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## STATUSES AND BRANCH-WEIGHTS OF WEIGHTED TREES

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*Abstract.* In this paper we show that in a tree with vertex weights the vertices with the second smallest status and those with the second smallest branch-weight are the same.

*Keywords:* tree, status, branch-weight, median, centroid, second median, second centroid

*MSC 2010:* 05C12

## 1. INTRODUCTION

All graphs considered in this paper are finite, simple, and without loops. If  $G$  is a graph and there exists a weight function  $w: V(G) \cup E(G) \rightarrow R^+$ , then  $(G, w)$  is a *weighted graph*.

For a connected weighted graph  $(G, w)$ , pertinent definitions and notation are given below.

For a path  $P$  in  $G$ , the *weight length* of  $P$ , denoted by  $l_w(P)$ , is defined by

$$l_w(P) = \sum_{e \in E(P)} w(e).$$

For vertices  $x, y$  in  $G$ , the *weight distance* between  $x$  and  $y$ , denoted by  $d_w(x, y)$ , is defined by

$$d_w(x, y) = \min l_w(P),$$

where the minimum is taken over all paths  $P$  joining  $x$  and  $y$ .

For any vertex  $x$  of  $G$ , the *status* of  $x$ , denoted by  $s(x)$ , is defined by

$$s(x) = \sum_{y \in V(G)} w(y)d_w(y, x).$$

The *median* of  $G$ , denoted by  $M_1(G)$ , is the set of vertices in  $G$  with the smallest status, i.e.,  $M_1(G) = \{t \in V(G) : s(t) \leq s(x) \text{ for all } x \in V(G)\}$ .

The *second median* of  $G$ , denoted by  $M_2(G)$ , is the set of vertices in  $G$  with the second smallest status, i.e.,  $M_2(G) = \{t \in V(G) - M_1(G) : s(t) \leq s(x) \text{ for all } x \in V(G) - M_1(G)\}$ .

The *weight* of  $G$ , denoted by  $w(G)$ , is defined by

$$w(G) = \sum_{x \in V(G)} w(x).$$

Note that by definition, the weight of a connected weighted graph is independent of the weights of its edges.

If  $T$  is a tree and  $(T, w)$  is a weighted graph, then we say that  $(T, w)$  is a *weighted tree*.

For a weighted tree  $(T, w)$ , we give the following definitions and notation.

For any vertex  $x$  of  $T$ , the *branch-weight* of  $x$ , denoted by  $bw(x)$ , is the maximum weight of any component of  $T - x$ .

The *centroid* of  $T$ , denoted by  $C_1(T)$ , is the set of vertices in  $T$  with the smallest branch-weight, i.e.,  $C_1(T) = \{t \in V(T) : bw(t) \leq bw(x) \text{ for all } x \in V(T)\}$ .

The *second centroid* of  $T$ , denoted by  $C_2(T)$ , is the set of vertices in  $T$  with the second smallest branch-weight, i.e.,  $C_2(T) = \{t \in V(T) - C_1(T) : bw(t) \leq bw(x) \text{ for all } x \in V(T) - C_1(T)\}$ .

B. Zelinka [4] showed that any tree with constant weight function has its median equal to its centroid. A. Kang and D. Ault [2] extended the result to any weighted tree with constant vertex weight. O. Kariv and S. L. Hakimi [3] extended the result further to any weighted tree.

**Proposition 1.1** [3, Lemma 3.1]. *Let  $(T, w)$  be a weighted tree and  $x$  a vertex in  $T$ . Then  $x$  is in the centroid of  $T$  if and only if  $bw(x) \leq \frac{1}{2}w(T)$ .*

**Proposition 1.2** [3, Theorem 3.1]. *Any weighted tree has its median equal to its centroid.*

The purpose of this paper is to prove the following result.

**Theorem 3.3.** *Let  $(T, w)$  be a weighted tree with  $w(e) = 1$  for each  $e \in E(T)$ . Then  $M_2(T) = C_2(T)$ .*

## 2. SOME REMARKS

In this section, we give some remarks which we need for our discussions. Let us begin with those about statuses and medians. Though the main result deals with weighted trees, we state the remarks for connected weighted graphs if possible.

**Remark 2.1.** Let  $(G, w)$  be a connected weighted graph. Suppose that  $x, y$  are vertices in  $G$  such that  $xy$  is a cut edge. Let  $G_x, G_y$  be the components of  $G - xy$  with  $x \in V(G_x), y \in V(G_y)$ . Then we have  $s(x) - s(y) = w(xy)(w(G_y) - w(G_x))$ .

*Proof.*

$$\begin{aligned}
 s(x) - s(y) &= \sum_{t \in V(G)} w(t)(d_w(t, x) - d_w(t, y)) \\
 &= \sum_{t \in V(G_x)} w(t)(d_w(t, x) - d_w(t, y)) + \sum_{t \in V(G_y)} w(t)(d_w(t, x) - d_w(t, y)) \\
 &= \sum_{t \in V(G_x)} w(t)(-w(xy)) + \sum_{t \in V(G_y)} w(t)(w(xy)) \\
 &= w(xy)(w(G_y) - w(G_x)).
 \end{aligned}$$

□

**Remark 2.2.** Let  $(G, w)$  be a connected weighted graph. Suppose that  $x_1, x_2, \dots, x_k$  ( $k \geq 2$ ) are vertices in  $G$  such that  $x_1x_2 \dots x_k$  is a path and, for  $i = 1, 2, \dots, k - 1$ , each  $x_i x_{i+1}$  is a cut edge of  $G$  with  $w(x_i x_{i+1}) = 1$ . Let  $G_1, G_2, \dots, G_k$  be the components of  $G - \{x_1x_2, x_2x_3, \dots, x_{k-1}x_k\}$  with  $x_i \in V(G_i)$  for  $i = 1, 2, \dots, k$ . Then

$$s(x_1) - s(x_k) = \sum_{i=1}^k (-k - 1 + 2i)w(G_i).$$

*Proof.* We prove the result by induction on  $k$ . By Remark 2.1 this is true for  $k = 2$ . Suppose the result holds for  $k \geq 2$ . Now  $x_1x_2 \dots x_k x_{k+1}$  is a path in  $G$  such that each  $x_i x_{i+1}$  is a cut edge of  $G$  with  $w(x_i x_{i+1}) = 1$  for  $i = 1, 2, \dots, k$ , and  $G_1, G_2, \dots, G_k, G_{k+1}$  are components of  $G - \{x_1x_2, x_2x_3, \dots, x_{k-1}x_k, x_k x_{k+1}\}$  with  $x_i \in V(G_i), i = 1, 2, \dots, k, k + 1$ . Applying the induction hypothesis to the path  $x_1x_2 \dots x_k$  we have  $s(x_1) - s(x_k) = \sum_{i=1}^{k-1} (-k - 1 + 2i)w(G_i) + (k - 1)(w(G_k) + w(G_{k+1}))$ . Considering the path  $x_k x_{k+1}$  we have  $s(x_k) - s(x_{k+1}) = -(w(G_1) + w(G_2) + \dots +$

$w(G_k) + w(G_{k+1})$ . Thus

$$\begin{aligned}
s(x_1) - s(x_{k+1}) &= (s(x_1) - s(x_k)) + (s(x_k) - s(x_{k+1})) \\
&= \sum_{i=1}^{k-1} (-k - 1 + 2i)w(G_i) + (k - 1)(w(G_k) + w(G_{k+1})) \\
&\quad - (w(G_1) + w(G_2) + \dots + w(G_k)) + w(G_{k+1}) \\
&= \sum_{i=1}^{k-1} (-k - 2 + 2i)w(G_i) + (k - 2)w(G_k) + k \cdot w(G_{k+1}) \\
&= \sum_{i=1}^{k+1} (-k - 2 + 2i)w(G_i) \\
&= \sum_{i=1}^{k+1} (-(k + 1) - 1 + 2i)w(G_i).
\end{aligned}$$

This completes the proof.  $\square$

**Remark 2.3.** Let  $(G, w)$  be a connected weighted graph. Suppose that  $x_1, x_2, \dots, x_k$  ( $k \geq 3$ ) are vertices in  $G$  such that  $x_1x_2 \dots x_k$  is a path, and for  $i = 1, 2, \dots, k - 1$ ,  $x_i x_{i+1}$  is a cut edge of  $G$ . If  $s(x_1) \leq s(x_2)$ , then  $s(x_2) < s(x_3) < s(x_4) < \dots < s(x_k)$ .

*Proof.* It suffices to show that  $s(x_2) < s(x_3)$ . Let  $G_1, G_2, G_3$  be the components of  $G - \{x_1x_2, x_2x_3\}$  such that  $x_i \in V(G_i)$ ,  $i = 1, 2, 3$ . By Remark 2.1,

$$\begin{aligned}
s(x_1) - s(x_2) &= w(x_1x_2)((w(G_2) + w(G_3)) - w(G_1)), \\
s(x_2) - s(x_3) &= w(x_2x_3)(w(G_3) - (w(G_1) + w(G_2))).
\end{aligned}$$

Since  $s(x_1) \leq s(x_2)$ , we have  $w(G_2) + w(G_3) - w(G_1) \leq 0$ , which implies that  $w(G_3) - w(G_1) - w(G_2) < 0$  for  $w(G_2) > 0$ . Thus  $s(x_2) < s(x_3)$ .  $\square$

The above remark for trees with constant weight functions appeared in [1, Theorem 3.3].

**Remark 2.4.** The median of a weighted tree consists either of one single vertex or two vertices which are adjacent.

*Proof.* This follows immediately from Remark 2.3.  $\square$

The following remarks concern branch-weights of weighted trees.

**Remark 2.5.** Let  $x$  be a vertex of a weighted tree  $(T, w)$  and  $B$  a component of  $T - x$  such that  $bw(x) = w(B)$ . Then

- (1)  $bw(y) > bw(x)$  for each  $y \in V(T) - (V(B) \cup \{x\})$ ,
- (2) if  $y \in V(T) - \{x\}$  and  $bw(y) \leq bw(x)$  then  $y \in V(B)$ .

**Proof.** (1) Let  $B'$  be the component of  $T - y$  such that  $x \in V(B')$ . Then  $V(B') \supset \{x\} \cup V(B)$ . Hence  $bw(y) \geq w(B') > w(B) = bw(x)$ .

(2) This follows from (1). □

**Remark 2.6.** Let  $x_1x_2 \dots x_k$  ( $k \geq 3$ ) be a path in a weighted tree  $(T, w)$  where  $bw(x_1) \leq bw(x_2)$ . Then  $bw(x_2) < bw(x_3) < bw(x_4) < \dots < bw(x_k)$ .

**Proof.** It suffices to show that  $bw(x_2) < bw(x_3)$ . Let  $B$  be a component of  $T - x_2$  such that  $bw(x_2) = w(B)$ . By Remark 2.5(2),  $x_1 \in V(B)$ . From  $x_1 \in V(B)$  and  $x_2 \notin V(B)$ , we see that  $x_3 \notin V(B)$ . Since  $B$  is a component of  $T - x_2$  such that  $bw(x_2) = w(B)$ , by Remark 2.5(1), we conclude that  $bw(x_3) > bw(x_2)$ . □

### 3. MAIN RESULT

For a graph  $G$  and  $A \subset V(G)$ , we use  $N(A)$  to denote the set

$$\{x \in V(G) - A : x \text{ is adjacent to some vertex in } A\}.$$

**Lemma 3.1.** Let  $(T, w)$  be a weighted tree with  $w(e) = 1$  for each  $e \in E(T)$ . For each  $x \in N(M_1(T))$ , let  $T_x$  denote the component of  $T - M_1(T)$  with  $x \in V(T_x)$ . Then we have

- (1)  $s(x) - s(y) = 2w(T_y) - 2w(T_x)$  if  $x, y \in N(M_1(T))$  and  $x \neq y$ ,
- (2)  $M_2(T) = \{x \in N(M_1(T)) : w(T_x) \geq w(T_y) \text{ for all } y \in N(M_1(T))\}$ .

**Proof.** (1) By Remark 2.4,  $M_1(T)$  consists either of a single vertex or of two adjacent vertices. Let  $x, y \in N(M_1(T))$  with  $x \neq y$ . We distinguish two cases.

*Case 1.*  $x, y$  are adjacent to the same vertex in  $M_1(T)$ , say  $x, y$  are adjacent to  $m \in M_1(T)$ .

We see that  $T_x, T_y$  are the components of  $T - \{xm, my\}$  such that  $x \in V(T_x), y \in V(T_y)$ . Applying Remark 2.2 to the path  $xmy$ , we obtain  $s(x) - s(y) = -2w(T_x) + 2w(T_y)$  since  $w(e) = 1$  for all  $e \in E(T)$ .

*Case 2.*  $x, y$  are adjacent to distinct vertices in  $M_1(T)$ , say,  $x$  is adjacent to  $m_1$ ,  $y$  is adjacent to  $m_2$  where  $m_1, m_2 \in M_1(T)$  and  $m_1 \neq m_2$ .

By Remark 2.4  $m_1, m_2$  are adjacent. Let  $T' = T - \{xm_1, m_1m_2, m_2y\}$ . We see that  $T_x, T_y$  are components of  $T'$  such that  $x \in V(T_x), y \in V(T_y)$ . Let  $T_1, T_2$  be the components of  $T'$  such that  $m_1 \in V(T_1), m_2 \in V(T_2)$ . Applying Remark 2.2 to the path  $xm_1m_2y$ , we obtain  $s(x) - s(y) = -3w(T_x) - w(T_1) + w(T_2) + 3w(T_y)$  again since  $w(e) = 1$  for all  $e \in E(T)$ . Since  $m_1, m_2$  are in the median of  $T$ , we have, by Remark 2.1,  $w(T_x) + w(T_1) = w(T_2) + w(T_y)$ . Thus  $s(x) - s(y) = -2w(T_x) + 2w(T_y)$ .

(2) From Remark 2.3, we see that  $M_2(T) \subset N(M_1(T))$ . Thus  $M_2(T) = \{x \in N(M_1(T)): s(x) \leq s(y) \text{ for all } y \in N(M_1(T))\}$ . By(1), for  $x, y \in N(M_1(T))$  with  $x \neq y$ , we have  $s(x) \leq s(y)$  if and only if  $w(T_x) \geq w(T_y)$ . Thus  $M_2(T) = \{x \in N(M_1(T)): w(T_x) \geq w(T_y) \text{ for all } y \in N(M_1(T))\}$ . This completes the proof.  $\square$

**Lemma 3.2.** *Let  $(T, w)$  be a weighted tree. For each  $x \in N(C_1(T))$ , let  $T_x$  be the component of  $T - C_1(T)$  with  $x \in V(T_x)$ . Then we have*

- (1)  $bw(x) = w(T) - w(T_x)$  if  $x \in N(C_1(T))$ ,
- (2)  $C_2(T) = \{x \in N(C_1(T)): w(T_x) \geq w(T_y) \text{ for all } y \in N(C_1(T))\}$ .

*Proof.* (1) Let  $x \in N(C_1(T))$ . Suppose that  $x$  is adjacent to  $c$  where  $c \in C_1(T)$ . We see that  $T_x$  is the component of  $T - c$  with  $x \in V(T_x)$ . By Proposition 1.1,  $bw(c) \leq \frac{1}{2}w(T)$ . Thus  $w(T_x) \leq bw(c) \leq \frac{1}{2}w(T)$ . Let  $A_0, A_1, \dots, A_k$  be the components of  $T - x$  where  $c \in V(A_0)$ . Then  $V(T_x) = \{x\} \cup V(A_1) \cup V(A_2) \cup \dots \cup V(A_k)$ , which implies that for  $i = 1, 2, \dots, k$  we have  $w(A_i) < w(T_x) \leq \frac{1}{2}w(T)$ . Note also that  $w(A_0) = w(T) - w(T_x) \geq \frac{1}{2}w(T)$ . Hence  $bw(x) = \max_{0 \leq i \leq k} w(A_i) = w(A_0) = w(T) - w(T_x)$ .

(2) From Remark 2.6, we see that  $C_2(T) \subset N(C_1(T))$ . Thus  $C_2(T) = \{x \in N(C_1(T)): bw(x) \leq bw(y) \text{ for all } y \in N(C_1(T))\}$ . By(1), for  $x, y \in N(C_1(T))$  with  $x \neq y$ , we have  $bw(x) \leq bw(y)$  if and only if  $w(T_x) \geq w(T_y)$ . Thus  $C_2(T) = \{x \in N(C_1(T)): w(T_x) \geq w(T_y) \text{ for all } y \in N(C_1(T))\}$ . This completes the proof.  $\square$

Since by Proposition 1.2  $M_1(T) = C_1(T)$  for any weighted tree  $T$ , the main result of this paper now follows from Lemmas 3.1(2) and 3.2(2).

**Theorem 3.3.** *Let  $(T, w)$  be a weighted tree with  $w(e) = 1$  for each  $e \in E(T)$ . Then  $M_2(T) = C_2(T)$ .*

The above theorem cannot be extended to trees the edge weights of which are not constant. Consider the following example. Let  $T$  be the tree in Fig. 1 with  $w(a) = w(b) = w(c) = w(d) = w(e) = 1$ ,  $w(ab) = w(bc) = w(de) = 1$ ,  $w(bd) = 4$ . Then  $s(a) = s(c) = 14$ ,  $s(b) = 11$ ,  $s(d) = 15$ ,  $s(e) = 18$ ; thus  $M_2(T) = \{a, c\}$ . Further,  $bw(a) = bw(c) = bw(e) = 4$ ,  $bw(d) = 3$ ,  $bw(b) = 2$ ; thus  $C_2(T) = \{d\}$ . We have  $M_2(T) \neq C_2(T)$ .

Also in a weighted tree, the vertices with the third smallest status need not be the same as those with the third smallest branch-weight, even if the weight function of the tree is a constant. Consider the following example. Let  $T$  be the tree in Fig. 2 with constant vertex weight 1 and constant edge weight 1. Then  $s(a) = 26$ ,  $s(b) = 19$ ,  $s(c) = 14$ ,  $s(d) = 11$ ,  $s(e) = s(f) = s(g) = s(h) = s(i) = 18$ , and  $bw(a) = bw(e) = bw(f) = bw(g) = bw(h) = bw(i) = 8$ ,  $bw(b) = 7$ ,  $bw(c) = 6$ ,

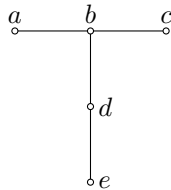


Fig. 1

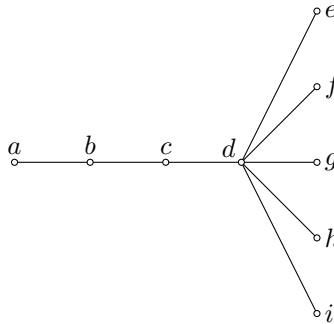


Fig. 2

$bw(d) = 3$ . Thus the vertices  $e, f, g, h, i$  are those with the third smallest status, and the vertex  $b$  is the one with the third smallest branch-weight.

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