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Czechoslovak Mathematical Journal, Vol. 60 (2010), No. 1, 85–94

Persistent URL: http://dml.cz/dmlcz/140551

Czechoslovak Mathematical Journal, 60 (135) (2010), 85–94

THE TRIADJOINT OF AN ORTHOSYMMETRIC BIMORPHISM

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(Received July 18, 2008)

Abstract. Let A and B be two Archimedean vector lattices and let $(A')'_n$ and $(B')'_n$ be their order continuous order biduals. If $\Psi\colon A\times A\to B$ is a positive orthosymmetric bimorphism, then the triadjoint $\Psi^{***}\colon (A')'_n\times(A')'_n\to(B')'_n$ of Ψ is inevitably orthosymmetric. This leads to a new and short proof of the commutativity of almost f-algebras.

Keywords: almost f-algebra orthosymmetric bimorphism

MSC 2010: 06F25, 47B65

1. INTRODUCTION

Let A, B, C be archimedean vector lattices and let $\Psi\colon A\times B\to C$ be an order bounded bilinear map. Denote by A' the order dual of A and by A'_n the order continuous order dual of A. In [2], Arens investigated (in a more general abstract setting) the process of forming the *adjoint operation*

$$\Psi^*\colon C'\times A\to B'$$

defined, for all $f\in C'$, $a\in A$, $b\in B$, by

$$\Psi^*(f,a)(b)=f(\Psi(a,b)).$$

This construction can be iterated in the following way:

$$\Psi^{**}\colon B''\times C'\to A'$$

defined, for all $F\in B''$, $f\in C'$, $a\in A$, by

$$\Psi^{**}(F,f)(a)=F(\Psi^*(f,a)),$$

and

$$\Psi^{***}\colon\ A'' \times B'' \to C''$$

defined, for all $F \in A''$, $G \in B''$, $f \in C'$, by

$$\Psi^{***}(F,G)(f) = F(\Psi^{**}(G,f)),$$

with $\Psi^{***}((A')'_n \times (B')'_n) \to (C')'_n$. Moreover, certain properties such as associativity, provided Ψ has them, are transmitted to Ψ^{***} (this makes sense only when $A = B = C$). The transmission of commutativity problem was considered by Arens in [2] and answered in the negative. However, it was shown by Grobler [8, Theorem 4] that if A is an abelian lattice ordered algebra, then $(A')'_n$ furnished with the Arens product is abelian. The question arises as to whether the restriction of Ψ^{***} to $(A')'_n \times (A')'_n$ is still a positive and orthosymmetric bimorphism when Ψ is only a positive orthosymmetric bimorphism. The answer is affirmative (Theorem 1). As an application, we prove that the restriction of Ψ^{***} to $(A')'_n \times (A')'_n$ (called also the triadjoint of Ψ) and, by the way Ψ, are inevitably symmetric.

In 1981 Scheffold in [13] proved that any normed almost f-algebra is commutative. Basly and Triki [3] make able to do a way with the norm condition. Both the proof of Scheffold and the proof of Basly and Triki make use of the Axiom of Choice by using non constructive representation theorems. Bernau and Huijsmans in [5] gave a constructive proof. It is long and quite involved. In 2000 Buskes and Van Rooij gave another proof (see [7]). The disadvantage of their approach is that the proof is not intrinsic, i.e., does not take place in the almost f-algebra itself. In this paper we present a new, short and constructive proof. Interestingly, it deals with positive orthosymmetric maps rather than with algebra multiplications and does not make use of associativity.

We take it for granted that the reader is familiar with the notions of vector lattices (or Riesz spaces) and operators between them. For terminology, notation and concepts that are not explained in the paper we refer to the standard monographs [1], [10], [11], [12] and [15].

2. Definitions and notations

We shall assume throughout this paper that all vector lattices (or Riesz spaces) under consideration are Archimedean.

Let A be a (real) vector lattice. A vector subspace I of A is called *an order ideal* (or *o-ideal*) whenever $|a| \leqslant |b|$ and $b \in I$ imply $a \in I$. Every o-ideal is a vector sublattice of A. The principal o-ideal generated by $0 \leqslant e \in A$ is denoted by

A_e and it is a sublattice of A. A linear mapping T defined on a vector lattice A with values in a vector lattice B is called *positive* if $T(A^+) \subset B^+$. A linear mapping $T \in \mathscr{L}(A, B)$ is called *a lattice* (or *Riesz*) *homomorphism* whenever $a \wedge b = 0$ implies $T(a) \wedge T(b) = 0$.

Let A be a vector lattice and let $0 \leqslant v \in A$. The sequence $\{a_n,\ n = 1, 2, \ldots\}$ in A is called (v) *relatively uniformly convergent* to $a \in A$ if for every real number $\varepsilon > 0$ there exists a natural number n_ε such that $|a_n - a| \leqslant \varepsilon v$ for all $n \geqslant n_\varepsilon$. This will be denoted by $a_n \to a\ (v)$. If $a_n \to a\ (v)$ for some $0 \leqslant v \in A$, then the sequence $\{a_{n,}\ n = 1, 2, \ldots\}$ is called *(relatively) uniformly convergent* to a, which is denoted by $a_n \to a(r \cdot u)$. The notion of a (v) *relatively uniformly Cauchy* sequence is defined in the obvious way. A vector lattice A is called *relatively uniformly complete* if every relatively uniformly Cauchy sequence in A has a relatively uniform limit. Relatively uniform limits are uniquely determined in archimedean vector lattices, see [10, Theorem 63.2]. Furthermore, If $A^\mathfrak{d}$ is the Dedekind completion of a vector lattice A, then the closure A^{ru} of A in $A^\mathfrak{d}$ with respect to the relatively uniform topology, is the relatively uniform completion of A.

Recall that if A is a real vector lattice, we denote by A^* the *algebraic dual* of A, that is, the vector space of all linear forms on A. A linear form $f \in A^*$ is called *order bounded* if for each order interval $[x, y] \subset A$, the set $f([x, y])$ is order bounded in $\mathbb{R}$. An order bounded linear form f on A is called *order continuous* if f converges to 0 along each filter that converges to 0. The vector space of all order bounded linear forms on A is called the order dual of A and is denoted by A'. Its order dual $(A')'$ is denoted by A''. The vector space of all order continuous linear forms on A' is called the order continuous order bidual of A and is denoted by $(A')'_n$. We define for every $x \in A$ an element $x'' \in A''$ by putting $x''(f) = f(x)$ for all $f \in A'$. The map $\sigma\colon\ A \to A''$ defined by $\sigma(x) = x''$ for all $x \in A$ is a lattice homomorphism from A into A''. Recall that $\sigma(A) \subset (A')'_n$ and the order ideal $I_{\sigma(A)}$ generated by $\sigma(A)$ in $(A')'_n$ is order dense in $(A')'_n$, that is, $\{\sigma(A)\}^{dd} = (A')'_n$. Hence for each $0 \leqslant F \in (A')'_n$ there exists an upwards directed net F_α in $I_{\sigma(A)}$ with $F_\alpha \nearrow F$. For more explanation, see [4], [8], [9].

In the next paragraphs, we recall definitions and some basic facts about almost f-algebras. For more information about this field, we refer the reader to [1], [5], [6], [7]. A vector lattice A which is simultaneously an associative algebra such that $ab \geqslant 0$ for each $0 \leqslant a, b \in A$ is called a *lattice ordered algebra* (*ℓ-algebra*). An ℓ-algebra A is called an *almost f-algebra* whenever it follows from $a \wedge b = 0$ that $ab = 0$.

We end this section with the following definition. Let A and B be vector lattices. A bilinear map Ψ from $A \times A$ into B is said to be *orthosymmetric* if $a \wedge b = 0$ implies $\Psi(a, b) = 0$, see [7].

3. The main results

The following proposition is important in the context of our problem. We are indebted to Bernau and Huijsmans [4] for some steps.

Proposition 1. *Let A, B be vector lattices, let $\Psi\colon A \times A \to B$ be a positive orthosymmetric bimorphism and let $\Psi^{***}\colon (A')'_n \times (A')'_n \to (B')'_n$ be the adjoint of Ψ^{**}. If $0 \leqslant G, H \in (A')'_n$ with $G \wedge H = 0$ and $G, H \leqslant x''$ for some $x \in A_+$, then*

$$\Psi^{***}(H, G) = \Psi^{***}(G, H) = 0.$$

P r o o f. Let $0 \leqslant f \in B'$ and let $k = \Psi^*(f, x) \vee \Psi^{**}(x'', f) \in A'$. It follows from [4, Corollary 1.2] that there exist $g, h \in A'$ with $g \wedge h = 0$ and $G(g) = 0 = H(h)$ such that $k = g + h$.

Hence

$$0 = (g \wedge h)(x) = \inf\{g(y) + h(z),\ x = y + z, y, z \in A_+\},$$

which implies that for $\varepsilon > 0$ there exist $y, z \in A_+$ such that $x = y + z$ and $g(y) < \varepsilon$, $h(z) < \varepsilon$. We now define linear functionals G_1 and H_1 on A' by $G_1 = G \wedge (y - y \wedge z)''$ and $H_1 = H \wedge (z - y \wedge z)''$.

It is obvious that $0 \leqslant G_1$,$H_1 \in (A')'_n$. However, the following relations hold:

$$\begin{aligned}
0 \leqslant G - G_1 &= G - G \wedge (y - y \wedge z)'' \\
&= 0 \vee [G - (y - y \wedge z)''] \\
&= [G - (y - y \wedge z)'']^+ \\
&\leqslant [x'' - (y - y \wedge z)'']^+ \\
&= [(y + z)'' - (y - y \wedge z)'']^+ \\
&= [(y + z - (y - y \wedge z))'']^+ \\
&= [(z + y \wedge z)'']^+ \\
&\leqslant ((2z)^+)'' = 2z'',
\end{aligned}$$

that is

$$0 \leqslant G - G_1 \leqslant 2z''.$$

Using the same argument, we have

$$0 \leqslant H - H_1 \leqslant 2y''.$$

Since $(y - y \wedge z) \wedge (z - y \wedge z) = 0$, then

$$\Psi^{***}((y - y \wedge z)'', (z - y \wedge z)'') = [\Psi((y - y \wedge z), (z - y \wedge z))]'' = 0.$$

Moreover,

$$\Psi^{***}(G_1, H_1) \leqslant \Psi^{***}((y - y \wedge z)'', (z - y \wedge z)'') = 0.$$

Since $0 \leqslant H, G \leqslant x''$, it follows that

$$\begin{aligned}
\Psi^{***}(G - G_1, H)(f) &\leqslant \Psi^{***}(G - G_1, x'')(f) \\
&= (G - G_1)(\Psi^{**}(x'', f)) \\
&\leqslant (G - G_1)(k) \\
&= (G - G_1)(g) + (G - G_1)(h) \\
&= 0 + (G - G_1)(h) \\
&\leqslant 0 + 2h(z) \\
&< 2\varepsilon.
\end{aligned}$$

Moreover,

$$\begin{aligned}
\Psi^{***}(G, H - H_1)(f) &\leqslant \Psi^{***}(x'', H - H_1)(f) \\
&= x''(\Psi^{**}(H - H_1, f)) \\
&= (\Psi^{**}(H - H_1, f))(x) \\
&= (H - H_1)(\Psi^{*}(f, x)) \\
&\leqslant (H - H_1)(k) \\
&= (H - H_1)(g + h) \\
&= 0 + (H - H_1)(g) \\
&\leqslant 2g(y) \\
&< 2\varepsilon.
\end{aligned}$$

Therefore

$$\begin{aligned}
\Psi^{***}(G, H)(f) &= \Psi^{***}(G - G_1 + G_1, H)(f) \\
&= \Psi^{***}(G - G_1, H)(f) + \Psi^{***}(G_1, H)(f) \\
&\leqslant 2\varepsilon + \Psi^{***}(G_1, H)(f) \\
&= 2\varepsilon + \Psi^{***}(G_1, H - H_1 + H_1)(f) \\
&\leqslant 2\varepsilon + \Psi^{***}(G_1, H - H_1)(f) + \Psi^{***}(G_1, H_1)(f) \\
&\leqslant 2\varepsilon + \Psi^{***}(G, H - H_1)(f) + \Psi^{***}(G_1, H_1)(f) \\
&\leqslant 4\varepsilon + \Psi^{***}(G_1, H_1)(f).
\end{aligned}$$

Since $\Psi^{***}(G_1, H_1)(f) = 0$, we have

$$\Psi^{***}(G, H)(f) \leqslant 4\varepsilon.$$

Since this holds for an arbitrary $\varepsilon > 0$, we have $\Psi^{***}(G,H)(f) = 0$. Using the same argument, we deduce that $\Psi^{***}(H,G)(f) = 0$. The result holds for all $f \in B'$ since $f = f^+ - f^-$ for any $f \in B'$ as required. $\square$

We have collected all prerequisites for the first main result of this paper.

Theorem 1. *Let A, B be vector lattices and let $\Psi \colon A \times A \to B$ be a positive orthosymmetric bimorphism. Then $\Psi^{***} \colon (A')'_n \times (A')'_n \to (B')'_n$ the adjoint of Ψ^{**}, is inevitably a positive orthosymmetric bimorphism.*

Proof. Let $0 \leqslant G, H \in (A')'_n$ with $G \wedge H = 0$. We have to show that $\Psi^{***}(G,H) = 0$.

Consider the order ideal $I_{\sigma(A)}$ generated by $\sigma(A) = \{x'', \forall x \in A\}$ in $(A')'_n$, that is $I_{\sigma(A)} = \{F \in (A')'_n, |F| \leqslant x'' \text{ for some } x \in A\}$. We recall that $I_{\sigma(A)}$ is order dense in $(A')'_n$ and hence there exist $G_\alpha, H_\beta \in I_{\sigma(A)}$ such that $0 \leqslant G_\alpha \nearrow G$ and $0 \leqslant H_\beta \nearrow H$ with $0 \leqslant G_\alpha \leqslant x''_\alpha$, $0 \leqslant H_\beta \leqslant y''_\beta$ for some $0 \leqslant x_\alpha, y_\beta \in A$. It follows from $G \wedge H = 0$ that $G_\alpha \wedge H_\beta = 0$ for all α, β. Furthermore, $0 \leqslant G_\alpha, H_\beta \leqslant (x_\alpha + y_\beta)''$. Then Proposition 1 yields that $\Psi^{***}(G_\alpha, H_\beta) = 0$ for all α, β.

Now let $0 \leqslant f \in B'$ and let $a \in A_+$. It follows from $0 \leqslant H_\beta \nearrow H$ that

$$\Psi^{**}(H_\beta, f)(a) = H_\beta(\Psi^*(f,a)) \nearrow H(\Psi^*(f,a)) = \Psi^{**}(H,f)(a).$$

It follows that $\Psi^{**}(H_\beta, f) \nearrow \Psi^{**}(H,f)$. Hence by the order continuity of G_α for each α we obtain that

$$\Psi^{***}(G_\alpha, H_\beta)(f) = G_\alpha(\Psi^{**}(H_\beta, f)) \nearrow G_\alpha(\Psi^{**}(H,f)) = \Psi^{***}(G_\alpha, H)(f).$$

Moreover,

$$\Psi^{***}(G_\alpha, H)(f) = G_\alpha(\Psi^{**}(H,f)) \nearrow G(\Psi^{**}(H,f)) = \Psi^{***}(G,H)(f)$$

for all $0 \leqslant f \in B'$. It follows that

$$\Psi^{***}(G_\alpha, H_\beta)(f) \nearrow \Psi^{***}(G,H)(f)$$

for all $0 \leqslant f \in B'$. Consequently,

$$\Psi^{***}(G_\alpha, H_\beta) \nearrow \Psi^{***}(G,H).$$

Hence

$$\Psi^{***}(G,H) = 0$$

and we are done. $\square$

Remark 1. We note that in all the above proofs, we have not used the fact that Ψ is symmetric.

Next, we will present an alternative proof of the commutativity of almost f-algebras.

Recall that a topological space X is called *Stonian* (or *extremally disconnected*) if X possesses the property that each open subset of X has an open closure.

To reach our aim, we need first the following result, which is a simple combination of a theorem of Kakutani [12, Chap. II, Theorem 7.4] with [12, Chap. II, Proposition 7.7]. For this reason its proof has been omitted.

Proposition 2. *Let X be a compact Hausdorff space. Then the Dedekind completion of $C(X)$ is a vector lattice of the form $C(Y)$ for some Stonian compact Hausdorff space Y.*

We are in position to present a new and short proof of the commutativity of almost f-algebras.

Theorem 2. *Let A, B be vector lattices and let $\Psi_0\colon A \times A \to B$ be a positive orthosymmetric bimorphism. Then Ψ_0 is symmetric, that is*

$$\Psi_0(f,g) = \Psi_0(g,f) \qquad \text{for all } f,g \in A$$

P r o o f. First, we note that Triki advocated in [14, Theorem 4] that any positive orthosymmetric $\Psi\colon A \times A \to B$ can be extended in a unique way to $A^{ru} \times A^{ru} \to B^{ru}$ (where A^{ru} (resp. B^{ru}) is the closure of A (resp. of B) with respect to the relatively uniform topology) in such a manner that the new extension (denoted also by Ψ_0) is also a positive orthosymmetric bimorphism. Then without loss of generality we can assume that A is a uniformly complete vector lattice. Let $f, g \in A$, $h = |f| + |g|$ and put $e = \Psi_0(h,h)$. Hence the order ideals I_h and I_e generated by h and e in A and B can be identified respectively with $C(X)$ and $C(Y)$ for some compact Hausdorff space X and Y, respectively (use the Kakutani Representation Theorem). We claim that the restriction of Ψ_0 to $C(X) \times C(X) \to C(Y)$, denoted by Ψ, is symmetric.

Recall that $C(X)$ and $C(Y)$ are Banach vector lattices and hence $C(X)$ and $C(Y)$ are naturally embedded in $(C(X)')'_n$ and $(C(Y)')'_n$ respectively. Then the triadjoint $\Psi^{***}\colon (C(X)')'_n \times (C(X)')'_n \to (C(Y)')'_n$ of Ψ is an extension of Ψ. Then without loss of generality we can assume that $C(X)$ is a Dedekind complete vector lattice. By the previous proposition, X is a Stonian compact Hausdorff space.

Let $L = \{k \in C(X);\ k(X) \text{ is a finite subset of } \mathbb{R}\}$. The fact that X is a Stonian compact Hausdorff space coupled with the M. H. Stone Theorem [12, Chap. II, Theorem 7.3] imply that L is a dense vector sublattice of $C(X)$.

We claim that the restriction of Ψ to $L \times L$, denoted also by Ψ, is symmetric. To this end, let $k \in L$, let $\{x_1, \ldots, x_m\} = k(X)$ and let $X_i = k^{-1}\{x_i\}$ for all $1 \leqslant i \leqslant m$. Then X_i is a closed subset of X. Since $x_i \neq x_j$ for all $i \neq j$, there exists a real number $\varepsilon > 0$ such that $]x_i - \varepsilon, x_i + \varepsilon[\cap]x_j - \varepsilon, x_j + \varepsilon[= \emptyset$ for all $i \neq j$. Then $X_i = k^{-1}\{]x_i - \varepsilon, x_i + \varepsilon[\}$. It follows that X_i is an open and closed subset of X. Then

$$k = \sum_{i=1}^{m} x_i \mathbf{1}_{X_i}$$

where $\mathbf{1}_{X_i}(t) = \begin{cases} 1 \text{ if } t \in X_i \\ 0 \text{ if } t \in X_i^c \end{cases}$ (here X_i^c is the complement of X_i in X). Let $k' \in L$, then there exist $\{y_1, \ldots, y_n\} \subset \mathbb{R}$ and $Y_1, \ldots, Y_n$ disjoint open and closed subsets of X such that $k' = \sum\limits_{j=1}^{n} y_j \mathbf{1}_{Y_j}$. It follows that,

$$\Psi_0(k, k') = \Psi_0\left(\sum_{i=1}^{m} x_i \mathbf{1}_{X_i}, \sum_{j=1}^{n} y_j \mathbf{1}_{Y_j}\right) = \sum_{\substack{1 \leqslant i \leqslant m \\ 1 \leqslant j \leqslant n}} x_i y_j \Psi(\mathbf{1}_{X_i}, \mathbf{1}_{Y_j}).$$

Let $Z_{ij} = X_i \cap Y_j$, for all $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n$. Then $\mathbf{1}_{X_i} = \mathbf{1}_{Z_{ij}} + \mathbf{1}_{X_i \setminus Z_{ij}}$ and $\mathbf{1}_{Y_j} = \mathbf{1}_{Z_{ij}} + \mathbf{1}_{Y_j \setminus Z_{ij}}$ for all $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n$. Then

$$\begin{aligned}\Psi_0(\mathbf{1}_{X_i}, \mathbf{1}_{Y_j}) = {} & \Psi_0(\mathbf{1}_{Z_{ij}}, \mathbf{1}_{Z_{ij}}) + \Psi_0(\mathbf{1}_{Z_{ij}}, \mathbf{1}_{Y_j \setminus Z_{ij}}) \\ & + \Psi_0(\mathbf{1}_{X_i \setminus Z_{ij}}, \mathbf{1}_{Z_{ij}}) + \Psi_0(\mathbf{1}_{X_i \setminus Z_{ij}}, \mathbf{1}_{Y_j \setminus Z_{ij}}).\end{aligned}$$

Since

$$X_i \setminus Z_{ij} \cap Y_j \setminus Z_{ij} = Z_{ij} \cap Y_j \setminus Z_{ij} = X_i \setminus Z_{ij} \cap Z_{ij} = \emptyset,$$

we deduce

$$\mathbf{1}_{X_i \setminus Z_{ij}} \wedge \mathbf{1}_{Y_j \setminus Z_{ij}} = \mathbf{1}_{Z_{ij}} \wedge \mathbf{1}_{Y_j \setminus Z_{ij}} = \mathbf{1}_{X_i \setminus Z_{ij}} \wedge \mathbf{1}_{Z_{ij}} = 0.$$

Then

$$\Psi_0(\mathbf{1}_{X_i \setminus Z_{ij}}, \mathbf{1}_{Y_j \setminus Z_{ij}}) = \Psi_0(\mathbf{1}_{Z_{ij}}, \mathbf{1}_{Y_j \setminus Z_{ij}}) = \Psi_0(\mathbf{1}_{X_i \setminus Z_{ij}}, \mathbf{1}_{Z_{ij}}) = 0,$$

which leads to

$$\Psi_0(k, k') = \sum_{\substack{1 \leqslant i \leqslant m \\ 1 \leqslant j \leqslant n}} x_i y_j \Psi_0(\mathbf{1}_{Z_{ij}}, \mathbf{1}_{Z_{ij}}) = \Psi_0(k', k).$$

Hence the restriction of Ψ_0 to $L \times L$ is symmetric. Now since $f, g \in C(X)$ and since L is dense in $C(X)$, there exists $f_n, g_n \in L$, for all $n \in \mathbb{N}$, such that $f_n \to f$ and $g_n \to g$. By the positivity of Ψ_0 we have

$$\Psi_0(f_n, g_n) \to \Psi_0(f, g)(r \cdot u)$$

and

$$\Psi_0(g_n, f_n) \to \Psi_0(g, f)(r \cdot u)$$

and since $\Psi_0(f_n, g_n) = \Psi_0(g_n, f_n)$, it follows that $\Psi_0(f, g) = \Psi_0(g, f)$, which gives the desired result. □

The above theorem yields the following corollary.

Corollary 1. *Every almost f-algebra is commutative.*

Proof. Let A be an almost f-algebra and let $\Psi\colon A \times A \to A$ be the bilinear map defined by $\Psi(x, y)) = xy$. It is an easy task to show that Ψ is a positive orthosymmetric bimorphism. In view of Theorem 2, we deduce that Ψ is symmetric, that is $xy = yx$ for all $x, y \in A$, which gives the desired result. □

Acknowledgement. The author thanks the referee for his careful reading of the paper and for his valuable suggestions.

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