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LAMBERT MULTIPLIERS BETWEEN $L^p$ SPACES

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Abstract. In this paper Lambert multipliers acting between $L^p$ spaces are characterized by using some properties of conditional expectation operator. Also, Fredholmness of corresponding bounded operators is investigated.

Keywords: conditional expectation, multipliers, multiplication operators, Fredholm operator

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1. Introduction and preliminaries

Let $L(X, \Sigma, \mu)$ be a $\sigma$-finite measure space. For any complete $\sigma$-finite sub-algebra $\mathcal{A} \subseteq \Sigma$ with $1 \leq p \leq \infty$, the $L^p$-space $L^p(X, \mathcal{A}, \mu|\mathcal{A})$ is abbreviated by $L^p(\mathcal{A})$, and its norm is denoted by $\| \cdot \|_p$. We view $L^p(\mathcal{A})$ as a Banach sub-space of $L^p(\Sigma)$. The support of a measurable function $f$ is defined by $\sigma(f) = \{x \in X: f(x) \neq 0\}$. All comparisons between two functions or two sets are to be interpreted as holding up to a $\mu$-null set.

To examine the weighted composition operators efficiently, Alan Lambert in [9] associated with each transformation $T$ the so-called conditional expectation operator $E(\cdot|\mathcal{A}) = E(\cdot)$ which is defined for each non-negative measurable function $f$ or for each $f \in L^p(\Sigma)$, and is uniquely determined by the conditions

(i) $E(f)$ is $\mathcal{A}$-measurable and

(ii) if $A$ is any $\mathcal{A}$-measurable set for which $\int_A f \, d\mu$ converges then

$$\int_A f \, d\mu = \int_A E(f) \, d\mu.$$
This operator will play a major role in our work, and we list here some of its useful properties:

- If \( g \) is \( \mathcal{A} \)-measurable then \( E(fg) = E(f)g \).
- \( |E(f)|^p \leq E(|f|^p) \).
- \( \|E(f)\|_p \leq \|f\|_p \).
- If \( f \geq 0 \) then \( E(f) \geq 0 \); if \( f > 0 \) then \( E(f) > 0 \).
- \( E(|f|^2) = |E(f)|^2 \) if and only if \( f \in L^p(\mathcal{A}) \).

As an operator on \( L^p(\Sigma) \), \( E(\cdot) \) is contractive idempotent and \( E(L^p(\Sigma)) = L^p(\mathcal{A}) \).

A real-valued \( \Sigma \)-measurable function \( f \) is said to be conditionable with respect to \( \mathcal{A} \) if \( \mu(\{ x \in X : E(f^+)(x) = E(f^-)(x) = \infty \}) = 0 \). In this case \( E(f) := E(f^+) - E(f^-) \).

If \( f \) is complex-valued, then \( f \) is conditionable if both the real and imaginary parts of \( f \) are conditionable and their respective expectations are not both infinite on the same set of positive measure. In this case, \( E(f) := E(\Re f) + iE(\Im f) \) (see [4]). We denote the linear space of all conditionable \( \Sigma \)-measurable functions on \( X \) by \( L^0(\Sigma) \).

Let \( 1 \leq p, q \leq \infty \). Define \( K^*_{p,q} \), the set of all Lambert multipliers from \( L^p(\Sigma) \) into \( L^q(\Sigma) \), as follows:

\[
K^*_{p,q} = \{ u \in L^0(\Sigma) : u \star L^p(\Sigma) \subseteq L^q(\Sigma) \}.
\]

\( K^*_{p,q} \) is a vector subspace of \( L^0(\Sigma) \). Put \( K^*_{p,p} = K^*_p \). In the following theorem we characterize the members of \( K^*_{p,q} \) in the case \( 1 \leq p = q < \infty \).

**Theorem 2.1.** Suppose \( 1 \leq p < \infty \) and \( u \in L^0(\Sigma) \). Then \( u \in K^*_p \) if and only if \( E(|u|^p) \in L^\infty(\mathcal{A}) \).
Therefore, we have the representation theorem, there exists a unique function \( \phi \) such that
\[
\| \phi \|^p = \int_X \| \phi \|^p \, d\mu \leq \int_X E(|u|^p)\| f \|^p \, d\mu \leq \| E(|u|^p) \|_\infty \| f \|^p.
\]
Hence \( \| E(u)f \|_p \leq \| E(|u|^p) \|_\infty \| f \|_p \). A similar argument, using the fact that \( E(f E(g)) = E(f)E(g) \), reveals that we also have
\[
\| E(u)f \|_p = \| uE(f) \|_p \leq \| E(|u|^p) \|_\infty \| f \|_p.
\]
Thus \( \| E(u)f \|_p = \| uE(f) \|_p \leq \| E(|u|^p) \|_\infty \| f \|_p \). Accordingly, we get that
\[
\| u \ast f \|_p \leq \| E(u)f \|_p + \| uE(f) \|_p + \| E(u)E(f) \|_p \leq 3\| E(|u|^p) \|_\infty \| f \|_p.
\]
It follows that \( u \ast f \in L^p(\Sigma) \) and hence \( u \in K_p^* \).

Now, suppose only that \( u \in K_p^* \). An easy consequence of the closed graph theorem and the result guaranteeing a pointwise convergent subsequence for each \( L^p(\Sigma) \) convergent sequence ensures that the operator \( T_u : L^p(\Sigma) \to L^p(\Sigma) \) given by \( T_u f = u \ast f \) is bounded. Define a linear functional \( \varphi \) on \( L^1(\mathcal{A}) \) by
\[
\varphi(f) = \int_X E(|u|^p) f \, d\mu, \quad f \in L^1(\mathcal{A}).
\]
We shall show that \( \varphi \) is bounded. To this end, since for each \( f \in L^1(\mathcal{A}) \), \( E(|f|^{1/p}) = |f|^{1/p} \in L^p(\mathcal{A}) \), we have
\[
|\varphi(f)| \leq \int_X E(|u|^p)|f| \, d\mu = \int_X (E(|u||f|^{1/p})^p \, d\mu
\]
\[
= \int_X |u||f|^{1/p}^p \, d\mu = \| T_u |f|^{1/p} \|_p^p
\]
\[
\leq \| T_u \|_p \| |f|^{1/p} \|_p = \| T_u \|_p \| f \|_1.
\]
Thus, \( \varphi \) is a bounded linear functional on \( L^1(\mathcal{A}) \) and \( \| \varphi \| \leq \| T_u \|_p \). By the Riesz representation theorem, there exists a unique function \( g \in L^\infty(\mathcal{A}) \) such that
\[
\varphi(f) = \int_X g f \, d\mu, \quad f \in L^1(\mathcal{A}).
\]
Therefore, we have \( g = E(|u|^p) \) a.e. on \( X \) and hence \( E(|u|^p) \in L^\infty(\mathcal{A}) \). \( \square \)
Let $\mathcal{S} := \{T_u: u \in K_p^*\}$ and let $\mathcal{S}'$ be the commutant of $\mathcal{S}$ in the algebra of all bounded linear operators. Still proceeding as in the proof of Theorem 6.6 given in [2] and Theorem 4.1 given in [6], one establishes that $\mathcal{S} = \mathcal{S}' = \mathcal{S}''$ (see also [3]). Thus $\mathcal{S}$ is maximal abelian and hence it is norm closed.

For $u \in K_p^*$ define $\|u\|_{K_p^*} = \|E(|u|^p)\|_{L^\infty}^{1/p}$. Then precisely the same calculation as that shown in the proof of Theorem 2.1 yields that

$$
\|u \ast f\|_p \leq 3(\|E(|u|^p)\|_{L^\infty}^{1/p}\|f\|_p) < \infty, \quad f \in L^p(\Sigma),
$$

and

$$
\int_X E(|u|^p)|f| \, d\mu \leq \|T_u\|_p\|f\|_1, \quad f \in L^1(A).
$$

It follows that

$$
\|T_u\| \leq 3\|E(|u|^p)\|_{L^\infty}^{1/p}
$$

and

$$
\|E(|u|^p)\|_{L^\infty} = \sup_{\|f\|_1 \leq 1} \int_X E(|u|^p)|f| \, d\mu \leq \|T_u\|^p.
$$

It follows from (2.1) and (2.2) that

$$
\|u\|_{K_p^*} \leq \|T_u\| \leq 3\|u\|_{K_p^*}.
$$

Consequently, $\|\cdot\|_{K_p^*}$ and the operator norm $\|\cdot\|$ are equivalent norms on $\mathcal{S}$. Also, since $\mathcal{S}$ is norm closed, it follows from (2.3) that $K_p^*$ is a Banach space with the norm $\|\cdot\|_{K_p^*}$.

Let $1 < q < p < \infty$. Our second task is the description of the members of $K_{p,q}^*$ in terms of the conditional expectation induced by $A$.

**Theorem 2.2.** Suppose $1 < q < p < \infty$ and $u \in L^0(\Sigma)$. Then $u \in K_{p,q}^*$ if and only if $(E(|u|^q))^{1/q} \in L^r(A)$, where $1/p + 1/r = 1/q$.

**Proof.** Suppose $(E(|u|^q))^{1/q} \in L^r(A)$. Let $f \in L^p(\Sigma)$. Using the same method as in the proof of Theorem 2.1, we have

$$
\|E(u)f\|_q^q \leq \int_X E(|u|^q)|f|^q \, d\mu = \|E(|u|^q)\|_{L^\infty}^{1/q} f\|_q^q \leq (E(|u|^q))^{1/q}\|f\|_p^q.
$$

By similar computation we obtain

$$
\|uE(f)\|_q^q \leq \int_X |u|^q E(|f|^q) \, d\mu = \int_X E(|u|^q)E(|f|^q) \, d\mu \\
\leq ((E(|u|^q))^{1/q}\|E(|f|^q)\|_{p/q}) \leq ((E(|u|^q))^{1/q}\|f\|_p^q.
$$

34
Therefore we have \( \|Tuf\| \leq 3\|(E(|u|^q))^{1/q}\|_r \|f\|_p \) for all \( f \in L^p(\Sigma) \). Consequently, \( T_u \) is bounded and hence \( u \in K^{*,p}_{p,q} \).

Now, suppose only that \( u \in K^{*,p}_{p,q} \). Define \( \varphi : L^{p/q}(A) \to \mathbb{C} \) given by \( \varphi(f) = \int_X E(|u|^q)f \, d\mu \). Clearly \( \varphi \) is a linear functional. We shall show that \( \varphi \) is bounded. For each \( f \in L^{p/q}(A) \) we get that

\[
|\varphi(f)| \leq \int_X E(|u|^q)|f| \, d\mu = \int_X E((|u||f|^{1/q})^q) \, d\mu = \|T_u|f|^{1/q}\|_q^q \leq \|T_u\|_q^q \|f\|_{p/q}.
\]

It follows that \( \|\varphi\| \leq \|T_u\|_q^q \) and hence \( \varphi \) is bounded. By the Riesz representation theorem, there exists a unique \( g \in L^{r/q}(A) \) such that \( \varphi(f) = \int_X gf \, d\mu \) for each \( f \in L^{p/q}(A) \). Hence \( g = E(|u|^q) \) a.e. on \( X \). That is, \((E(|u|^q))^{1/q} \in L^r(A) \) and hence the proof is complete. \( \square \)

Recall that an \( A \)-atom of the measure \( \mu \) is an element \( A \in A \) with \( \mu(A) > 0 \) such that for each \( F \in \Sigma \), if \( F \subseteq A \) then either \( \mu(F) = 0 \) or \( \mu(F) = \mu(A) \). A measure with no atoms is called non-atomic. It is a well-known fact that every \( \sigma \)-finite measure space \((X, A, \mu|_A)\) can be partitioned uniquely as

\[ X = \left( \bigcup_{n \in \mathbb{N}} A_n \right) \cup B, \]

where \( \{A_n\}_{n \in \mathbb{N}} \) is a countable collection of pairwise disjoint \( A \)-atoms and \( B \), being disjoint from each \( A_n \), is non-atomic (see [13]).

In the following theorem we characterize the members of \( K^{*,p}_{p,q} \) in the case \( 1 \leq p < q < \infty \).

**Theorem 2.3.** Suppose \( 1 \leq p < q < \infty \) and \( u \in L^0(\Sigma) \). Then \( u \in K^{*,p}_{p,q} \) if and only if

(i) \( E(|u|^q) = 0 \) a.e. on \( B \);

(ii) \( M := \sup_{n \in \mathbb{N}} \frac{E(|u|^q)(A_n)}{\mu(A_n)^{q/r}} < \infty \), where \( \frac{1}{q} + \frac{1}{r} = \frac{1}{p} \).

35
Proof. Suppose that both (i) and (ii) hold. Then, for each \( f \in L^p(\Sigma) \) with \( \|f\|_p \leq 1 \) we have

\[
\|E(u)f\|_q^q \leq \int_X E(|u|^q)|f|^q \, d\mu = \left( \int_B + \int_{\bigcup A_n} \right) (E(|u|^q)|f|^q) \, d\mu
\]

\[
= \sum_{n \in \mathbb{N}} \int_{A_n} E(|u|^q)|f|^q \, d\mu = \sum_{n \in \mathbb{N}} E(|u|^q)(A_n)|f(A_n)|^q \mu(A_n)
\]

\[
= \sum_{n \in \mathbb{N}} \frac{E(|u|^q)(A_n)}{\mu(A_n)^{q/p}} (|f(A_n)|^p \mu(A_n))^{q/p} \leq M \|f\|_p^q \leq M,
\]

where we have used the fact that \( E(|u|^q) \) is a constant \( \mathcal{A}\)-measurable function on each \( A_n \) (see [5, Theorem I.7.3]). Consequently, \( \|E(u)f\|_q \leq M^{1/q} \). Since the conditional expectation operator \( E \) is a contraction, similar computation shows that \( \|uE(f)\|_q \leq M^{1/q} \) and \( \|E(u)E(f)\|_q \leq M^{1/q} \). It follows that \( \|T_u\| \leq 3M^{1/q} < \infty \) and hence \( u \in K_{p,q}^* \).

Conversely, suppose that \( u \in K_{p,q}^* \). First we show that \( E(|u|^q) = 0 \) a.e. on \( B \). Assuming the contrary, we can find some \( \delta > 0 \) such that \( \mu(\{ x \in B: E(|u|^q)(x) \geq \delta \}) > 0 \). Put \( F = \{ x \in B: E(|u|^q)(x) \geq \delta \} \). Since \( (X, \mathcal{A}, \mu|_\mathcal{A}) \) is a \( \sigma \)-finite measure space, we can suppose that \( \mu(F) < \infty \). Also, since \( F \) is non-atomic so for all \( n \in \mathbb{N} \) there exists \( F_n \subset F \) such that \( \mu(F_n) = \mu(F)/2^n \). For any \( n \in \mathbb{N} \), put \( f_n = 1/(\mu(F_n))^{1/p} \chi_{F_n} \). It is clear that \( f_n \in L^p(\mathcal{A}) \) and \( \|f_n\|_p = 1 \). Since \( q/p > 1 \), we have

\[
\infty > \|T_u\| \geq \|T_u f_n\|_q^q = \|u * f_n\|_q^q = \|u f_n\|_q^q
\]

\[
= \int_X |u f_n|^q \, d\mu = 1/(\mu(F_n)^{q/p}) \int_{F_n} |u|^q \, d\mu = 1/(\mu(F_n)^{q/p}) \int_{F_n} E(|u|^q) \, d\mu
\]

\[
\geq \delta \mu(F_n)/(\mu(F_n)^{q/p}) = \delta \left( \frac{2^n}{\mu(F)} \right)^{q/p-1} \rightarrow \infty \text{ as } n \rightarrow \infty,
\]

which is a contradiction. Hence we conclude that \( \mu(\{ x \in B: E(|u|^q)(x) \neq 0 \}) = 0 \).

Next, we examine the supremum in (ii). For any \( n \in \mathbb{N} \), put \( f_n = 1/(\mu(A_n)^{1/p}) \chi_{A_n} \). Then it is clear that \( f_n \in L^p(\mathcal{A}) \) and \( \|f_n\|_p = 1 \). Hence we have

\[
\infty > \|T_u\| \geq \|T_u f_n\|_q^q = \frac{1}{\mu(A_n)^{q/p}} \int_{A_n} E(|u|^q) \, d\mu
\]

\[
= \frac{1}{\mu(A_n)^{q/p}} E(|u|^q)(A_n) \mu(A_n) = \frac{E(|u|^q)(A_n)}{\mu(A_n)^{q/r}}.
\]

Since this holds for any \( n \in \mathbb{N} \), we get that \( M < \infty \).
Theorem 2.4.
(i) $u \in K^*_\infty$ if and only if $u \in L^\infty(\Sigma)$.
(ii) If $1 \leq q < \infty$, then $u \in K^*_{\infty,q}$ if and only if $|u| \in L^q(\Sigma)$.
(iii) If $1 \leq p < \infty$, then $u \in K^*_{p,\infty}$ if and only if $u = 0$ a.e. on $B$ and

$$\sup_{n \in \mathbb{N}}(|u|^p(A_n)/\mu(A_n)) < \infty.$$ 

Proof. (i) Suppose that for each $f \in L^\infty(\Sigma)$, $u \ast f \in L^\infty(\Sigma)$. Since the conditional expectation operator $E$ is a contraction, we obtain

$$|u| = |u\chi_X| = |Tu\chi_X| \leq \|T_u\| < \infty.$$ 

Conversely, suppose that $u \in L^\infty(\Sigma)$. Then for each $f \in L^\infty(\Sigma)$ we have $\|T_u f\|_\infty \leq 3\|u\|_\infty\|f\|_\infty$. Thus $\|T_u\| \leq 3\|u\|_\infty$ and hence $u \in K^*_\infty$. Consequently, we get (i).

(ii) Let $|u| \in L^q(\Sigma)$ and $f \in L^\infty(\Sigma)$. Then we have

$$\|uE(f)\|_q^q = \int_X |uE(f)|^q d\mu \leq \|f\|_\infty^q \int_X |u|^q d\mu = \|f\|_\infty^q \|u\|_q^q.$$ 

Hence, $\|uE(f)\|_q \leq \|f\|_\infty\|u\|_q$. Similarly, we get $\|uE(f)\|_q \leq \|f\|_\infty\|u\|_q$ and $\|E(u)E(f)\|_q \leq \|f\|_\infty\|u\|_q$. Thus $\|T_u\| \leq 3\|u\|_q$ and hence $u \in K^*_{\infty,q}$. Conversely, suppose that $T_u(L^\infty(\Sigma)) \subseteq L^q(\Sigma)$. Since $T_u\chi_X \in L^q(\Sigma)$, it follows that

$$\infty > \|T_u\chi_X\|_q^q = \int_X |T_u\chi_X|^q d\mu = \int_X |u|^q d\mu = \|u\|_q^q.$$ 

Thus we get (ii).

(iii) Suppose that $u = 0$ a.e. on $B$ and $M := \sup_{n \in \mathbb{N}}(|u|^p(A_n)/\mu(A_n)) < \infty$. Then for each $f \in L^p(\Sigma)$ with $\|f\|_p \leq 1$ we have

$$\|uE(f)\|_\infty^p = \inf \{b \geq 0 : |uE(f)|^p \leq b\} = \inf \{b \geq 0 : |u|^p|E(f)|^p \leq b\} = \inf \{b \geq 0 : |u|^p(A_n)|E(f)(A_n)|^p \leq b, n \in \mathbb{N}\} \leq \inf \{b \geq 0 : |u|^p(A_n)(E|f|^p)(A_n) \leq b, n \in \mathbb{N}\} \leq \sup_{n \in \mathbb{N}} \frac{|u|^p(A_n)}{\mu(A_n)} = M < \infty.$$ 

Consequently, $\|uE(f)\|_\infty \leq M^{1/p}$. Similarly, since

$$|u(A_n)|^p = \frac{1}{\mu(A_n)} \int_{A_n} |u|^p d\mu = \frac{1}{\mu(A_n)} \int_{A_n} E(|u|^p) d\mu = (E(|u|^p))(A_n),$$
we get that \( \|f E(u)\|_{\infty} \leq M^{1/p} \) and \( \|E(u)E(f)\|_{\infty} \leq M^{1/p} \). Therefore \( \|T_u\| \leq 3M^{1/p} \) and hence \( u \in K^*_p \).

Conversely, suppose that \( u \in K^*_p \). First we show that \( u = 0 \) a.e. on \( B \). Assuming the contrary, we can find \( \delta > 0 \) such that \( \mu(\{x \in X: |u(x)| \geq \delta\}) > 0 \). Put \( F = \{x \in X: |u(x)| \geq \delta\} \). Since \( F \) is non atomic, choose a number \( a \) such that \( 0 < a < \mu(F) \) and a sequence \( F_1, F_2, \ldots \in A \) of disjoint subsets of \( F \) such that \( \mu(F_k) = a/2^{2k} \) for all \( k \in \mathbb{N} \). We define a function \( f_0 \) on \( X \) by

\[
f_0 = \sum_{k=1}^{\infty} 2^{k/2} \chi_{F_k}.
\]

It is easy to show that \( f_0 \in L^p(A) \), but that it is not in \( L^\infty(A) \). It follows that

\[
\infty = \delta^{1/p} \|f_0\|_{L^\infty(A)} = \|\delta^{1/p} f_0\|_{L^\infty(A)} \leq \|T_u f_0\|_{L^\infty(A)} \leq \|T_u\| \|f_0\|_{L^p(A)} < \infty,
\]

which is a contradiction. Hence \( \mu(\{x \in X: |u(x)| \neq 0\}) = 0 \), in other words, \( u = 0 \) a.e. on \( B \).

Now, for any \( n \in \mathbb{N} \), put \( f_n = 1/(\mu(A_n)^{1/p}) \chi_{A_n} \). It is clear that for all \( n \in \mathbb{N} \), \( f_n \in L^p(A) \) and \( \|f_n\|_p = 1 \). Then we obtain

\[
\infty > \|T_u\|_p \geq \|T_u f_n\|_\infty = \|u f_n\|_\infty \geq \frac{|u|^p(A_n)}{\mu(A_n)}.
\]

Therefore \( M < \infty \). This complete the proof.

\[\square\]

3. Fredholmness of *-multiplication operators

Proposition 3.1. Let \( 1 \leq p < \infty \), \( 1/p + 1/q = 1 \), and \( u \in K^*_p \). Then, for each \( g \in L^p(\Sigma) \), \( f \in L^q(\Sigma) \) and \( n \in \mathbb{N} \) we have

(i) \( T_u^n g = (E(u))^{n-1} (E(u)g + nu E(g) - n E(u) E(g)) \),

(ii) \( T_u^n f = (\overline{E(u)})^{n-1} \{n E(\overline{u}f) + E(u)(f - n E(f))\} \).

Proof. (i) is trivial.

(ii) We will prove the result by induction. Since \( E(g)f = f E(g) \) for each \( g \in L^p(\Sigma) \) and \( f \in L^q(\Sigma) \), we have

\[
(g, T_u^n f) = (T_u^n g, f) = \int (u E(g) - g E(u) - E(g) E(u)) \overline{f} \, d\mu
\]

\[
= \int (g E(u \overline{f}) + E(u) g \overline{f} - g E(u) E(\overline{f})) \, d\mu
\]

\[
= \int g (E(\overline{u}f) + E(u) f - E(u) E(f)) \, d\mu
\]

\[
= (g, E(\overline{u}f) + E(u) f - E(u) E(f)).
\]
which shows that the result holds for \( n = 1 \). Assume now that it holds for \( n = k \) and calculate

\[
T_u^{*(k+1)} f = T_u^{*} \left( (E(u))^{k-1} \{ kE(\bar{u}f) + E(u)(f - kE(f)) \} \right) \\
= (E(u))^k \{ (k + 1)E(\bar{u}f) - kE(f)E(u) \} \\
+ (E(u))^k \{ kE(\bar{u}f) + E(u)(f - kE(f)) \} \\
- (E(u))^k \{ kE(\bar{u}f) - (k - 1)E(u)E(f) \} \\
= (E(u))^k \{ (k + 1)E(\bar{u}f) + E(u)(f - (k + 1)E(f)) \}.
\]

Thus the proposition is proved. \( \square \)

In what follows we use the symbols \( \mathcal{N}(T_u) \) and \( \mathcal{R}(T_u) \) to denote the kernel and the range of \( T_u \), respectively. We recall that \( T_u \) is said to be a Fredholm operator if \( \mathcal{R}(T_u) \) is closed, \( \dim \mathcal{N}(T_u) < \infty \), and \( \text{codim} \mathcal{R}(T_u) < \infty \).

The next result gives a necessary and sufficient condition for a \( \star \)-multiplication operator \( T_u \) on \( L^p(\Sigma) \) to be a Fredholm operator, thereby generalizing the result in [11] for multiplication operators.

**Theorem 3.2.** Suppose that \( u \in K_p^\star \) and \( \mathcal{A} \) is a non-atomic measure space. Then the operator \( T_u \) is Fredholm on \( L^p(\Sigma) \) \((1 \leq p < \infty)\) if and only if \( |E(u)| \geq \delta \) almost everywhere on \( X \) for some \( \delta > 0 \).

**Proof.** Suppose that \( T_u \) is a Fredholm operator. We first claim that \( T_u \) is onto. Suppose the contrary. Then there exists \( f_0 \in L^p(\Sigma) \setminus \mathcal{R}(T_u) \). Since \( \mathcal{R}(T_u) \) is closed, there exists \( g_0 \in L^q(\Sigma) \), the dual space of \( L^p(\Sigma) \), such that

(3.1) \[
(g_0, f_0) = \int \bar{f}_0 g_0 \, d\mu = 1
\]

and

(3.2) \[
(g_0, T_u f) = \int g_0 \bar{T_u f} \, d\mu = 0, \quad f \in L^p(\Sigma).
\]

Now (3.1) yields that the set \( B_r = \{ x \in X : |E(\bar{f}_0 g_0)(x)| \geq r \} \) has positive measure for some \( r > 0 \). As \( \mathcal{A} \) is non-atomic, we can choose a sequence \( \{ A_n \} \) of subsets of \( B_r \) with \( 0 < \mu(A_n) < \infty \) and \( A_m \cap A_n = \emptyset \) for \( m \neq n \). Put \( g_n = \chi_{A_n} g_0 \). Clearly, \( g_n \in L^q(\Sigma) \) and is nonzero, because

\[
\int_X |\bar{f}_0 g_n| \, d\mu \geq \int_{A_n} |\bar{f}_0 g_n| \, d\mu = \int_{A_n} E(|\bar{f}_0 g_n|) \geq \int_{A_n} |E(\bar{f}_0 g_0)| \, d\mu \geq r \mu(A_n) > 0
\]
for each $n$. Also, for each $f \in L^p(\Sigma)$, $\chi_{A_n} f \in L^p(\Sigma)$ and so (3.2) implies that

$$(T_u^* g_n, f) = (g_n, T_u f) = \int_{A_n} g_0 T_u f \, d\mu = \int_x g_0 T_u (\chi_{A_n} f) \, d\mu = (g_0, T_u (\chi_{A_n} f)),$$

which implies that $T_u^* g_n = 0$ and so $g_n \in \mathcal{N}(T_u^*)$. Since all the sets in $\{A_n\}$ are disjoint, the sequence $\{g_n\}$ forms a linearly independent subset of $\mathcal{N}(T_u^*)$. This contradicts the fact that $\dim \mathcal{N}(T_u^*) = \text{codim} \mathcal{R}(T_u) < \infty$. Hence $T_u$ is onto. Let $Z(E(u)) := \sigma(E(u))^c = \{x \in \chi: E(u)(x) = 0\}$. Then $\mu(Z(E(u))) = 0$. Since $\mu(Z(E(u))) > 0$, there is an $F \subseteq Z(E(u))$ with $0 < \mu(F) < \infty$. If $\chi_F \in \mathcal{R}(T_u)$, then there exists $f \in L^p(\Sigma)$ such that $T_u f = \chi_F$. Then

$$\mu(F) = \int_{\chi_F} T_u f \, d\mu = \int_{\chi_F} E(T_u f) \, d\mu = \int_{\chi_F} E(u) E(f) \, d\mu = 0,$$

and this is a contradiction. So $\chi_F \in L^p(\Sigma) \setminus \mathcal{R}(T_u)$, which contradicts the fact that $T_u$ is onto. For each $n = 1, 2, \ldots$, let

$$H_n = \left\{ x \in \chi: \frac{||E(|u|^p)||_\infty}{(n+1)^2} < |E(u)|^p(x) \leq \frac{||E(|u|^p)||_\infty}{n^2} \right\}$$

and $H = \{ n \in \mathbb{N}: \mu(H_n) > 0 \}$. Then the $H_n$'s are pairwise disjoint, $X = \bigcup_{n=1}^{\infty} H_n$ and $\mu(H_n) < \infty$ for each $n \geq 1$. Take

$$f(x) = \begin{cases} \frac{|E(u)|}{\mu(H_n)^{1/p}}, & x \in H_n, \ n \in H, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\int_X |f|^p \, d\mu = \sum_{n \in H} \int_{H_n} \frac{|E(u)|^p}{\mu(H_n)} \, d\mu \leq \sum_{n \in H} \frac{||E(|u|^p)||_\infty}{n^2} \mu(H_n) < \infty.$$ 

Therefore $f \in L^p(\mathcal{A})$ and so there exist $g \in L^p(\Sigma)$ such that $T_u g = f$. Hence $E(u) E(g) = E(T_u g) = f$. Since $E(g) = f / E(u)$ off $Z(E(u))$ and $\mu(Z(E(u))) = 0$, it follows that

$$\int_X |g|^p \, d\mu = \int_X |E(|g|^p)| \, d\mu \geq \int_X |E(g)|^p \, d\mu = \int_X \frac{|f|^p}{E(u)|^p} \, d\mu = \sum_{n \in H} \frac{1}{\mu(H_n)} \, d\mu = \sum_{n \in H} 1.$$
This implies that $H$ must be a finite set. So there is an $n_0$ such that $n \geq n_0$ implies $\mu(H_n) = 0$. Together with $\mu(Z(E(u))) = 0$, we obtain

$$
\mu \left( \left\{ x \in X : |E(u)|^p(x) \leq \frac{\|E(|u|^p)|\infty}{n_0^2} \right\} \right) = \mu \left( \bigcup_{n=n_0}^{\infty} H_n \cup Z(E(u)) \right) = 0,
$$

that is $|E(u)| \geq \left(\left(\|E(|u|^p)|\infty/n_0^2\right)^{1/p} = \delta$ almost everywhere on $X$.

Conversely, suppose that $|E(u)| \geq \delta$ a.e. on $X$ for some $\delta > 0$. Let $f \in N(T_u^*)$. We have $T_u^* f = E(\bar{u}f) + E(u)(f - E(f)) = 0$ and so $E(\bar{u}f) = E(T_u^* f) = 0$. Thus

$$
\int_X \bar{u}f \, d\mu = \int_X E(\bar{u}f) \, d\mu = 0,
$$

which implies that

$$
N(T_u^*) \subseteq \left\{ f \in L^p(\Sigma) : \int_X \bar{u}f \, d\mu = 0 \right\} \subseteq L^p(Z(u), \Sigma_Z(u), \mu|Z(u)).
$$

Also, since $E(|u|) \geq |E(u)| \geq \delta$ and $X$ is a $\sigma$-finite measure space, we have $|u| \geq \delta$ and hence $\mu(Z(u)) = 0$. It follows that

$$
codim \mathcal{R}(T_u) = \dim N(T_u^*) = 0.
$$

Now, we shall show that $T_u$ has closed range. Let $\{T_u f_n\}$ be an arbitrary sequence in $\mathcal{R}(T_u)$ and let $\|T_u f_n - g\|_p \to 0$ for some $g \in L^p(\Sigma)$. Hence we have $E(u) E(f_n) = E(T_u f_n) \xrightarrow{L^p} E(g)$. Since by hypothesis $|E(u)| \geq \delta$, it follows that $E(g)/E(u) \in L^p(A)$ and $E(f_n) \xrightarrow{L^p} E(g)/E(u)$. Consequently,

$$
f_n \xrightarrow{L^p} \frac{1}{E(u)} \left\{ g + E(g) - \frac{uE(g)}{E(u)} \right\} := f
$$

and hence $T_u f_n \xrightarrow{L^p} T_u f$. Therefore $g = T_u f$, which implies that $T_u$ has closed range. Thus the theorem is proved. $\square$

Now, we consider the particular case when $p = 2$. An operator $T$ on a Hilbert space $H$ is normal if $TT^* = T^*T$, and $T$ is self-adjoint if $T = T^*$. 41
Proposition 3.3. Let $u \in K^*_2$. Then the following claims are true:

(i) $T_u$ is a normal operator if and only if $u \in L^\infty(A)$.

(ii) $T_u$ is a self-adjoint operator if and only if $u \in L^\infty(A)$ is real valued.

Proof. (i) Assume $T_u$ is normal. Then for each $f \in L^2(\Sigma)$ we have $E(T_u T_u^* f) = E(u)E(\bar{u} f)$ and $E(T_u^* T_u f) = E(f)E(|u|^2) + E(u)E(\bar{u} f) - E(\bar{u})E(u)E(f)$. Therefore we obtain that $E(|u|^2) = |E(u)|^2$. Consequently $u \in L^\infty(A)$. Conversely, suppose that $u \in L^\infty(A)$ and take $f \in L^2(\Sigma)$. Then $T_u^* T_u f = T_u T_u^* f = |u|^2 f$, and hence $T_u$ is normal. (ii) follows from (i).

Example 3.4. Let $X = [-1,1]$, $d\mu = dx$, let $\Sigma$ be the Lebesgue sets, and $A$ the $\sigma$-subalgebra generated by the sets symmetric about the origin. Put $0 < a \leq 1$. Then for each $f \in L^2(\Sigma)$ we have

$$
\int_{-a}^{a} E(f)(x) \, dx = \int_{-a}^{a} f(x) \, dx = \int_{-a}^{a} \left\{ \frac{f(x) + f(-x)}{2} + \frac{f(x) - f(-x)}{2} \right\} \, dx = \int_{-a}^{a} \frac{f(x) + f(-x)}{2} \, dx.
$$

Consequently, $(Ef)(x) = (f(x) + f(-x))/2$. Now, if we take $u(x) = \cos x + \sin x$, then the $\star$-multiplication operator $T_u: L^2(\Sigma) \to L^2(\Sigma)$ has the form

$$(T_u f)(x) = \left( \cos x + \frac{1}{2} \sin x \right) f(x) + \frac{1}{2} \sin x f(-x).$$

Direct computation shows that $(T_u^* f)(x) = (\cos x + \sin x/2) f(x) - \sin x/2 f(-x)$ and $|E(u)| \geq \cos 1$. Therefore, $T_u$ is a Fredholm but not a normal operator. □

References


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