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LAMBERT MULTIPLIERS BETWEEN $L^p$ SPACES

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Abstract. In this paper Lambert multipliers acting between $L^p$ spaces are characterized by using some properties of conditional expectation operator. Also, Fredholmness of corresponding bounded operators is investigated.

Keywords: conditional expectation, multipliers, multiplication operators, Fredholm operator

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1. Introduction and preliminaries

Let $L(X, \Sigma, \mu)$ be a $\sigma$-finite measure space. For any complete $\sigma$-finite sub-algebra $\mathcal{A} \subseteq \Sigma$ with $1 \leq p \leq \infty$, the $L^p$-space $L^p(X, \mathcal{A}, \mu|\mathcal{A})$ is abbreviated by $L^p(\mathcal{A})$, and its norm is denoted by $\| \cdot \|_p$. We view $L^p(\mathcal{A})$ as a Banach sub-space of $L^p(\Sigma)$. The support of a measurable function $f$ is defined by $\sigma(f) = \{ x \in X : f(x) \neq 0 \}$. All comparisons between two functions or two sets are to be interpreted as holding up to a $\mu$-null set.

To examine the weighted composition operators efficiently, Alan Lambert in [9] associated with each transformation $T$ the so-called conditional expectation operator $E(\cdot | \mathcal{A}) = E(\cdot)$ which is defined for each non-negative measurable function $f$ or for each $f \in L^p(\Sigma)$, and is uniquely determined by the conditions

(i) $E(f)$ is $\mathcal{A}$-measurable and
(ii) if $A$ is any $\mathcal{A}$-measurable set for which $\int_A f \, d\mu$ converges then

$$
\int_A f \, d\mu = \int_A E(f) \, d\mu.
$$
This operator will play a major role in our work, and we list here some of its useful properties:

- If \( g \) is \( \mathcal{A} \)-measurable then \( E(fg) = E(f)g \).
- \( |E(f)|^p \leq E(|f|^p) \).
- \( \|E(f)\|_p \leq \|f\|_p \).
- If \( f \geq 0 \) then \( E(f) \geq 0 \); if \( f > 0 \) then \( E(f) > 0 \).
- \( E(|f|^2) = |E(f)|^2 \) if and only if \( f \in L^p(\mathcal{A}) \).

As an operator on \( L^p(\Sigma) \), \( E(\cdot) \) is contractive idempotent and \( E(L^p(\Sigma)) = L^p(\mathcal{A}) \).

A real-valued \( \Sigma \)-measurable function \( f \) is said to be conditionable with respect to \( \mathcal{A} \) if
\[
\mu(\{ x \in X : E(f_+)(x) = E(f_-)(x) = \infty \}) = 0.
\]
In this case \( E(f) := E(f_+) - E(f_-) \).

If \( f \) is complex-valued, then \( f \) is conditionable if both the real and imaginary parts of \( f \) are conditionable and their respective expectations are not both infinite on the same set of positive measure. In this case, \( E(f) := E(\Re f) + iE(\Im f) \) (see [4]). We denote the linear space of all conditionable \( \Sigma \)-measurable functions on \( X \) by \( L^0(\Sigma) \).

For \( f \) and \( g \) in \( L^0(\Sigma) \), we define \( f \star g = fE(g) + gE(f) - E(fE(g)) \). Let \( 1 \leq p, q \leq \infty \). A measurable function \( u \in L^0(\Sigma) \) for which \( u \star f \in L^q(\Sigma) \) for each \( f \in L^p(\Sigma) \) is called a Lambert multiplier. In other words, \( u \in L^0(\Sigma) \) is a Lambert multiplier if and only if the corresponding \( \star \)-multiplication operator \( T_u : L^p(\Sigma) \rightarrow L^q(\Sigma) \) defined as \( T_u f = u \star f \) is bounded. Note that if \( u \) is a \( \mathcal{A} \)-measurable function or \( \mathcal{A} = \Sigma \), then \( u \in K^\star_p \) if and only if the multiplication operator \( M_u : L^p(\Sigma) \rightarrow L^q(\Sigma) \) is bounded.

In the next section, Lambert multipliers acting between two different \( L^p(\Sigma) \) spaces are characterized by using some properties of the conditional expectation operator.

In Section 3, Fredholmness of the corresponding \( \star \)-multiplication operators will be investigated.

### 2. Characterization of Lambert multipliers

Let \( 1 \leq p, q \leq \infty \). Define \( K^\star_{p,q} \), the set of all Lambert multipliers from \( L^p(\Sigma) \) into \( L^q(\Sigma) \), as follows:

\[
K^\star_{p,q} = \{ u \in L^0(\Sigma) : u \star L^p(\Sigma) \subseteq L^q(\Sigma) \}.
\]

\( K^\star_{p,q} \) is a vector subspace of \( L^0(\Sigma) \). Put \( K^\star_{p,p} = K^\star_p \). In the following theorem we characterize the members of \( K^\star_{p,q} \) in the case \( 1 \leq p = q < \infty \).

**Theorem 2.1.** Suppose \( 1 \leq p < \infty \) and \( u \in L^0(\Sigma) \). Then \( u \in K^\star_p \) if and only if \( E(|u|^p) \in L^\infty(\mathcal{A}) \).
Therefore, we have a representation theorem, there exists a unique function $g$ such that $\|g\|_p \leq \|E(|u|^p)\|_\infty \|f\|_p$. Hence $\|E(u)f\|_p \leq \|E(|u|^p)\|_\infty^{1/p} \|f\|_p$. A similar argument, using the fact that $E(fE(g)) = E(f)E(g)$, reveals that we also have

$$\|E(u)E(f)\|_p = \|uE(f)\|_p \leq \|E(|u|^p)\|_\infty^{1/p} \|f\|_p.$$ 

Thus $\|E(u)E(f)\|_p = \|uE(f)\|_p \leq \|E(|u|^p)\|_\infty^{1/p} \|f\|_p$. Accordingly, we get that $\|u \ast f\|_p \leq \|E(u)f\|_p + \|uE(f)\|_p + \|E(u)E(f)\|_p \leq 3\|E(|u|^p)\|_\infty^{1/p} \|f\|_p$.

It follows that $u \ast f \in L^p(\Sigma)$ and hence $u \in K^*_p$.

Now, suppose only that $u \in K^*_p$. An easy consequence of the closed graph theorem and the result guaranteeing a pointwise convergent subsequence for each $L^p(\Sigma)$ convergent sequence ensures that the operator $T_u : L^p(\Sigma) \rightarrow L^p(\Sigma)$ given by $T_u f = u \ast f$ is bounded. Define a linear functional $\varphi$ on $L^1(A)$ by

$$\varphi(f) = \int_A E(|u|^p)f \, d\mu, \quad f \in L^1(A).$$

We shall show that $\varphi$ is bounded. To this end, since for each $f \in L^1(A)$, $E(|f|^{1/p}) = |f|^{1/p} \in L^p(A)$, we have

$$|\varphi(f)| \leq \int_A E(|u|^p)|f| \, d\mu = \int_A (E(|u||f|^{1/p}) \, d\mu$$

$$= \int_A (|u||f|^{1/p}) \, d\mu = \|T_u|f|^{1/p}\|_p$$

$$\leq \|T_u\|_p \|f\|_1.$$ 

Thus, $\varphi$ is a bounded linear functional on $L^1(A)$ and $\|\varphi\| \leq \|T_u\|_p$. By the Riesz representation theorem, there exists a unique function $g \in L^\infty(A)$ such that $\varphi(f) = \int_A g f \, d\mu, \quad f \in L^1(A)$. Therefore, we have $g = E(|u|^p)$ a.e. on $X$ and hence $E(|u|^p) \in L^\infty(A)$.
Let \( \mathfrak{S} := \{ T_u : u \in K^*_p \} \) and let \( \mathfrak{S}' \) be the commutant of \( \mathfrak{S} \) in the algebra of all bounded linear operators. Still proceeding as in the proof of Theorem 6.6 given in [2] and Theorem 4.1 given in [6], one establishes that \( \mathfrak{S} = \mathfrak{S}' = \mathfrak{S}'' \) (see also [3]). Thus \( \mathfrak{S} \) is maximal abelian and hence it is norm closed.

For \( u \in K^*_p \) define \( \| u \|_{K^*_p} = \| E(|u|^p) \|^{1/p} \). Then precisely the same calculation as that shown in the proof of Theorem 2.1 yields that

\[
\| u \ast f \|_p \leq 3(\| E(|u|^p) \|^{1/p} \| f \|_p) < \infty, \quad f \in L^p(\Sigma),
\]

and

\[
\int_X E(|u|^p) |f| \, d\mu \leq \| T_u \|_p \| f \|_1, \quad f \in L^1(A).
\]

It follows that

\[
(2.1) \quad \| T_u \| \leq 3\| E(|u|^p) \|^{1/p}.
\]

and

\[
(2.2) \quad \| E(|u|^p) \|_\infty = \sup_{\| f \|_1 \leq 1} \int_X E(|u|^p) |f| \, d\mu \leq \| T_u \|_p.
\]

It follows from (2.1) and (2.2) that

\[
(2.3) \quad \| u \|_{K^*_p} \leq \| T_u \| \leq 3\| u \|_{K^*_p}.
\]

Consequently, \( \| \cdot \|_{K^*_p} \) and the operator norm \( \| \cdot \| \) are equivalent norms on \( \mathfrak{S} \). Also, since \( \mathfrak{S} \) is norm closed, it follows from (2.3) that \( K^*_p \) is a Banach space with the norm \( \| \cdot \|_{K^*_p} \).

Let \( 1 \leq q < p < \infty \). Our second task is the description of the members of \( K^*_{p,q} \) in terms of the conditional expectation induced by \( A \).

**Theorem 2.2.** Suppose \( 1 \leq q < p < \infty \) and \( u \in L^0(\Sigma) \). Then \( u \in K^*_{p,q} \) if and only if \( (E(|u|^q))^{1/q} \in L^r(A) \), where \( 1/p + 1/r = 1/q \).

**Proof.** Suppose \( (E(|u|^q))^{1/q} \in L^r(A) \). Let \( f \in L^p(\Sigma) \). Using the same method as in the proof of Theorem 2.1, we have

\[
\| E(u)f \|_q^q \leq \int_X E(|u|^q) |f| \, d\mu = \| E(|u|^q) \|^{1/q} \| f \|_p^q \leq \| (E(|u|^q))^{1/q} \|_r \| f \|_p^q.
\]

By similar computation we obtain

\[
\| uE(f) \|_q^q \leq \int_X |u|^q E(|f|^q) \, d\mu = \int_X E(|u|^q) E(|f|^q) \, d\mu \leq \| (E(|u|^q))^{1/q} \|_r \| E(|f|^q) \|_{p/q} \leq \| (E(|u|^q))^{1/q} \|_r \| f \|_p^q.
\]
and
\[ \|E(u)E(f)\|_q^q \leq \int_X E(|u|^q)E(|f|^q)\,d\mu \leq \|(E(|u|^q))^{1/q}\|_r^q \|(E(|f|^q))^{1/q}\|_p^q \leq \|(E(|u|^q))^{1/q}\|_r \|f\|_p. \]

Therefore we have \( \|Tu\| \leq 3\|(E(|u|^q))^{1/q}\|_r \|f\|_p \) for all \( f \in L^p(\Sigma) \). Consequently, \( Tu \) is bounded and hence \( u \in K_{p,q}^* \).

Now, suppose only that \( u \in K_{p,q}^* \). Define \( \varphi: L^{p/q}(A) \to \mathbb{C} \) given by \( \varphi(f) = \int_X E(|u|^q)f\,d\mu \). Clearly \( \varphi \) is a linear functional. We shall show that \( \varphi \) is bounded. For each \( f \in L^{p/q}(A) \) we get that
\[ |\varphi(f)| \leq \int_X E(|u|^q)|f|\,d\mu = \int_X E((|u||f|^{1/q})^q)\,d\mu = \|Tu|^{1/q}\|_q^q \leq \|Tu\|^q \|f\|_{p/q}. \]

It follows that \( \|\varphi\| \leq \|Tu\|^q \) and hence \( \varphi \) is bounded. By the Riesz representation theorem, there exists a unique \( g \in L^{r/q}(A) \) such that \( \varphi(f) = \int_X gf\,d\mu \) for each \( f \in L^{p/q}(A) \). Hence \( g = E(|u|^q) \) a.e. on \( X \). That is, \( (E(|u|^q)^{1/q} \in L^r(A) \) and hence the proof is complete. \( \square \)

Recall that an \( A \)-atom of the measure \( \mu \) is an element \( A \in A \) with \( \mu(A) > 0 \) such that for each \( F \in \Sigma \), if \( F \subseteq A \) then either \( \mu(F) = 0 \) or \( \mu(F) = \mu(A) \). A measure with no atoms is called non-atomic. It is a well-known fact that every \( \sigma \)-finite measure space \((X, A, \mu|_A)\) can be partitioned uniquely as
\[ X = \left( \bigcup_{n \in \mathbb{N}} A_n \right) \cup B, \]
where \( \{A_n\}_{n \in \mathbb{N}} \) is a countable collection of pairwise disjoint \( A \)-atoms and \( B \), being disjoint from each \( A_n \), is non-atomic (see [13]).

In the following theorem we characterize the members of \( K_{p,q}^* \) in the case \( 1 \leq p < q < \infty \).

**Theorem 2.3.** Suppose \( 1 \leq p < q < \infty \) and \( u \in L^0(\Sigma) \). Then \( u \in K_{p,q}^* \) if and only if
\begin{enumerate}
  \item \( E(|u|^q) = 0 \) a.e. on \( B \);
  \item \( M := \sup_{n \in \mathbb{N}} \frac{E(|u|^q)(A_n)}{\mu(A_n)^{q/r}} < \infty \), where \( \frac{1}{q} + \frac{1}{r} = \frac{1}{p} \).
\end{enumerate
Proof. Suppose that both (i) and (ii) hold. Then, for each \( f \in L^p(\Sigma) \) with \( \| f \|_p \leq 1 \) we have

\[
\| E(u)f \|_q^q \leq \int_X E(|u|^q)|f|^q \, d\mu = \left( \int_B + \int_{\bigcup A_n} \right) (E(|u|^q)|f|^q) \, d\mu \\
= \sum_{n \in \mathbb{N}} \int_{A_n} E(|u|^q)|f|^q \, d\mu = \sum_{n \in \mathbb{N}} E(|u|^q)(A_n)|f(A_n)|^q \mu(A_n) \\
= \sum_{n \in \mathbb{N}} \frac{(E(|u|^q)(A_n))}{\mu(A_n)^{q/p}} \| f(A_n) \|^q \leq M \| f \|_p^q \leq M,
\]

where we have used the fact that \( E(|u|^q) \) is a constant \( \mathcal{A} \)-measurable function on each \( A_n \) (see [5, Theorem I.7.3]). Consequently, \( \| E(u)f \|_q \leq M^{1/q} \). Since the conditional expectation operator \( E \) is a contraction, similar computation shows that \( \| uE(f) \|_q \leq M^{1/q} \) and \( \| E(u)E(f) \|_q \leq M^{1/q} \). It follows that \( \| T_u \| \leq 3M^{1/q} < \infty \) and hence \( u \in K_{p,q}^* \).

Conversely, suppose that \( u \in K_{p,q}^* \). First we show that \( E(|u|^q) = 0 \) a.e. on \( B \). Assuming the contrary, we can find some \( \delta > 0 \) such that \( \mu(\{ x \in B : E(|u|^q)(x) \geq \delta \}) > 0 \). Let \( F = \{ x \in B : E(|u|^q)(x) \geq \delta \} \). Since \( (X, \mathcal{A}, \mu|_{\mathcal{A}}) \) is a \( \sigma \)-finite measure space, we can suppose that \( \mu(F) < \infty \). Also, since \( F \) is non-atomic so for all \( n \in \mathbb{N} \) there exists \( F_n \subseteq F \) such that \( \mu(F_n) = \mu(F)/2^n \). For any \( n \in \mathbb{N} \), put \( f_n = 1/(\mu(F_n))^{1/p} \chi_{F_n} \). It is clear that \( f_n \in L^p(\mathcal{A}) \) and \( \| f_n \|_p = 1 \). Since \( q/p > 1 \), we have

\[
\infty > \| T_u \|_q^q \geq \| T_u f_n \|_q^q = \| u * f_n \|_q^q = \| u f_n \|_q^q \\
= \int_X |u f_n|^q \, d\mu = 1/(\mu(F_n)^{q/p}) \int_{F_n} |u|^q \, d\mu = 1/(\mu(F_n)^{q/p}) \int_{F_n} E(|u|^q) \, d\mu \\
\geq 2^\delta \mu(F_n)/(\mu(F_n)^{q/p}) = \delta \left( \frac{\mu(F)}{2^n} \right)^{1-q/p} = \delta \left( \frac{2^n}{\mu(F)} \right)^{q/p-1} \to \infty \quad \text{as} \quad n \to \infty,
\]

which is a contradiction. Hence we conclude that \( \mu(\{ x \in B : E(|u|^q)(x) \neq 0 \}) = 0 \).

Next, we examine the supremum in (ii). For any \( n \in \mathbb{N} \), put \( f_n = 1/(\mu(A_n)^{1/p}) \chi_{A_n} \). Then it is clear that \( f_n \in L^p(\mathcal{A}) \) and \( \| f_n \|_p = 1 \). Hence we have

\[
\infty > \| T_u \|_q^q \geq \| T_u f_n \|_q^q = \frac{1}{\mu(A_n)^{q/p}} \int_{A_n} E(|u|^q) \, d\mu \\
= \frac{1}{\mu(A_n)^{q/p}} E(|u|^q)(A_n) \mu(A_n) = \frac{E(|u|^q)(A_n)}{\mu(A_n)^{q/p}}.
\]

Since this holds for any \( n \in \mathbb{N} \), we get that \( M < \infty \). \qed
Theorem 2.4.
(i) \( u \in K^*_\infty \) if and only if \( u \in L^\infty(\Sigma) \).
(ii) If \( 1 \leq q < \infty \), then \( u \in K^*_{\infty,q} \) if and only if \( |u| \in L^q(\Sigma) \).
(iii) If \( 1 \leq p < \infty \), then \( u \in K^*_{p,\infty} \) if and only if \( u = 0 \) a.e. on \( B \) and
\[
\sup_{n \in \mathbb{N}} (|u|^p(A_n)/\mu(A_n)) < \infty.
\]

Proof. (i) Suppose that for each \( f \in L^\infty(\Sigma) \), \( u \ast f \in L^\infty(\Sigma) \). Since the conditional expectation operator \( E \) is a contraction, we obtain
\[
\|u\|_\infty = \|u\chi_X\|_\infty = \|Tu\chi_X\|_\infty \leq \|Tu\|_\infty < \infty.
\]

Conversely, suppose that \( u \in L^\infty(\Sigma) \). Then for each \( f \in L^\infty(\Sigma) \) we have \( \|Tu\|_\infty \leq 3\|u\|_\infty \|f\|_\infty \). Thus \( \|Tu\| \leq 3\|u\|_\infty \) and hence \( u \in K^*_\infty \). Consequently, we get (i).

(ii) Let \( |u| \in L^q(\Sigma) \) and \( f \in L^\infty(\Sigma) \). Then we have
\[
\|uE(f)\|_q = \int_X |uE(f)|^q d\mu \leq \|f\|_\infty^q \int_X |u|^q d\mu = \|f\|_\infty^q \|u\|_q^q.
\]

Hence, \( \|uE(f)\|_q \leq \|f\|_\infty \|u\|_q \). Similarly, we get \( \|uE(f)\|_q \leq \|f\|_\infty \|u\|_q \) and
\[
\|E(u)E(f)\|_q \leq \|f\|_\infty \|u\|_q. \quad \text{Thus } \|Tu\| \leq 3\|u\|_q \text{ and hence } u \in K^*_{\infty,q}.
\]

Conversely, suppose that \( Tu(L^\infty(\Sigma)) \subseteq L^q(\Sigma) \). Since \( Tu\chi_X \in L^q(\Sigma) \), it follows that
\[
\infty > \|Tu\chi_X\|_q^q = \int_X |Tu\chi_X|^q d\mu = \int_X |u|^q d\mu = \|u\|_q^q.
\]

Thus we get (ii).

(iii) Suppose that \( u = 0 \) a.e. on \( B \) and \( M := \sup_{n \in \mathbb{N}} (|u|^p(A_n)/\mu(A_n)) < \infty \). Then for each \( f \in L^p(\Sigma) \) with \( \|f\|_p \leq 1 \) we have
\[
\|uE(f)\|_\infty = \inf \{b \geq 0 : \|uE(f)\|_p \leq b\} = \inf \{b \geq 0 : \|u|^p|E(f)|_p \leq b\} = \inf \{b \geq 0 : \|u|^p(A_n)|E(f)(A_n)|_p \leq b, n \in \mathbb{N}\} \leq \inf \{b \geq 0 : \|u|^p(A_n)|E(f)|^p(A_n) \leq b, n \in \mathbb{N}\} \leq \sup_{n \in \mathbb{N}} \frac{|u|^p(A_n)}{\mu(A_n)} = M < \infty.
\]

Consequently, \( \|uE(f)\|_\infty \leq M^{1/p} \). Similarly, since
\[
|u(A_n)|^p = \frac{1}{\mu(A_n)} \int_{A_n} |u|^p d\mu = \frac{1}{\mu(A_n)} \int_{A_n} E(|u|^p) d\mu = (E(|u|^p))(A_n),
\]

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we get that \( \|fE(u)\|_{\infty} \leq M^{1/p} \) and \( \|E(u)E(f)\|_{\infty} \leq M^{1/p} \). Therefore \( \|T_u\| \leq 3M^{1/p} \) and hence \( u \in K_{p,\infty}^* \).

Conversely, suppose that \( u \in K_{p,\infty}^* \). First we show that \( u = 0 \) a.e. on \( B \). Assuming the contrary, we can find \( \delta > 0 \) such that \( \mu(\{x \in X: |u(x)| \geq \delta\}) > 0 \). Put \( F = \{x \in X: |u(x)| \geq \delta\} \). Since \( F \) is non atomic, choose a number \( a \) such that \( 0 < a < \mu(F) \) and a sequence \( F_1, F_2, \ldots \in \mathcal{A} \) of disjoint subsets of \( F \) such that \( \mu(F_k) = a/2^k \) for all \( k \in \mathbb{N} \). We define a function \( f_0 \) on \( X \) by

\[
f_0 = \sum_{k=1}^{\infty} 2^{k/2p} x_{F_k}.
\]

It is easy to show that \( f_0 \in L^p(\mathcal{A}) \), but that it is not in \( L^\infty(\mathcal{A}) \). It follows that

\[
\infty = \delta^{1/p} \|f_0\|_{L^\infty(\mathcal{A})} = \|\delta^{1/p} f_0\|_{L^\infty(\mathcal{A})} \leq \|T_u f_0\|_{L^\infty(\mathcal{A})} \leq \|T_u\| \|f_0\|_{L^p(\mathcal{A})} < \infty,
\]

which is a contradiction. Hence \( \mu(\{x \in X: |u(x)| \neq 0\}) = 0 \), in other words, \( u = 0 \) a.e. on \( B \).

Now, for any \( n \in \mathbb{N} \), put \( f_n = 1/(\mu(A_n)^{1/p}) \chi_{A_n} \). It is clear that for all \( n \in \mathbb{N} \), \( f_n \in L^p(\mathcal{A}) \) and \( \|f_n\|_p = 1 \). Then we obtain

\[
\infty > \|T_u\|^p \geq \|T_u f_n\|^p_\infty = \|u f_n\|^p_\infty \geq \frac{\|u\|^p(A_n)}{\mu(A_n)}.
\]

Therefore \( M < \infty \). This complete the proof. \hfill \Box

3. FREDHOLMNESS OF \( * \)-MULTIPLICATION OPERATORS

**Proposition 3.1.** Let \( 1 \leq p < \infty \), \( 1/p + 1/q = 1 \), and \( u \in K_{p}^* \). Then, for each \( g \in L^p(\Sigma) \), \( f \in L^q(\Sigma) \), and \( n \in \mathbb{N} \) we have

- (i) \( T_u^n g = (E(u))^n E(g) - nE(u)E(g) \),
- (ii) \( T_u^n f = (E(u))^{n-1} \{nE(\bar{u} f) + E(u)(f - nE(f))\} \).

**Proof.** (i) is trivial.

(ii) We will prove the result by induction. Since \( E(g) f = f E(g) \) for each \( g \in L^p(\Sigma) \) and \( f \in L^q(\Sigma) \), we have

\[
(g, T_u^n f) = (T_u^n g, f) = \int (u E(g) + g E(u) - E(g) E(u)) \overline{f} \, d\mu
\]

\[
= \int (g E(u \overline{f}) + E(u \overline{g} f) - g E(u) E(\overline{f})) \, d\mu
\]

\[
= \int g \left( E(\bar{u} f) + E(u \overline{f}) - E(u) E(f) \right) \, d\mu
\]

\[
= (g, E(\bar{u} f) + E(u \overline{f}) - E(u) E(f)),
\]

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which shows that the result holds for \( n = 1 \). Assume now that it holds for \( n = k \) and calculate

\[
T_u^{*}(k+1)f = T_u^{*}\left(\left(\bar{E}(u)\right)^{k-1}\left\{ kE(\bar{u}f) + \bar{E}(u)(f - kE(f)) \right\}\right)
\]

\[
= \left(\bar{E}(u)\right)^k\left( (k + 1)E(\bar{u}f) - kE(f)\bar{E}(u) \right)
\]

\[
+ \left(\bar{E}(u)\right)^k\left\{ kE(\bar{u}f) + \bar{E}(u)(f - kE(f)) \right\}
\]

\[
- \left(\bar{E}(u)\right)^k\left( kE(\bar{u}f) - (k - 1)\bar{E}(u)E(f) \right)
\]

\[
= \left(\bar{E}(u)\right)^k\left\{ (k + 1)E(\bar{u}f) + \bar{E}(u)(f - (k + 1)E(f)) \right\}.
\]

Thus the proposition is proved. \( \square \)

In what follows we use the symbols \( N(T_u) \) and \( R(T_u) \) to denote the kernel and the range of \( T_u \), respectively. We recall that \( T_u \) is said to be a Fredholm operator if \( R(T_u) \) is closed, \( \dim N(T_u) < \infty \), and \( \text{codim} R(T_u) < \infty \).

The next result gives a necessary and sufficient condition for a \( \star \)-multiplication operator \( T_u \) on \( L^p(\Sigma) \) to be a Fredholm operator, thereby generalizing the result in [11] for multiplication operators.

**Theorem 3.2.** Suppose that \( u \in K_p^* \) and \( A \) is a non-atomic measure space. Then the operator \( T_u \) is Fredholm on \( L^p(\Sigma) \) (1 \( \leqslant p \leqslant \infty \)) if and only if \( |E(u)| \geqslant \delta \) almost everywhere on \( X \) for some \( \delta > 0 \).

**Proof.** Suppose that \( T_u \) is a Fredholm operator. We first claim that \( T_u \) is onto. Suppose the contrary. Then there exists \( f_0 \in L^p(\Sigma) \setminus R(T_u) \). Since \( R(T_u) \) is closed, there exists \( g_0 \in L^q(\Sigma) \), the dual space of \( L^p(\Sigma) \), such that

\[
(g_0, f_0) = \int f_0g_0 \, d\mu = 1
\]

and

\[
(g_0, T_u f) = \int g_0\overline{T_u f} \, d\mu = 0, \quad f \in L^p(\Sigma).
\]

Now (3.1) yields that the set \( B_r = \{ x \in X : |E(\bar{f_0}g_0)(x)| \geqslant r \} \) has positive measure for some \( r > 0 \). As \( A \) is non-atomic, we can choose a sequence \( \{ A_n \} \) of subsets of \( B_r \) with \( 0 < \mu(A_n) < \infty \) and \( A_m \cap A_n = \emptyset \) for \( m \neq n \). Put \( g_n = \chi_{A_n}g_0 \). Clearly, \( g_n \in L^q(\Sigma) \) and is nonzero, because

\[
\int_X |\bar{f_0}g_n| \, d\mu \geqslant \int_{A_n} |\bar{f_0}g_n| \, d\mu = \int_{A_n} E(|\bar{f_0}g_0|) \geqslant \int_{A_n} |E(\bar{f_0}g_0)| \, d\mu \geqslant r\mu(A_n) > 0
\]
for each \( n \). Also, for each \( f \in L^p(\Sigma) \), \( \chi_{A_n} f \in L^p(\Sigma) \) and so (3.2) implies that

\[
(T_u^*g_n, f) = (g_n, T_u f) = \int_{A_n} g_0 \overline{T_u f} \, d\mu = \int_X g_0 T_u(\chi_{A_n} f) \, d\mu = (g_0, T_u(\chi_{A_n} f)),
\]

which implies that \( T_u^*g_n = 0 \) and so \( g_n \in \mathcal{N}(T_u^*) \). Since all the sets in \( \{A_n\} \) are disjoint, the sequence \( \{g_n\} \) forms a linearly independent subset of \( \mathcal{N}(T_u^*) \). This contradicts the fact that \( \dim \mathcal{N}(T_u^*) = \text{codim} \mathcal{R}(T_u) < \infty \). Hence \( T_u \) is onto. Let

\[
Z(E(u)) := \sigma(E(u))^c = \{ x \in X : E(u)(x) = 0 \}.
\]

Then \( \mu(Z(E(u))) = 0 \). Since \( \mu(Z(E(u))) > 0 \), there is an \( F \subseteq Z(E(u)) \) with \( 0 < \mu(F) < \infty \). If \( \chi_F \in \mathcal{R}(T_u) \), then there exists \( f \in L^p(\Sigma) \) such that \( T_u f = \chi_F \). Then

\[
\mu(F) = \int_X \chi_F \, d\mu = \int_F T_u f \, d\mu = \int_F E(T_u f) \, d\mu = \int_F E(u) E(f) \, d\mu = 0,
\]

and this is a contradiction. So \( \chi_F \in L^p(\Sigma) \setminus \mathcal{R}(T_u) \), which contradicts the fact that \( T_u \) is onto. For each \( n = 1, 2, \ldots \), let

\[
H_n = \left\{ x \in X : \frac{\|E(|u|^p)\|_\infty}{(n+1)^2} < |E(u)|^p(x) \leq \frac{\|E(|u|^p)\|_\infty}{n^2} \right\}
\]

and \( H = \{ n \in \mathbb{N} : \mu(H_n) > 0 \} \). Then the \( H_n \)'s are pairwise disjoint, \( X = \bigcup_{n=1}^{\infty} H_n \) and \( \mu(H_n) < \infty \) for each \( n \geq 1 \). Take

\[
f(x) = \begin{cases} \frac{|E(u)|}{\mu(H_n)^{1/p}}, & x \in H_n, \ n \in H, \\ 0, & \text{otherwise.} \end{cases}
\]

Then

\[
\int_X |f|^p \, d\mu = \sum_{n \in H} \int_{H_n} \frac{|E(u)|^p}{\mu(H_n)} \, d\mu \leq \sum_{n \in H} \frac{\|E(|u|^p)\|_\infty}{n^2} < \infty.
\]

Therefore \( f \in L^p(\mathcal{A}) \) and so there exist \( g \in L^p(\Sigma) \) such that \( T_u g = f \). Hence \( E(u)E(g) = E(T_u g) = f \). Since \( E(g) = f / E(u) \) off \( Z(E(u)) \) and \( \mu(Z(E(u))) = 0 \), it follows that

\[
\int_X |g|^p \, d\mu = \int_X |E(|g|^p)| \, d\mu \geq \int_X |E(g)|^p \, d\mu \\
= \int_X |f|^p \, d\mu = \sum_{n \in H} \int_{H_n} \frac{1}{\mu(H_n)} \, d\mu = \sum_{n \in H} 1.
\]
This implies that $H$ must be a finite set. So there is an $n_0$ such that $n \geq n_0$ implies $\mu(H_n) = 0$. Together with $\mu(Z(E(u))) = 0$, we obtain

$$
\mu\left( \left\{ x \in X : |E(u)|^p(x) \leq \frac{\|E(|u|^p)\|_\infty}{n_0^2} \right\} \right) = \mu\left( \bigcup_{n=n_0}^\infty H_n \cup Z(E(u)) \right) = 0,
$$

that is $|E(u)| \geq \left( \|E(|u|^p)\|_\infty / n_0^2 \right)^{1/p} := \delta$ almost everywhere on $X$.

Conversely, suppose that $|E(u)| \geq \delta$ a.e. on $X$ for some $\delta > 0$. Let $f \in \mathcal{N}(T_u^*)$. We have $T_u^* f = E(\bar{u} f) + E(u)(f - E(f)) = 0$ and so $E(\bar{u} f) = E(T_u^* f) = 0$. Thus

$$
\int_X \bar{u} f \, d\mu = \int_X E(\bar{u} f) \, d\mu = 0,
$$

which implies that

$$
\mathcal{N}(T_u^*) \subseteq \left\{ f \in L^p(\Sigma) : \int_X \bar{u} f \, d\mu = 0 \right\} \subseteq L^p(Z(u), \Sigma_{Z(u)}, \mu|_{Z(u)}).
$$

Also, since $E(|u|) \geq |E(u)| \geq \delta$ and $X$ is a $\sigma$-finite measure space, we have $|u| \geq \delta$ and hence $\mu(Z(u)) = 0$. It follows that

$$
codim \mathcal{R}(T_u) = \dim \mathcal{N}(T_u^*) = 0.
$$

Now, we shall show that $T_u$ has closed range. Let $\{T_u f_n\}$ be an arbitrary sequence in $\mathcal{R}(T_u)$ and let $\|T_u f_n - g\|_p \rightarrow 0$ for some $g \in L^p(\Sigma)$. Hence we have $E(u)E(f_n) = E(T_u f_n) \xrightarrow{L^p} E(g)$. Since by hypothesis $|E(u)| \geq \delta$, it follows that $E(g)/E(u) \in L^p(A)$ and $E(f_n) \xrightarrow{L^p} E(g)/E(u)$. Consequently,

$$
f_n \xrightarrow{L^p} \frac{1}{E(u)} \left\{ g + E(g) - \frac{u E(g)}{E(u)} \right\} := f
$$

and hence $T_u f_n \xrightarrow{L^p} T_u f$. Therefore $g = T_u f$, which implies that $T_u$ has closed range. Thus the theorem is proved. \qed

Now, we consider the particular case when $p = 2$. An operator $T$ on a Hilbert space $H$ is normal if $TT^* = T^*T$, and $T$ is self-adjoint if $T = T^*$. 

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Proposition 3.3. Let $u \in K^*_2$. Then the following claims are true:

(i) $T_u$ is a normal operator if and only if $u \in L^\infty(A)$.

(ii) $T_u$ is a self-adjoint operator if and only if $u \in L^\infty(A)$ is real valued.

Proof. (i) Assume $T_u$ is normal. Then for each $f \in L^2(\Sigma)$ we have $E(T_u T_u^* f) = E(u) E(uf)$ and $E(T_u^* T_u f) = E(f) E(|u|^2) + E(u) E(\bar{u}f) - E(\bar{u}) E(u) E(f)$. Therefore we obtain that $E(|u|^2) = |E(u)|^2$. Consequently $u \in L^\infty(A)$. Conversely, suppose that $u \in L^\infty(A)$ and take $f \in L^2(\Sigma)$. Then $T_u^* T_u f = T_u T_u^* f = |u|^2 f$, and hence $T_u$ is normal.

(ii) follows from (i). 

Example 3.4. Let $X = [-1,1]$, $d\mu = dx$, let $\Sigma$ be the Lebesgue sets, and $A$ the $\sigma$-subalgebra generated by the sets symmetric about the origin. Put $0 < a \leq 1$. Then for each $f \in L^2(\Sigma)$ we have

$$\int_{-a}^{a} E(f(x)) \, dx = \int_{-a}^{a} f(x) \, dx$$

$$= \int_{-a}^{a} \left\{ \frac{f(x) + f(-x)}{2} + \frac{f(x) - f(-x)}{2} \right\} \, dx$$

$$= \int_{-a}^{a} \frac{f(x) + f(-x)}{2} \, dx.$$

Consequently, $(Ef)(x) = (f(x) + f(-x))/2$. Now, if we take $u(x) = \cos x + \sin x$, then the $\ast$-multiplication operator $T_u: L^2(\Sigma) \to L^2(\Sigma)$ has the form

$$(T_u f)(x) = \left( \cos x + \frac{1}{2} \sin x \right) f(x) + \frac{1}{2} \sin x f(-x).$$

Direct computation shows that $(T_u^* f)(x) = (\cos x + \sin x/2) f(x) - \sin x/2 f(-x)$ and $|E(u)| \geq \cos 1$. Therefore, $T_u$ is a Fredholm but not a normal operator. 

References


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