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EXISTENCE OF POSITIVE SOLUTIONS FOR SINGULAR FOUR-POINT BOUNDARY VALUE PROBLEM WITH A p-LAPLACIAN

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Abstract. In this paper we deal with the four-point singular boundary value problem

$$\begin{cases} (\varphi_p(u'(t)))' + q(t)f(t, u(t), u'(t)) = 0, & t \in (0,1), \\ u'(0) - \alpha u(\xi) = 0, \quad u'(1) + \beta u(\eta) = 0, \end{cases}$$

where $\varphi_p(s) = |s|^{p-2}s$, $p > 1$, $0 < \xi < \eta < 1$, $\alpha, \beta > 0$, $q \in C[0,1]$, $q(t) > 0$, $t \in (0,1)$, and $f \in C([0,1] \times (0,+\infty) \times \mathbb{R}, (0,+\infty))$ may be singular at $u = 0$. By using the well-known theory of the Leray-Schauder degree, sufficient conditions are given for the existence of positive solutions.

Keywords: singular, four-point, positive solution, p-Laplacian

MSC 2010: 34B10, 34B16, 34B18

1. Introduction

Singular boundary value problems (BVPs) arise in applied mathematics and physics such as gas dynamics, nuclear physics, chemical reactions, studies of atomic structure and atomic calculation [7]. They also appear in the study of positive radial solutions of nonlinear elliptic equations. Therefore, they have been extensively studied in recent years, see, for instance, [1]–[5], [8], [13] and references therein. After studying singular two-point BVPs in detail, some authors began to pay attention to singular multi-point BVPs [9]–[12], [14]–[17]. They studied multi-point BVPs with

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several types of boundary conditions such as

$$u(0)=0,\ u(1)=\beta u(\eta); \qquad u(0)=\alpha u(\xi),\ u(1)=0;$$

$$u(0)=0,\ u(1)=\sum_{i=1}^{m-2}\beta_i u(\eta_i); \qquad u(0)=\sum_{i=1}^{m-2}\alpha_i u(\xi_i),\ u(1)=0;$$

$$u'(0)=0,\ u(1)=\sum_{i=1}^{m-2}\beta_i u(\eta_i); \qquad u(0)=\sum_{i=1}^{m-2}\alpha_i u(\xi_i),\ u'(1)=0;$$

$$u'(0)=0,\ u(1)=u(\eta); \qquad u(0)=\sum_{i=1}^{m-2}\alpha_i u(\xi_i),\ u'(1)=\sum_{i=1}^{m-2}\beta_i u'(\eta_i);$$

$$u(0)=\alpha u(\xi),\ u(1)=\beta u(\eta); \qquad u(0)=\sum_{i=1}^{m-2}\alpha_i u(\xi_i),\ u(1)=\sum_{i=1}^{m-2}\beta_i u(\eta_i),$$

where $\alpha,\beta,\alpha_i,\beta_i>0$, $0<\xi,\eta,\xi_i,\eta_i<1$ $(i=1,2,\ldots,m-1)$.

All the above multi-point boundary conditions are generalizations of the classical Dirichlet boundary, Neumann and mixed conditions. Due to its difficulty, few work has been done concerning the Sturm-Liouville-type multi-point boundary condition. It is an interesting problem to establish similar results for Sturm-Liouville-type BVP.

In this paper we aim at investigating the singular four-point BVP

$$\begin{cases} (\varphi_p(u'(t)))'+q(t)f(t,u(t),u'(t))=0, & t\in(0,1),\\ u'(0)-\alpha u(\xi)=0, \quad u'(1)+\beta u(\eta)=0, \end{cases} \tag{1.1}$$

where $\varphi_p(s)=|s|^{p-2}s$, $p>1$, $0<\xi<\eta<1$, $\alpha,\beta>0$, $q\in C[0,1]$, $q(t)>0$, $t\in(0,1)$, and $f\in C([0,1]\times(0,+\infty)\times\mathbb{R},(0,+\infty))$ may be singular at $u=0$. Sufficient conditions are given to guarantee the existence of positive solutions.

The method we use mainly depends on the theory of the Leray-Schauder degree. First, the positive solutions are considered for a constructed nonsingular BVP, then using the Arzelà-Ascoli theorem, we obtain positive solutions for the singular problem which is approximated by the family of solutions to the nonsingular BVPs. The key for finding a pseudo-lower-bound is by no means an easy task.

In this paper we consider the Banach space $X=C^1[0,1]$ equipped with the norm $\|u\|=\max\{|u|_0,|u'|_0\}$, where $|u|_0=\max\limits_{0\leqslant t\leqslant 1}|u(t)|$.

We say a function $u(t)$ is a positive solution to problem (1.1) if $u\in C^1[0,1]$, $\varphi_p(u')\in C^1[0,1]$, $u>0$ on $[0,1]$, the differential equation is satisfied for all $t\in(0,1)$ and the boundary conditions hold.

The following hypotheses are adopted throughout this paper:

(H_1) $0<\xi<\eta<1$, $0<\alpha\leqslant 1/\xi$, $0<\beta\leqslant 1/(1-\eta)$, $q\in C[0,1]$, $q(t)>0$, $t\in(0,1)$;

(H_2) $f\colon [0,1]\times(0,+\infty)\times\mathbb{R}\to(0,+\infty)$ is continuous, there are functions f_1, f_2 and h such that $0<f(t,y,z)\leqslant h(z)[f_1(y)+f_2(y)]$ on $(0,1)\times(0,+\infty)\times\mathbb{R}$ where f_1 is continuous, positive and nonincreasing on $(0,+\infty)$ and such that $\int_0^r f_1(s)\,\mathrm{d}s<+\infty$ for all $r>0$, f_2 is continuous, nonnegative and nondecreasing on $[0,+\infty)$ and h is continuous, positive and nondecreasing on $\mathbb{R}$;

(H_3) for given $H>0$ and $L>0$, there are a function $\psi_{H,L}$ and a constant $\gamma\in[0,1)$ such that $\psi_{H,L}$ is continuous on $[0,1]$, positive on $(0,1)$ and the inequality

$$f(t,y,z)\geqslant\psi_{H,L}(t)(\varphi_p(|z|))^{\gamma}$$

holds for $t\in[0,1]$, $y\in(0,H]$ and $z\in[-L,L]$;

(H_4) $I_1(x)=\int_0^x(\varphi_p^{-1}(u))/(h(\varphi_p^{-1}(u)))\,\mathrm{d}u<+\infty$, $x>0$.

2. Preliminaries

In this section we give some lemmas which are important in the proof of our main results.

Lemma 2.1. *Suppose that $e\in C[0,1]$, $e(t)>0$, $t\in(0,1)$, $A\geqslant0$ is a constant. Then the BVP*

$$(2.1)\qquad \begin{cases}(\varphi_p(u'(t)))'+e(t)=0, & t\in(0,1),\\ u'(0)-\alpha u(\xi)=-A, \quad u'(1)+\beta u(\eta)=\dfrac{\beta}{\alpha}A,\end{cases}$$

has a unique solution. Moreover, this solution can be expressed by

$$(2.2)\quad u(t)=\begin{cases}\dfrac{1}{\alpha}\varphi_p^{-1}\Big(\displaystyle\int_0^{\sigma}e(\tau)\,\mathrm{d}\tau\Big)+\int_{\xi}^{t}\varphi_p^{-1}\Big(\int_s^{\sigma}e(\tau)\,\mathrm{d}\tau\Big)\,\mathrm{d}s+\dfrac{A}{\alpha}, & 0\leqslant t\leqslant\sigma,\\ \dfrac{1}{\beta}\varphi_p^{-1}\Big(\displaystyle\int_{\sigma}^{1}e(\tau)\,\mathrm{d}\tau\Big)+\int_{t}^{\eta}\varphi_p^{-1}\Big(\int_{\sigma}^{s}e(\tau)\,\mathrm{d}\tau\Big)\,\mathrm{d}s+\dfrac{A}{\alpha}, & \sigma\leqslant t\leqslant1,\end{cases}$$

where σ satisfies

$$(2.3)\quad \frac{1}{\alpha}\varphi_p^{-1}\Big(\int_0^{\sigma}e(\tau)\,\mathrm{d}\tau\Big)+\int_{\xi}^{\sigma}\varphi_p^{-1}\Big(\int_s^{\sigma}e(\tau)\,\mathrm{d}\tau\Big)\,\mathrm{d}s = \frac{1}{\beta}\varphi_p^{-1}\Big(\int_{\sigma}^{1}e(\tau)\,\mathrm{d}\tau\Big)+\int_{\sigma}^{\eta}\varphi_p^{-1}\Big(\int_{\sigma}^{s}e(\tau)\,\mathrm{d}\tau\Big)\,\mathrm{d}s.$$

P r o o f. First, we show (2.3) has a unique solution. Set

$$v_1(t) := \frac{1}{\alpha}\varphi_p^{-1}\Big(\int_0^t e(\tau)\,\mathrm{d}\tau\Big) + \int_\xi^t \varphi_p^{-1}\Big(\int_s^t e(\tau)\,\mathrm{d}\tau\Big)\,\mathrm{d}s,$$
$$v_2(t) := \frac{1}{\beta}\varphi_p^{-1}\Big(\int_t^1 e(\tau)\,\mathrm{d}\tau\Big) + \int_t^\eta \varphi_p^{-1}\Big(\int_t^s e(\tau)\,\mathrm{d}\tau\Big)\,\mathrm{d}s.$$

Clearly, v_1 is continuous and strictly increasing on $[0,1]$, v_2 is continuous and strictly decreasing on $[0,1]$, and $v_1(0) < v_2(0)$, $v_1(1) > v_2(1)$, so $v_1(t) = v_2(t)$ has a unique solution, and we denote it by $\sigma \in (0,1)$.

Then it is easy to verify that (2.2) is a solution of (2.1). On the other hand, if u is a solution of (2.1), then $(\varphi_p(u'(t)))' = -e(t) < 0$ on $(0,1)$. Since $u'(0) - \alpha u(\xi) = -A$, $u'(1) + \beta u(\eta) = \beta\alpha^{-1}A$, there exists a unique $\hat\sigma \in (0,1)$ such that $u'(\hat\sigma) = 0$. Integrating the equation in (2.1) on $[0,\hat\sigma]$, we arrive at

$$u'(t) = \varphi_p^{-1}\Big(\int_t^{\hat\sigma} e(s)\,\mathrm{d}s\Big), \quad t \in [0,\hat\sigma], \tag{2.4}$$

which implies $u'(0) = \varphi_p^{-1}\big(\int_0^{\hat\sigma} e(\tau)\,\mathrm{d}\tau\big)$. Integrating (2.4) from 0 to t one obtains

$$u(t) = u(0) + \int_0^t \varphi_p^{-1}\Big(\int_s^{\hat\sigma} e(\tau)\,\mathrm{d}\tau\Big)\,\mathrm{d}s, \tag{2.5}$$

and then $u(\xi) = u(0) + \int_0^\xi \varphi_p^{-1}\big(\int_s^{\hat\sigma} e(\tau)\,\mathrm{d}\tau\big)\,\mathrm{d}s$. Together with the boundary conditions we have

$$u(t) = \frac{1}{\alpha}\varphi_p^{-1}\Big(\int_0^{\hat\sigma} e(\tau)\,\mathrm{d}\tau\Big) + \int_\xi^t \varphi_p^{-1}\Big(\int_s^{\hat\sigma} e(\tau)\,\mathrm{d}\tau\Big)\,\mathrm{d}s + \frac{A}{\alpha}, \quad 0 \leqslant t \leqslant 1,$$

which is, evidently, the unique solution to (2.1).

Similarly, we obtain

$$u(t) = \frac{1}{\beta}\varphi_p^{-1}\Big(\int_{\hat\sigma}^1 e(\tau)\,\mathrm{d}\tau\Big) + \int_t^\eta \varphi_p^{-1}\Big(\int_{\hat\sigma}^s e(\tau)\,\mathrm{d}\tau\Big)\,\mathrm{d}s + \frac{A}{\alpha}, \quad 0 \leqslant t \leqslant 1.$$

Let $t = \hat\sigma$, then $v_1(\hat\sigma) = v_2(\hat\sigma)$. Having in mind the definition of σ we can see that $\hat\sigma = \sigma$. Therefore the unique solution to (2.1) can be expressed by (2.2). The proof is complete. □

In order to solve (1.1), we consider the nonsingular problem

$$\begin{cases} (\varphi_p(u'(t)))' + q(t)F(t,u(t),u'(t)) = 0, & t \in (0,1), \\ u'(0) - \alpha u(\xi) = -A, \quad u'(1) + \beta u(\eta) = \dfrac{\beta}{\alpha}A, \end{cases} \tag{2.6}$$

where φ_p, q are the same as in (1.1), $F \in C([0,1]\times\mathbb{R}^2, (0,+\infty))$, $A \geqslant 0$.

Let $u \in X$ and define the operator $T\colon X \to X$ by

$$
(2.7) \qquad (Tu)(t) = \begin{cases} \dfrac{1}{\alpha}\varphi_p^{-1}\Big(\displaystyle\int_0^\sigma q(\tau)F(\tau,u(\tau),u'(\tau))\,\mathrm{d}\tau\Big) \\ \qquad + \displaystyle\int_\xi^t \varphi_p^{-1}\Big(\int_s^\sigma q(\tau)F(\tau,u(\tau),u'(\tau))\,\mathrm{d}\tau\Big)\,\mathrm{d}s + \frac{A}{\alpha}, \\ \hfill 0 \leqslant t \leqslant \sigma, \\ \dfrac{1}{\beta}\varphi_p^{-1}\Big(\displaystyle\int_\sigma^1 q(\tau)F(\tau,u(\tau),u'(\tau))\,\mathrm{d}\tau\Big) \\ \qquad + \displaystyle\int_t^\eta \varphi_p^{-1}\Big(\int_\sigma^s q(\tau)F(\tau,u(\tau),u'(\tau))\,\mathrm{d}\tau\Big)\,\mathrm{d}s + \frac{A}{\alpha}, \\ \hfill \sigma \leqslant t \leqslant 1, \end{cases}
$$

where σ is determined by (2.3) with $e(t)$ replaced by $q(t)F(t,u(t),u'(t))$. □

Lemma 2.2. *$T\colon X \to X$ is completely continuous.*

Proof. It is easy to prove that $T\colon X \to X$ is well defined. T is completely continuous if it is continuous and maps bounded subsets of X into relatively compact ones.

Now we show that T is continuous. Let $\lim\limits_{n\to+\infty} \|u_n - u\| = 0$. By Lemma 2.2, for any $n = 1,2,\ldots$ there exists a unique $\sigma_n \in (0,1)$ such that $A_{1,n}(\sigma_n) = A_{2,n}(\sigma_n)$, where

$$
\begin{aligned}
A_{1,n}(t) &= \frac{1}{\alpha}\varphi_p^{-1}\Big(\int_0^{\sigma_n} q(\tau)F(\tau,u_n(\tau),u_n'(\tau))\,\mathrm{d}\tau\Big) \\
&\quad + \int_\xi^t \varphi_p^{-1}\Big(\int_s^{\sigma_n} q(\tau)F(\tau,u_n(\tau),u_n'(\tau))\,\mathrm{d}\tau\Big)\,\mathrm{d}s, \\
A_{2,n}(t) &= \frac{1}{\beta}\varphi_p^{-1}\Big(\int_{\sigma_n}^1 q(\tau)F(\tau,u_n(\tau),u_n'(\tau))\,\mathrm{d}\tau\Big) \\
&\quad + \int_t^\eta \varphi_p^{-1}\Big(\int_{\sigma_n}^s q(\tau)F(\tau,u_n(\tau),u_n'(\tau))\,\mathrm{d}\tau\Big)\,\mathrm{d}s
\end{aligned}
$$

for $t \in [0,1]$. Since the sequence $\{\sigma_n\} \subset (0,1)$ is bounded, it contains a converging subsequence. Replacing, if necessary, $\{\sigma_n\}$ by such a subsequence, we denote $\sigma_0 =$

$\lim\limits_{n\to+\infty}\sigma_n$ and

$$\begin{aligned} A_{1,0}(t) &= \frac{1}{\alpha}\varphi_p^{-1}\biggl(\int_0^{\sigma_0} q(\tau)F(\tau,u_n(\tau),u_n'(\tau))\,\mathrm{d}\tau\biggr)\\ &\quad+\int_\xi^t \varphi_p^{-1}\biggl(\int_s^{\sigma_0} q(\tau)F(\tau,u_n(\tau),u_n'(\tau))\,\mathrm{d}\tau\biggr)\,\mathrm{d}s,\\ A_{2,0}(t) &= \frac{1}{\beta}\varphi_p^{-1}\biggl(\int_{\sigma_0}^1 q(\tau)F(\tau,u_n(\tau),u_n'(\tau))\,\mathrm{d}\tau\biggr)\\ &\quad+\int_t^\eta \varphi_p^{-1}\biggl(\int_{\sigma_0}^s q(\tau)F(\tau,u_n(\tau),u_n'(\tau))\,\mathrm{d}\tau\biggr)\,\mathrm{d}s \end{aligned}$$

for $t\in[0,1]$. Then $\lim\limits_{n\to+\infty}|A_{i,n}-A_{i,0}|_0=0$ for $i=1,2$. Let $\underline{\sigma}_n=\min\{\sigma_n,\sigma_0\}$ and $\overline{\sigma}_n=\max\{\sigma_n,\sigma_0\}$, $n=1,2,\ldots$. Of course, $\lim\limits_{n\to+\infty}t_n=\sigma_0$ holds for each sequence $\{t_n\}$ such that $\underline{\sigma}_n\leqslant t_n\leqslant\overline{\sigma}_n$ for all $n\in\mathbb{N}$.

Noticing that

$$\begin{aligned} \max_{t\in[\underline{\sigma}_n,\overline{\sigma}_n]}|A_{i,n}(t)-A_{j,0}(t)| &\leqslant \max_{t\in[\underline{\sigma}_n,\overline{\sigma}_n]}|A_{i,n}(t)-A_{i,n}(\sigma_n)|+|A_{j,n}(\sigma_n)-A_{j,0}(\sigma_0)|\\ &\quad+\max_{t\in[\underline{\sigma}_n,\overline{\sigma}_n]}|A_{j,0}(\sigma_0)-A_{j,0}(t)|\to 0\\ &\qquad\text{as } n\to+\infty,\ i,j=1,2,\ i\neq j, \end{aligned}$$

we have

$$\begin{aligned} |Tu_n-Tu_0|_0 \leqslant \max\Bigl\{&\max_{t\in[0,\underline{\sigma}_n]}|A_{1,n}(t)-A_{1,0}(t)|,\ \max_{t\in[\overline{\sigma}_n,1]}|A_{2,n}(t)-A_{2,0}(t)|,\\ &\max_{t\in[\underline{\sigma}_n,\overline{\sigma}_n]}|A_{1,n}(t)-A_{2,0}(t)|,\ \max_{t\in[\underline{\sigma}_n,\overline{\sigma}_n]}|A_{2,n}(t)-A_{1,0}(t)|\Bigr\}\to 0\\ &\qquad\text{as } n\to+\infty. \end{aligned}$$

Also,

$$\begin{aligned} A_{1,n}'(t) &= \varphi_p^{-1}\biggl(\int_t^{\sigma_n} q(\tau)F(\tau,u_n(\tau),u_n'(\tau))\,\mathrm{d}\tau\biggr),\quad 0\leqslant t\leqslant\sigma_n,\\ A_{2,n}'(t) &= -\varphi_p^{-1}\biggl(\int_{\sigma_n}^t q(\tau)F(\tau,u_n(\tau),u_n'(\tau)\biggr)\,\mathrm{d}\tau,\quad \sigma_n\leqslant t\leqslant 1. \end{aligned}$$

We have

$$\begin{aligned} |(Tu_n)'-(Tu_0)'|_0 \leqslant \max\Bigl\{&\max_{t\in[0,\underline{\sigma}_n]}|A_{1,n}'(t)-A_{1,0}'(t)|,\ \max_{t\in[\overline{\sigma}_n,1]}|A_{2,n}'(t)-A_{2,0}'(t)|,\\ &\max_{t\in[\underline{\sigma}_n,\overline{\sigma}_n]}|A_{1,n}'(t)-A_{2,0}'(t)|,\ \max_{t\in[\underline{\sigma}_n,\overline{\sigma}_n]}|A_{2,n}'(t)-A_{1,0}'(t)|\Bigr\}\to 0\\ &\qquad\text{as } n\to+\infty, \end{aligned}$$

so T is continuous.

Suppose $D \subset X$ is a bounded set. Then there exists $r > 0$ such that $\|u\| \leqslant r$ for all $u \in D$. When $u \in D$, we have

$$\begin{aligned}
|Tu|_0 = \frac{1}{2}\max_{t\in[0,1]}\bigg|&\frac{1}{\alpha}\varphi_p^{-1}\Big(\int_0^\sigma q(\tau)F(\tau,u(\tau),u'(\tau))\,\mathrm{d}\tau\Big)\\
&+\int_\xi^t \varphi_p^{-1}\Big(\int_s^\sigma q(\tau)F(\tau,u(\tau),u'(\tau))\,\mathrm{d}\tau\Big)\,\mathrm{d}s\\
&+\frac{1}{\beta}\varphi_p^{-1}\Big(\int_\sigma^1 q(\tau)F(\tau,u(\tau),u'(\tau))\,\mathrm{d}\tau\Big)\\
&+\int_t^\eta \varphi_p^{-1}\Big(\int_\sigma^s q(\tau)F(\tau,u(\tau),u'(\tau))\,\mathrm{d}\tau\Big)\,\mathrm{d}s\bigg| + \frac{A}{\alpha}\\
\leqslant \frac{1}{2}\varphi_p^{-1}&\Big(\max_{t\in[0,1],\,|y|_0\leqslant r,\,|z|_0\leqslant r} F(t,y,z)\Big)\Big(\frac{1}{\alpha}+\frac{1}{\beta}+2\Big)\varphi_p^{-1}\Big(\int_0^1 q(s)\,\mathrm{d}s\Big)+\frac{A}{\alpha}
\end{aligned}$$

and

$$|(Tu)'|_0 \leqslant \varphi_p^{-1}\Big(\max_{t\in[0,1],\,|y|_0\leqslant r,\,|z|_0\leqslant r} F(t,y,z)\Big)\varphi_p^{-1}\Big(\int_0^1 q(s)\,\mathrm{d}s\Big) =: \Gamma,$$

so $T(D)$ is bounded.

Moreover, for any $t_1, t_2 \in [0,1]$ we have

$$|(Tu)(t_1)-(Tu)(t_2)| = \bigg|\int_{t_1}^{t_2}(Tu)'(s)\,\mathrm{d}s\bigg| \leqslant \Gamma|t_1-t_2| \to 0 \quad \text{uniformly as } t_1 \to t_2,$$

and

$$|\varphi_p((Tu)'(t_1)) - \varphi_p((Tu)'(t_2))| = \bigg|\int_{t_1}^{t_2} q(s)F(s,u(s),u'(s))\,\mathrm{d}s\bigg| \to 0 \quad \text{as } t_1 \to t_2.$$

Since φ_p^{-1} is continuous, so $|(Tu)'(t_1)-(Tu)'(t_2)| \to 0$ uniformly as $t_1 \to t_2$.

By the Arzelà-Ascoli theorem, $T(D)$ is relatively compact. Therefore, T is completely continuous. □

Now we give a existence principle which is important to the proof of the main results.

Consider the BVP

$$(2.8)_\lambda \qquad \begin{cases} (\varphi_p(u'(t)))' + \lambda q(t)F(t,u(t),u'(t)) = 0, \quad t \in (0,1),\\ u'(0) - \alpha u(\xi) = -A, \quad u'(1) + \beta u(\eta) = \dfrac{\beta}{\alpha}A \end{cases}$$

where $\lambda \in (0,1)$, F, q, A are defined as before.

Lemma 2.3 (Existence principle). *Assume that there exists $M > A/\alpha$ such that for all $\lambda \in (0,1)$ and all solutions u of problem $(2.8)_\lambda$ the relation*

$$\|u\| \neq M$$

holds. Then problem $(2.8)_1$ has a solution u such that $\|u\| \leqslant M$.

P r o o f. For any $\lambda \in [0,1]$ define the operator

$$(T_\lambda u)(t) = \begin{cases} \lambda \dfrac{1}{\alpha}\varphi_p^{-1}\Big(\displaystyle\int_0^\sigma q(\tau)F(\tau,u(\tau),u'(\tau))\,\mathrm{d}\tau\Big) \\ \quad +\lambda \displaystyle\int_\xi^t \varphi_p^{-1}\Big(\int_s^\sigma q(\tau)F(\tau,u(\tau),u'(\tau))\,\mathrm{d}\tau\Big)\,\mathrm{d}s + \dfrac{A}{\alpha}, \quad 0 \leqslant t \leqslant \sigma, \\ \lambda \dfrac{1}{\beta}\varphi_p^{-1}\Big(\displaystyle\int_\sigma^1 q(\tau)F(\tau,u(\tau),u'(\tau))\,\mathrm{d}\tau\Big) \\ \quad +\lambda \displaystyle\int_t^\eta \varphi_p^{-1}\Big(\int_\sigma^s q(\tau)F(\tau,u(\tau),u'(\tau))\,\mathrm{d}\tau\Big)\,\mathrm{d}s + \dfrac{A}{\alpha}, \quad \sigma \leqslant t \leqslant 1. \end{cases}$$

Then by Lemma 2.2, $T_\lambda\colon X \to X$ is completely continuous. It is easy to verify that $u(t)$ is a solution to $(2.8)_\lambda$ if and only if u is a fixed point of T_λ in X. Let $\Omega = \{u \in X\colon \|u\| < M\}$, then Ω is an open set in X.

If there exists $u \in \partial\Omega$ such that $T_1 u = u$, then $u(t)$ is a solution of $(2.8)_1$ and the conclusion follows. Otherwise, for any $u \in \partial\Omega$ we have $T_1 u \neq u$. If $\lambda = 0$ and $u \in \partial\Omega$, then $(I - T_0)u(t) = u(t) - T_0 u(t) = u(t) - A/\alpha \not\equiv 0$, so $T_0 u \neq u$ for any $u \in \partial\Omega$. For $\lambda \in (0,1)$ and $u \in \partial\Omega$, the inequality $T_\lambda u \neq u$ follows directly from our assumptions.

By the property of the Leray-Schauder degree, we get

$$\deg\{I - T_1, \Omega, \theta\} = \deg\{I - T_0, \Omega, \theta\} = 1,$$

so T_1 has a fixed point u in Ω. That is, $(2.8)_1$ has a solution u satisfying $\|u\| \leqslant M$. The proof is completed. □

Lemma 2.4. *Suppose* (H_1) *and* (H_2) *hold. If u is a solution to problem* (2.6)*, then*

(i) *$u(t)$ is concave on $[0,1]$;*
(ii) *there exists a unique $\sigma \in (0,1)$ such that $u'(\sigma) = 0$, $u'(t) \geqslant 0$, $t \in [0,\sigma]$, $u'(t) \leqslant 0$, $t \in [\sigma,1]$;*
(iii) *$u(t) \geqslant A/\alpha$ on $[0,1]$;*
(iv) *$u(t) \geqslant t(1-t)|u|_0$ on $[0,1]$;*
(v) *$|u|_0 \leqslant K|u'|_0 + A/\alpha$, where $K = \max\big\{1/\alpha + 1, 1/\beta + 1\big\}$.*

Proof. Suppose $u(t)$ is a solution to BVP (2.6), then

(i) $(\varphi_p(u'(t)))' = -q(t)F(t,u(t),u'(t)) \leqslant 0$, $t \in (0,1)$, so $\varphi_p(u')$ is nonincreasing, therefore u' is nonincreasing, which implies the concavity of $u(t)$.

(ii) By the proof of Lemma 2.1, we know that there exists a unique $\sigma \in (0,1)$ such that $u'(\sigma) = 0$, $u'(t) \geqslant 0$, $t \in [0,\sigma]$, $u'(t) \leqslant 0$, $t \in [\sigma,1]$.

(iii) By Lemma 2.1 and $0 < \alpha \leqslant 1/\xi$, we have for $t \in [0,\sigma]$

$$\begin{aligned}
u(t) &= \frac{1}{\alpha}\varphi_p^{-1}\Big(\int_0^\sigma q(\tau)F(\tau,u(\tau),u'(\tau))\,\mathrm{d}\tau\Big) \\
&\quad + \int_\xi^t \varphi_p^{-1}\Big(\int_s^\sigma q(\tau)F(\tau,u(\tau),u'(\tau))\,\mathrm{d}\tau\Big)\,\mathrm{d}s + \frac{A}{\alpha} \\
&= \frac{1}{\alpha}\varphi_p^{-1}\Big(\int_0^\sigma q(\tau)F(\tau,u(\tau),u'(\tau))\,\mathrm{d}\tau\Big) \\
&\quad - \int_0^\xi \varphi_p^{-1}\Big(\int_s^\sigma q(\tau)F(\tau,u(\tau),u'(\tau))\,\mathrm{d}\tau\Big)\,\mathrm{d}s \\
&\quad + \int_0^t \varphi_p^{-1}\Big(\int_s^\sigma q(\tau)F(\tau,u(\tau),u'(\tau))\,\mathrm{d}\tau\Big)\,\mathrm{d}s + \frac{A}{\alpha} \\
&\geqslant \frac{1}{\alpha}\varphi_p^{-1}\Big(\int_0^\sigma q(\tau)F(\tau,u(\tau),u'(\tau))\,\mathrm{d}\tau\Big) \\
&\quad - \xi\varphi_p^{-1}\Big(\int_0^\sigma q(\tau)F(\tau,u(\tau),u'(\tau))\,\mathrm{d}\tau\Big) \\
&\quad + \int_0^t \varphi_p^{-1}\Big(\int_s^\sigma q(\tau)F(\tau,u(\tau),u'(\tau))\,\mathrm{d}\tau\Big)\,\mathrm{d}s + \frac{A}{\alpha} \\
&\geqslant \int_0^t \varphi_p^{-1}\Big(\int_s^\sigma q(\tau)F(\tau,u(\tau),u'(\tau))\,\mathrm{d}\tau\Big)\,\mathrm{d}s + \frac{A}{\alpha} \geqslant \frac{A}{\alpha}.
\end{aligned}$$

Similarly, by $0 < \beta \leqslant 1/(1-\eta)$, we can also obtain $u(t) \geqslant A/\alpha$ for $t \in [\sigma,1]$. Therefore, $u(t) \geqslant A/\alpha$ for $t \in [0,1]$.

(iv) Since u is concave and $u(t) \geqslant A/\alpha$ on $[0,1]$, we have

$$\begin{aligned}
\frac{u(t)}{t} &\geqslant \frac{u(\sigma)}{\sigma} \geqslant |u|_0 \Rightarrow u(t) \geqslant t|u|_0 \geqslant t(1-t)|u|_0, \quad t \in [0,\sigma], \\
\frac{u(t)}{1-t} &\geqslant \frac{u(\sigma)}{1-\sigma} \geqslant |u|_0 \Rightarrow u(t) \geqslant (1-t)|u|_0 \geqslant t(1-t)|u|_0, \quad t \in [\sigma,1],
\end{aligned}$$

thus, $u(t) \geqslant t(1-t)|u|_0$ for all $t \in [0,1]$.

(v) By the boundary condition, we have

$$\begin{aligned}
|u|_0 &= \max_{0\leqslant t\leqslant 1} |u(t)| = |u(\sigma)| \\
&= \Big|u(\xi) + \int_\xi^\sigma u'(t)\,\mathrm{d}t\Big| = \Big|\frac{1}{\alpha}u'(0) + \frac{A}{\alpha} + \int_\xi^\sigma u'(t)\,\mathrm{d}t\Big| \leqslant \Big(1+\frac{1}{\alpha}\Big)|u'|_0 + \frac{A}{\alpha};
\end{aligned}$$

similarly, we can obtain $|u|_0 \leqslant (1+1/\beta)|u'|_0 + A/\alpha$. Let $K = \max\{1+1/\alpha, 1+1/\beta\}$, then $|u|_0 \leqslant K|u'|_0 + A/\alpha$. The proof is complete. $\square$

3. Existence results

In this section we present some new existence results for positive solutions of the singular four-point BVP (1.1).

Theorem 3.1. *Assume* (H_1)–(H_4) *hold and*

$$(\mathrm{H}_5) \qquad \sup_{0<c<+\infty} \frac{c}{K\varphi_p^{-1}(I_1^{-1}(|q|_0 f_2(c)c + |q|_0 \int_0^c f_1(s)\,\mathrm{d}s))} > 1,$$

$$\text{where } K = \max\Big\{1+\frac{1}{\alpha}, 1+\frac{1}{\beta}\Big\}.$$

Then (1.1) *has a positive solution* u.

P r o o f. Choose $M_0 > 0$ and $0 < \varepsilon < M_0$ with

$$\frac{M_0}{\varepsilon + K\varphi_p^{-1}(I_1^{-1}(|q|_0 f_2(M_0)M_0 + |q|_0 \int_0^{M_0} f_1(s)\,\mathrm{d}s))} > 1. \tag{3.1}$$

Let $n_0 \in \{1,2,3,\ldots\}$ be chosen so that $1/n_0 \leqslant \varepsilon$ and let $N_0 = \{n_0, n_0+1, n_0+2, \ldots\}$.

In what follows, we show that

$$(3.2)^m \qquad \begin{cases} (\varphi_p(u'(t)))' + q(t)f(t,u(t),u'(t)) = 0, & t \in (0,1),\\ u'(0) - \alpha u(\xi) = -\dfrac{\alpha}{m}, \quad u'(1) + \beta u(\eta) = \dfrac{\beta}{m}, \end{cases}$$

has a positive solution for each $m \in N_0$.

To this end, we consider

$$(3.3)^m \qquad \begin{cases} (\varphi_p(u'(t)))' + q(t)f^*(t,u(t),u'(t)) = 0, & t \in (0,1),\\ u'(0) - \alpha u(\xi) = -\dfrac{\alpha}{m}, \quad u'(1) + \beta u(\eta) = \dfrac{\beta}{m}, \end{cases}$$

where

$$f^*(t,y,z) = \begin{cases} f(t,y,z), & y \geqslant \dfrac{1}{m}, \ z \in \mathbb{R},\\ f\Big(t,\dfrac{1}{m},z\Big), & y < \dfrac{1}{m}, \ z \in \mathbb{R}; \end{cases}$$

then $f^*(t,y,z) \in C([0,1]\times\mathbb{R}^2, (0,+\infty))$.

Consider

$$(3.3)_\lambda^m \qquad \begin{cases} (\varphi_p(u'(t)))' + \lambda q(t) f^*(t, u(t), u'(t)) = 0, & t \in (0,1), \\ u'(0) - \alpha u(\xi) = -\dfrac{\alpha}{m}, \quad u'(1) + \beta u(\eta) = \dfrac{\beta}{m}. \end{cases}$$

Let $u \in X$ be a solution of $(3.3)_\lambda^m$. From Lemma 2.4 we know that $u''(t) \leqslant 0$ on $(0,1)$, $u(t) \geqslant 1/m$ on $[0,1]$, and there exists $\sigma \in (0,1)$ such that $u'(\sigma) = 0$, $u'(t) \geqslant 0$, $t \in [0,\sigma]$ and $u'(t) \leqslant 0$, $t \in [\sigma, 1]$.

Now, for $t \in [0,\sigma]$, by (H_2) we have

$$(3.4) \qquad \begin{aligned} 0 \leqslant -(\varphi_p(u'(t)))' &= \lambda q(t) f^*(t, u(t), u'(t)) \\ &= \lambda q(t) f(t, u(t), u'(t)) \\ &\leqslant q(t) h(u'(t))[f_1(u(t)) + f_2(u(t))]. \end{aligned}$$

Multiplying (3.4) by u' one obtains

$$(3.5) \qquad -(\varphi_p(u'(t)))' \varphi_p^{-1}(\varphi_p(u'(t))) \leqslant q(t) h(u'(t))[f_1(u(t)) + f_2(u(t))]u'(t).$$

Integrating (3.5) from t to σ yields that

$$\begin{aligned} \int_0^{\varphi_p(u'(t))} \frac{\varphi_p^{-1}(s)}{h(\varphi_p^{-1}(s))} \, \mathrm{d}s &\leqslant |q|_0 \int_{u(t)}^{u(\sigma)} [f_1(s) + f_2(s)] \, \mathrm{d}s \\ &\leqslant |q|_0 f_2(u(\sigma)) u(\sigma) + |q|_0 \int_0^{u(\sigma)} f_1(s) \, \mathrm{d}s, \end{aligned}$$

i.e.

$$(3.6) \qquad I_1(\varphi_p(u'(t))) \leqslant |q|_0 f_2(u(\sigma)) u(\sigma) + |q|_0 \int_0^{u(\sigma)} f_1(s) \, \mathrm{d}s.$$

Similarly, for $t \in [\sigma, 1]$, let $I_2(x) = I_1(-x)$, $x < 0$. By (H_2) and (H_4) we have

$$(3.7) \qquad I_1(-\varphi_p(u'(t))) = I_2(\varphi_p(u'(t))) \leqslant |q|_0 f_2(u(\sigma)) u(\sigma) + |q|_0 \int_0^{u(\sigma)} f_1(s) \, \mathrm{d}s.$$

By (3.6) and (3.7) we obtain that

$$0 \leqslant |u'(t)| \leqslant \varphi_p^{-1}\biggl(I_1^{-1}\biggl(|q|_0 f_2(u(\sigma)) u(\sigma) + |q|_0 \int_0^{u(\sigma)} f_1(s) \, \mathrm{d}s\biggr)\biggr).$$

Considering Lemma 2.4 (v), we get

$$u(\sigma) \leqslant \frac{1}{m} + K\varphi_p^{-1}\left(I_1^{-1}\left(|q|_0 f_2(u(\sigma))u(\sigma) + |q|_0 \int_0^{u(\sigma)} f_1(s)\,\mathrm{d}s\right)\right)$$
$$\leqslant \varepsilon + K\varphi_p^{-1}\left(I_1^{-1}\left(|q|_0 f_2(u(\sigma))u(\sigma) + |q|_0 \int_0^{u(\sigma)} f_1(s)\,\mathrm{d}s\right)\right)$$

and

$$\frac{u(\sigma)}{\varepsilon + K\varphi_p^{-1}(I_1^{-1}(|q|_0 f_2(u(\sigma))u(\sigma) + |q|_0 \int_0^{u(\sigma)} f_1(s)\,\mathrm{d}s))} \leqslant 1. \tag{3.8}$$

Now (3.1) together with (3.8) implies

$$0 < u(\sigma) = |u|_0 < M_0. \tag{3.9}$$

Next, we notice that any solution u of $(3.3)_\lambda^m$ with $1/m \leqslant u(t) \leqslant M_0$ for $t \in [0,1]$ also satisfies

$$|u'(t)| < \varphi_p^{-1}\left(I_1^{-1}\left(|q|_0 f_2(M)M + |q|_0 \int_0^M f_1(s)\,\mathrm{d}s\right)\right) + 1 =: M_1, \quad t \in [0,1]. \tag{3.10}$$

Let $M = \max\{M_0, M_1\}$. From (3.9) and (3.10) we have

$$\|u\| \neq M.$$

Thus Lemmas 2.3 and 2.4 imply that for any $m \in N_0$, $(3.3)^m$ has a positive solution $u_m \in C^1[0,1]$ and there exists $\sigma_m \in (0,1)$ such that $u_m'(\sigma_m) = 0$, $u_m'(t) \geqslant 0$ on $[0,\sigma_m]$ and $u_m'(t) \leqslant 0$ on $[\sigma_m, 1]$.

In fact,

$$\frac{1}{m} \leqslant u_m(t) \leqslant M_0, \quad |u_m'(t)| < M_1 \quad \text{for } t \in [0,1] \tag{3.11}$$

and $u_m(t)$ satisfies

$$\begin{cases} (\varphi_p(u_m'(t)))' + q(t)f(t, u_m(t), u_m'(t)) = 0, & t \in (0,1), \\ u_m'(0) - \alpha u_m(\xi) = -\dfrac{\alpha}{m}, \quad u_m'(1) + \beta u_m(\eta) = \dfrac{\beta}{m}. \end{cases} \tag{3.12}$$

Next we will give a sharper lower bound on u_m, i.e., we will show that there exists a constant $k > 0$ independent of m such that $u_m(t) \geqslant kt(1-t)$ for $t \in [0,1]$.

Notice that (H_3) guarantees the existence of a function $\psi_{M_0,M_1}(t)$ which is continuous on $[0,1]$ and positive on $(0,1)$ with $f(t,u_m(t),u_m'(t)) \geqslant \psi_{M_0,M_1}(t)[\varphi_p(|u_m'(t)|)]^\gamma$ for $(t,u_m(t),u_m'(t)) \in [0,1]\times(0,M_0]\times[-M_1,M_1]$. For $t\in[0,\sigma_m)$ we have

$$-(\varphi_p(u_m'(t)))' \geqslant q(t)\psi_{M_0,M_1}(t)[\varphi_p(u_m'(t))]^\gamma,$$

thus,

$$-\frac{\mathrm{d}(\varphi_p(u_m'(t)))}{[\varphi_p(u_m'(t))]^\gamma} \geqslant q(t)\psi_{M_0,M_1}(t). \tag{3.13}$$

Integrating (3.13) from t to σ_m one gets

$$u_m'(t) \geqslant \varphi_p^{-1}\biggl(\biggl[(1-\gamma)\int_t^{\sigma_m} q(s)\psi_{M_0,M_1}(s)\,\mathrm{d}s\biggr]^{1/(1-\gamma)}\biggr). \tag{3.14}$$

By integrating (3.14) from 0 to t one obtains

$$u_m(t) \geqslant \int_0^t \varphi_p^{-1}\biggl(\biggl[(1-\gamma)\int_s^{\sigma_m} q(\tau)\psi_{M_0,M_1}(\tau)\,\mathrm{d}\tau\biggr]^{1/(1-\gamma)}\biggr)\,\mathrm{d}s. \tag{3.15}$$

Similarly, for $t\in(\sigma_m,1]$ we have

$$-u_m'(t) \geqslant \varphi_p^{-1}\biggl(\biggl[(1-\gamma)\int_{\sigma_m}^{t} q(s)\psi_{M_0,M_1}(s)\,\mathrm{d}s\biggr]^{1/(1-\gamma)}\biggr) \tag{3.16}$$

and

$$u_m(t) \geqslant \int_t^1 \varphi_p^{-1}\biggl(\biggl[(1-\gamma)\int_{\sigma_m}^{s} q(\tau)\psi_{M_0,M_1}(\tau)\,\mathrm{d}\tau\biggr]^{1/(1-\gamma)}\biggr)\,\mathrm{d}s. \tag{3.17}$$

Case 1. If $\xi<\sigma_m$, by (3.15) we have

$$u_m(\xi) \geqslant \int_0^\xi \varphi_p^{-1}\biggl(\biggl[(1-\gamma)\int_s^{\xi} q(\tau)\psi_{M_0,M_1}(\tau)\,\mathrm{d}\tau\biggr]^{1/(1-\gamma)}\biggr)\,\mathrm{d}s =: \theta_1 > 0.$$

By the concavity of $u_m(t)$ on (0,1) we have

$$\begin{aligned}
\frac{u_m(t)}{t} &\geqslant \frac{u_m(\xi)}{\xi} \Rightarrow u_m(t) \geqslant \frac{\theta_1}{\xi}t \geqslant \frac{\theta_1}{\xi}t(1-t) \quad \text{for } t\in[0,\xi],\\
\frac{u_m(t)}{1-t} &\geqslant \frac{u_m(\xi)}{1-\xi} \Rightarrow u_m(t) \geqslant \frac{\theta_1}{1-\xi}(1-t) \geqslant \frac{\theta_1}{1-\xi}t(1-t) \quad \text{for } t\in[\xi,1].
\end{aligned}$$

Let $k_0=\min\{\theta_1/\xi,\theta_1/(1-\xi)\}$, then $u_m(t)\geqslant k_0t(1-t)$ for $t\in[0,1]$.

Case 2. If $\eta > \sigma_m$, by (3.17) we have

$$u_m(\eta) \geqslant \int_\eta^1 \varphi_p^{-1}\biggl(\biggl[(1-\gamma)\int_\eta^s q(\tau)\psi_{M_0,M_1}(\tau)\,\mathrm{d}\tau\biggr]^{1/(1-\gamma)}\biggr)\,\mathrm{d}s =: \theta_2 > 0.$$

By the concavity of $u_m(t)$ on (0,1) we have

$$\begin{aligned}\frac{u_m(t)}{t} &\geqslant \frac{u_m(\eta)}{\eta} \Rightarrow u_m(t) \geqslant \frac{\theta_2}{\eta}t \geqslant \frac{\theta_2}{\eta}t(1-t) \quad \text{for } t \in [0,\eta],\\ \frac{u_m(t)}{1-t} &\geqslant \frac{u_m(\eta)}{1-\eta} \Rightarrow u_m(t) \geqslant \frac{\theta_2}{1-\eta}(1-t) \geqslant \frac{\theta_2}{1-\eta}t(1-t) \quad \text{for } t \in [\eta,1].\end{aligned}$$

Let $k_1 = \min\{\theta_2/\eta, \theta_2/(1-\eta)\}$, then $u_m(t) \geqslant k_1 t(1-t)$ for $t \in [0,1]$.

Consequently, there exists a constant $k = \min\{k_0, k_1\} > 0$ with

$$u_m(t) \geqslant kt(1-t), \quad t \in [0,1]. \tag{3.18}$$

First, we show that both $\{u_m\}_{m=1}^\infty$, $\{u_m'\}_{m=1}^\infty$ are bounded and equi-continuous on [0,1]. We need only to check the equi-continuity of $\{u_m'\}_{m=1}^\infty$ since (3.11) holds. For any $t \in [0,1]$ we have

$$\begin{aligned}-(\varphi_p(u_m'(t)))' &\leqslant q(t)h(u_m'(t))[f_1(u_m(t)) + f_2(u_m(t))]\\ &\leqslant h(M_1)[f_2(M_0) + f_1(kt(1-t))]|q|_0,\end{aligned} \tag{3.19}$$

which implies $\{u_m'\}_{m=1}^\infty$ is equi-continuous.

From (3.11), (3.18), (3.19) and (H_2) we get that both $\{u_m\}_{m=1}^\infty$, $\{u_m'\}_{m=1}^\infty$ are bounded and equi-continuous on [0,1].

The Arzelà-Ascoli theorem guarantees that there is a subsequence $N^* \subset N_0$ and a function $z(t) \in X$ with $u_m^{(j)}(t) \to z^{(j)}(t)$ uniformly on $[0,1]$ as $m \to +\infty$ through N^*. So $z'(0) - \alpha z(\xi) = 0$, $z'(1) + \beta z(\eta) = 0$ with $z(t) \geqslant kt(1-t)$, $t \in [0,1]$. Taking into account that $u_m(t)$ is the solution of $(3.2)^m$ and applying Lemma 2.1, we have

$$u_m(t) = \begin{cases} \dfrac{1}{\alpha}\varphi_p^{-1}\biggl(\displaystyle\int_0^{\sigma_m} q(\tau)f(\tau,u_m(\tau),u_m'(\tau))\,\mathrm{d}\tau\biggr)\\ \quad + \displaystyle\int_\xi^t \varphi_p^{-1}\biggl(\int_s^{\sigma_m} q(\tau)f(\tau,u_m(\tau),u_m'(\tau))\,\mathrm{d}\tau\biggr)\,\mathrm{d}s + \frac{1}{m},\\ \hfill 0 \leqslant t \leqslant \sigma_m,\\ \dfrac{1}{\beta}\varphi_p^{-1}\biggl(\displaystyle\int_{\sigma_m}^1 q(\tau)f(\tau,u_m(\tau),u_m'(\tau))\,\mathrm{d}\tau\biggr)\\ \quad + \displaystyle\int_t^\eta \varphi_p^{-1}\biggl(\int_{\sigma_m}^s q(\tau)f(\tau,u_m(\tau),u_m'(\tau))\,\mathrm{d}\tau\biggr)\,\mathrm{d}s + \frac{1}{m},\\ \hfill \sigma_m \leqslant t \leqslant 1.\end{cases} \tag{3.20}$$

Since the sequence $\{\sigma_m\} \subset (0,1)$ is bounded, it contains a converging subsequence. Replacing $\{\sigma_m\}$ by such a subsequence, if necessary, we denote $\sigma_0 = \lim\limits_{m\to+\infty} \sigma_m$. Let $m \to +\infty$ through N^* in (3.20). Then by Lemma 2.2, one has

$$z(t) = \begin{cases} \dfrac{1}{\alpha}\varphi_p^{-1}\Big(\displaystyle\int_0^{\sigma_0} q(\tau) f(\tau, z(\tau), z'(\tau))\,\mathrm{d}\tau\Big) \\ \quad + \displaystyle\int_\xi^t \varphi_p^{-1}\Big(\int_s^{\sigma_0} q(\tau) f(\tau, z(\tau), z'(\tau))\,\mathrm{d}\tau\Big)\,\mathrm{d}s, \quad 0 \leqslant t \leqslant \sigma_0, \\ \dfrac{1}{\beta}\varphi_p^{-1}\Big(\displaystyle\int_{\sigma_0}^{1} q(\tau) f(\tau, z(\tau), z'(\tau))\,\mathrm{d}\tau\Big) \\ \quad + \displaystyle\int_t^\eta \varphi_p^{-1}\Big(\int_{\sigma_0}^{s} q(\tau) f(\tau, z(\tau), z'(\tau))\,\mathrm{d}\tau\Big)\,\mathrm{d}s, \quad \sigma_0 \leqslant t \leqslant 1. \end{cases} \tag{3.21}$$

From (3.21) we deduce immediately that $z \in X$ and $(\varphi_p(z'(t))'+q(t)f(t,z(t),z'(t)) = 0$, $t \in (0,1)$. The proof of Theorem 3.1 is complete. □

4. Examples

In this section we give some explicit examples to illustrate our results.

Example 4.1. Consider the singular four-point BVP with p-Laplacian

$$\begin{cases} (\varphi_p(u'))' + \mu \mathrm{e}^{u'}[u^{-b} + \lambda_0 u^l + \lambda_1] = 0, \quad 0 < t < 1, \\ u'(0) - u\Big(\dfrac{1}{4}\Big) = 0, \quad u'(1) + u\Big(\dfrac{3}{4}\Big) = 0, \end{cases} \tag{4.1}$$

where $p > 1$, $0 < b < 1$, $\lambda_0 \geqslant 0$, $\lambda_1 \geqslant 0$, $l \geqslant 0$, $\mu > 0$. If μ satisfies

$$\sup_{0<c<+\infty} \frac{c}{2\varphi_p^{-1}(I_1^{-1}(\mu \mathrm{e}^c c + \mu(1-b)^{-1} c^{1/(1-b)}))} > 1 \tag{4.2}$$

then the BVP (4.1) has at least one positive solution.

P r o o f. Obviously, $\alpha = \beta = 1$, $\xi = \frac{1}{4}$, $\eta = \frac{3}{4}$, $q(t) = \mu > 0$ and $q \in C[0,1]$, $f(t,y,z) = \mathrm{e}^z(y^{-b} + \lambda_0 y^l + \lambda_1) \in C([0,1] \times (0,+\infty) \times \mathbb{R}, (0,+\infty))$. It is easy to verify

- (H_1) $0 < \alpha = 1 < 1/\xi = 4$, $\quad 0 < \beta = 1 < 1/(1-\eta) = 4$;
- (H_2) $0 < f(t,y,z) = \mathrm{e}^z(y^{-b} + \lambda_0 y^l + \lambda_1) \leqslant h(z)[f_1(y) + f_2(y)]$, where $f_1(y) = y^{-b} > 0$ is continuous, nonincreasing on $(0,+\infty)$ and for any $x > 0$, $\int_0^x f_1(u)\,\mathrm{d}u = \int_0^x u^{-b}\,\mathrm{d}u < +\infty$, $f_2(y) = \lambda_0 y^l + \lambda_1 > 0$ is continuous on $[0,+\infty)$, $h(z) = \mathrm{e}^z > 0$ is continuous and nondecreasing on $\mathbb{R}$;
- (H_3) for constants $H > 0$, $L > 0$ there exists a function $\psi_{H,L}(t) = H^{-b} > 0$ continuous on $[0,1]$ and a constant $\gamma = 1$ with $f(t,y,z) \geqslant \mathrm{e}^z H^{-b} \geqslant$

$\psi_{H,L}(t)\varphi_p(|z|)$ on $[0,1]\times(0,H]\times[-L,L]$, where L satisfies the equation $|z|^{p-1}=\mathrm{e}^z$.

By (4.2), we know (H_4) holds. Therefore, by Theorem 3.1 we can obtain that (4.1) has at least one positive solution $u(t)$. □

Example 4.2. Consider the singular four-point BVP

$$
(4.3)\qquad \begin{cases} u''+\dfrac{1}{9}(u^{-1/3}+1)=0, & 0<t<1,\\[2mm] u'(0)-u\Big(\dfrac{1}{4}\Big)=0, & u'(1)+u\Big(\dfrac{3}{4}\Big)=0. \end{cases}
$$

Then the BVP (4.3) has at least one positive solution.

P r o o f. Let $p=2$, $\alpha=\beta=1$, $\xi=\frac{1}{4}$, $\eta=\frac{3}{4}$, $q(t)=\frac{1}{9}$, $f(t,y,z)=y^{-1/3}+1$. Clearly (H_1) holds and $f_1(y)=y^{-1/3}>0$ is continuous, nonincreasing on $(0,+\infty)$, $f_2(y)=y+1>0$ is continuous on $[0,+\infty)$, $h(z)=1>0$ is continuous and nondecreasing on $\mathbb{R}$. So (H_2) holds. Take $\psi_{H,L}(t)=H^{-1/3}$, $\gamma=1$, then (H_3) holds. From $I_1(x)=\int_0^x s\,\mathrm{d}s=\frac{1}{2}x^2$, $x>0$, $I_2(x)=I_1(-x)=\frac{1}{2}x^2$, $x<0$ we obtain that (H_4) holds. By $q(t)=\frac{1}{9}$, $\sup\limits_{0<c<+\infty} c/(K\varphi_p^{-1}(I_1^{-1}(f_2(c)c+\int_0^c f_1(s)\,\mathrm{d}s)))=\sup\limits_{0<c<+\infty} c/(2(2c(c+1)+3c^{2/3})^{1/2}=1/\big(2\sqrt{2}\big)>\frac{1}{3}=(|q|_0)^{1/2}$, ($\mathrm{H}_5$) holds, too. By Theorem 3.1 we conclude that (4.3) has at least one positive solution $u(t)$. □

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