(x,y_1,\mathscr{U}))\colon\ x\in E(\mathscr{U})\},$$

where $r_{\langle x,y_1\rangle}$ is a monotone operator at $\langle x,y_1\rangle$ for GOTP$(X*Y)$.

For every $x\in X$, $\{x\}*Y$ is monotonically Lindelöf because $\{x\}*Y$ is homeomorphic to Y. Let $r_{\{x\}*Y}$ be a monotone Lindelöf operator for $\{x\}*Y$. Define

$$r_4(\mathscr{U})=\{V\cap(\langle x,y_0\rangle,\langle x,y_1\rangle)\colon\ V\in r_{\{x\}*Y}(\mathscr{U}_x)\text{ and }x\in E(\mathscr{U})\},$$

where $\mathscr{U}_x=\{U\cap(\{x\}*Y)\colon\ U\in\mathscr{U}\}$.

Put

$$r(\mathscr{U})=r_1(\mathscr{U})\cup r_2(\mathscr{U})\cup r_3(\mathscr{U})\cup r_4(\mathscr{U}).$$

Then $r(\mathscr{U})$ is a countable family of open subsets of GOTP$(X*Y)$ and refines $\mathscr{U}$.

Claim 2. $r(\mathscr{U})$ is a cover of GOTP$(X*Y)$. Let $z=\langle x,y\rangle\in X*Y$. If $x\notin E(\mathscr{U})$, then there exist $\langle x',y''\rangle$, $\langle x'',y''\rangle\in V$ and $V\in r_{X*\{y_0,y_1\}}(\mathscr{W}_{\mathscr{U}})$ such that $x'<x<x''$. By Lemma 2.5, $z\in\{x\}*Y\subseteq V^{\nabla}\in r_1(\mathscr{U})$. If $x\in E(\mathscr{U})$, then there are two cases: (i) $y\neq y_0$ and $y\neq y_1$. Then z is covered by $r_4(\mathscr{U})$. (ii) $y=y_0$ or $y=y_1$. By Claim 1, z is covered by $r_2(\mathscr{U})$ or $r_3(\mathscr{U})$, respectively.

Claim 3. r is a monotone operator for GOTP$(X*Y)$. Suppose that $\mathscr{V}$ and $\mathscr{U}$ are open covers of GOTP$(X*Y)$ and $\mathscr{V}$ refines $\mathscr{U}$. Let $A\in r(\mathscr{V})$. Then there are four cases as follows.

(I) $A\in r_1(\mathscr{V})$.

There exists an $O\in r_{X*\{y_0,y_1\}}(\mathscr{W}_{\mathscr{V}})$ with $|O|>1$ such that $A=O^{\nabla}$ by the definition of $r_1(\mathscr{V})$. Moreover, there exists an $H\in r_{X*\{y_0,y_1\}}(\mathscr{W}_{\mathscr{U}})$ such that $O\subseteq H$ since $\mathscr{V}$ refines $\mathscr{U}$ and $r_{X*\{y_0,y_1\}}(\mathscr{W}_{\mathscr{V}})$ refines $r_{X*\{y_0,y_1\}}(\mathscr{W}_{\mathscr{V}})$. Hence $|H|>1$; let $B=H^{\nabla}$. Then $A\subseteq B\in r_1(\mathscr{U})$.

(II) $A \in r_2(\mathscr{V})$.

Then there exists an $x \in E(\mathscr{V})$ such that $A \in r_{\langle x,y_0\rangle}(\mathscr{O}(x,y_0,\mathscr{V}))$. By the definition of $\mathscr{O}(x,y_0,\mathscr{V})$, there exists a $W \in r_{X*\{y_0,y_1\}}(\mathscr{W}_{\mathscr{V}})$ with $\langle x,y_0\rangle \in W$ and an $O \in \mathscr{V}$ with $\langle x,y_0\rangle \in O$ such that $A \subseteq O \cap (W^{\nabla} \cup [\langle x,y_0\rangle,\langle x,y_1\rangle))$. Moreover, $E(\mathscr{U}) \subseteq E(\mathscr{V})$ since $\mathscr{V}$ refines $\mathscr{U}$. Then there are two possibilities to consider:

(II-1) $x \in E(\mathscr{U})$.

Then $\mathscr{O}(x,y_0,\mathscr{V})$ refines $\mathscr{O}(x,y_0,\mathscr{U})$ since $\mathscr{V}$ refines $\mathscr{U}$ and $r_{X*\{y_0,y_1\}}(\mathscr{W}_{\mathscr{V}})$ refines $r_{X*\{y_0,y_1\}}(\mathscr{W}_{\mathscr{U}})$. Hence, $A \prec r_{\langle x,y_0\rangle}(\mathscr{O}(x,y_0,\mathscr{U})) \subseteq r_2(\mathscr{U})$.

(II-2) $x \notin E(\mathscr{U})$.

In this case, neither $\langle x,y_0\rangle$ nor $\langle x,y_1\rangle$ is an endpoint of any member of $r_{X*\{y_0,y_1\}}(\mathscr{W}_{\mathscr{U}})$. Hence, for every member $D \in r_{X*\{y_0,y_1\}}(\mathscr{W}_{\mathscr{U}})$ with $\langle x,y_0\rangle \in D$, $\{x\}*Y \subseteq D^{\nabla}$. In addition, there exists an $H \in r_{X*\{y_0,y_1\}}(\mathscr{W}_{\mathscr{U}})$ with $|H| > 1$ such that $W \subseteq H$ since $r_{X*\{y_0,y_1\}}(\mathscr{W}_{\mathscr{V}})$ refines $r_{X*\{y_0,y_1\}}(\mathscr{W}_{\mathscr{U}})$. Consequently, $A \subseteq W^{\nabla} \cup [\langle x,y_0\rangle,\langle x,y_1\rangle) \subseteq H^{\nabla} \in r_1(\mathscr{U})$.

(III) $A \in r_3(\mathscr{V})$.

The proof for this case is similar to (II).

(IV) $A \in r_4(\mathscr{V})$.

In this case, there exist an $x \in E(\mathscr{V})$ and $W \in r_{\{x\}*Y}(\mathscr{V}_x)$ such that $A = W \cap (\langle x,y_0\rangle,\langle x,y_1\rangle)$.

(IV-1) $x \in E(\mathscr{U})$.

Because $r_{\{x\}*Y}(\mathscr{V}_x)$ refines $r_{\{x\}*Y}(\mathscr{U}_x)$, there exists an $H \in r_{\{x\}*Y}(\mathscr{U}_x)$ such that $W \subseteq H$. So $A \subseteq H \cap (\langle x,y_0\rangle,\langle x,y_1\rangle) \in r_4(\mathscr{U})$.

(IV-2) $x \notin E(\mathscr{U})$.

Similarly to (II-2), there exists $H \in r_{X*\{y_0,y_1\}}(\mathscr{W}_{\mathscr{U}})$ with $|H| > 1$ and $\langle x,y_0\rangle \in H$ such that $A \subseteq (\langle x,y_0\rangle,\langle x,y_1\rangle) \subseteq H^{\nabla} \in r_1(\mathscr{U})$. □

In Theorem 2.11, we do not know whether "GOTP($X * \{0,1\}$) is monotonically Lindelöf" can be replaced by "X is monotonically Lindelöf". So the following questions are raised.

Question 1. Let X be a monotonically Lindelöf GO-space. Is GOTP($X * \{0,1\}$) monotonically Lindelöf?

Notice that if X is a separable GO-space, then GOTP($X * \{0,1\}$) is separable (see [2, Proposition 3.1]), so by [2, Proposition 3.1], GOTP($X * \{0,1\}$) is monotonically Lindelöf. Therefore to find a counterexample for Question 1, a candidate GO-space should not be separable.

Question 2. Is there a non-separable monotonically Lindelöf GO-space X such that GOTP($X * \{0,1\}$) is monotonically Lindelöf?

In [2], a branch space is constructed from the Aronszajn tree which is non-separable and monotonically Lindelöf. We also do not know whether the double arrow of the branch space described in [2, Example 2.6] is monotonically Lindelöf.

Finally, we consider the cases that Y in $\mathrm{GOTP}(X*Y)$ does not have the maximal point or the minimal point. Obviously if Y has neither the maximal point nor the minimal point, then $\mathrm{GOTP}(X*Y)$ is the topological sum of $|X|$ many copies of Y's so that $\mathrm{GOTP}(X*Y)$ even is not Lindelöf when $|X|>\omega$, which is irrelative to the topology on X.

Theorem 2.12. *Suppose that $X=(X,\tau,<)$ is a GO-space, Y is a monotonically Lindelöf GO-space and Y has the maximal (minimal) point but not the other one. Then $\mathrm{GOTP}(X*Y)$ is monotonically Lindelöf if and only if the GO-space $X'=(X,\tau',<)$ is monotonically Lindelöf, where τ' is the topology on X with the subbase $\tau\cup\{[x,\rightarrow)\colon x\in X\}$ $(\tau\cup\{(\leftarrow,x]\colon x\in X\})$.*

P r o o f. The necessity is obvious since X' is a closed subspace of $\mathrm{GOTP}(X*Y)$.

Now let $\mathscr{U}$ be an open cover of $\mathrm{GOTP}(X*Y)$ and assume that Y has the maximal point y_1. Without loss of generality we may assume that every element of $\mathscr{U}$ is convex. For each convex open subset U of $\mathrm{GOTP}(X*Y)$ with $|p(U)|>1$, let

$$U'=\{x\in X\colon \text{ there are } \langle x',y'\rangle,\langle x'',y''\rangle\in U \text{ such that } x'\leqslant x<x''\}.$$

Then $\mathscr{U}'=\{U'\colon U\in\mathscr{U}$ with $|p(U)|>1\}\cup\{\{x\}\colon x$ is isolated in $X'\}$ is an open convex cover of X' and satisfies that if $\mathscr{U}$ refines $\mathscr{V}$ then $\mathscr{U}'$ refines $\mathscr{V}'$. Let $r_{X'}$ be the monotonically Lindelöf operator on X'. For each convex open subset W of X', put

$$W^{\triangledown}=\begin{cases}\bigcup\{\{x\}*Y\colon & \text{there exist } x',x''\in W \text{ such that } x'<x\leqslant x''\}\\ & \text{if } W \text{ has no maximal point, or the maximal}\\ & \text{point } x_1\in L_X, \text{ or the maximal point of } W\\ & \text{is just the maximal point of } X;\\ \bigcup\{\{x\}*Y\colon & \text{there exist } x',x''\in W \text{ such that } x'<x\leqslant x''\}\setminus\{\langle x_1,y_1\rangle\}\\ & \text{if } W \text{ has the maximal point } x_1\notin L_X.\end{cases}$$

Then it is easy to see that for convex open subsets W_1 of X' with $W\subset W_1$, we have $W^{\triangledown}\subset W_1^{\triangledown}$ and if $W\subset U'$ for a convex open subset U of $\mathrm{GOTP}(X*Y)$, then $W^{\triangledown}\subset U$. Put $r_1\mathscr{U}=\{W^{\triangledown}\colon W\in r_{X'}\mathscr{U}'\}$. Notice that $r_1\mathscr{U}$ possibly does not cover $\mathrm{GOTP}(X*Y)$ since if, for example, $W\in r_{X'}\mathscr{U}'$ is a singleton then $W^{\triangledown}=\emptyset$. It is easy to see that for an $x\in X'$, if $\{x\}*Y$ is not covered by $r_1\mathscr{U}$, then either $(\{x\} * Y) \cap \bigcup r_1\mathscr{U} = \emptyset$ or $\{\langle x, y_1\rangle\} = (\{x\} * Y) \setminus \bigcup r_1\mathscr{U}$, and that $S = \{x \in X\colon \{x\} * Y \text{ is not covered by } r_1\mathscr{U}\}$ is at most countable. The set S can be divided to three parts as follows:

(1) $S_1 = \{x \in S\colon x \in L_X \text{ or } x \text{ is the maximal point of } X\}$;

(2) $S_2 = \{x \in S\colon \{\langle x, y_1\rangle\} = (\{x\} * Y) \setminus \bigcup r_1\mathscr{U}\}$;

(3) $S_3 = S \setminus (S_1 \cup S_2)$.

For $x \in S_1$, the subset $\{x\} * Y$ is open in $\mathrm{GOTP}(X * Y)$. Let $r_{\{x\}*Y}$ be the monotonically Lindelöf operator on $\{x\}*Y$. Put $r_2\mathscr{U} = \bigcup\{r_{\{x\}*Y}\mathscr{U}|_{\{x\}*Y}\colon x \in S_1\}$, where $\mathscr{U}|_{\{x\}*Y} = \{U \cap (\{x\} * Y)\colon U \in \mathscr{U}\}$. Then $r_2\mathscr{U}$ is a countable open family refining $\mathscr{U}$.

For $x \in S_2 \cup S_3$, the subset $\{x\} * Y$ is not open since the maximal point $\langle x, y_1\rangle$ of $\{x\} * Y$ has no immediate successor in $X * Y$ and $x \notin L_X$. Notice that the left and right cofinalities at $\langle x, y_1\rangle$ in $\mathrm{GOTP}(X * Y)$ are not larger than ω_1 since $\{x\} * Y$ and X' are monotonically Lindelöf. By Lemma 2.10, $\mathrm{GOTP}(X * Y)$ is monotonically Lindelöf at $\langle x, y_1\rangle$. Let $r_{\langle x,y_1\rangle}$ be the monotonically Lindelöf operator at $\langle x, y_1\rangle$ and let $\mathscr{U}_{\langle x,y_1\rangle} = \{U \in \mathscr{U}\colon \langle x, y_1\rangle \in U\}$. Put

$$r_3\mathscr{U} = \bigcup\{r_{\langle x,y_1\rangle}\mathscr{U}_{\langle x,y_1\rangle}\colon x \in S_2 \cup S_3\}.$$

For $x \in S_3$, put $\mathscr{V}_x = \{U \cap (\{x\} * Y) \setminus \{\langle x, y_1\rangle\}\colon U \in \mathscr{U}\}$. Then $\mathscr{V}'_x = \mathscr{V}_x \cup r_{\langle x,y_1\rangle}\mathscr{U}_{\langle x,y_1\rangle}|_{\{x\}*Y}$ is an open cover of $\{x\} * Y$. Put $r_x\mathscr{V}'_x = \{V \setminus \{\langle x, y_1\rangle\}\colon V \in r_{\{x\}*Y}\mathscr{V}'_x\}$. Define

$$r_4\mathscr{U} = \bigcup\{r_x\mathscr{V}'_x\colon x \in S_3\}.$$

Put $r\mathscr{U} = r_1\mathscr{U} \cup r_2\mathscr{U} \cup r_3\mathscr{U} \cup r_4\mathscr{U}$. It is easy to check that r is a monotonically Lindelöf operator on $\mathrm{GOTP}(X * Y)$.

For the case that Y has the minimal point but not the maximal one, the proof is similar. □

Remark. In Theorem 2.12, the assumption that X' is monotonically Lindelöf cannot be replaced by X is monotonically Lindelöf since, in general, the monotone Lindelöfness of X is not equivalent to the monotone Lindelöfness of X'. For example, let X be the GO-space constructed by deleting all limit ordinals less than ω_1 from the LOTS $[0, \omega_1]$. Then X is homeomorphic to the space constructed in [2, Example 2.2] so that X is a monotonically Lindelöf GO-space. Let $Y = (0, 1]$. Then Y is monotonically Lindelöf which has the maximal point but not the minimal one. Then X' is an uncountable discrete space, thus X' is even not Lindelöf.

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