In-Jae Kim; Charles Waters
Symmetric sign patterns with maximal inertias

Czechoslovak Mathematical Journal, Vol. 60 (2010), No. 1, 101–104

Czechoslovak Mathematical Journal, 60 (135) (2010), 101–104

SYMMETRIC SIGN PATTERNS WITH MAXIMAL INERTIAS

In-Jae Kim, Charles Waters, Mankato

(Received August 7, 2008)

Abstract. The inertia of an n by n symmetric sign pattern is called maximal when it is not a proper subset of the inertia of another symmetric sign pattern of order n. In this note we classify all the maximal inertias for symmetric sign patterns of order n, and identify symmetric sign patterns with maximal inertias by using a rank-one perturbation.

Keywords: eigenvalue, inertia, maximal inertia, rank-one perturbation, symmetric sign pattern

MSC 2010: 15A18

1. Introduction

An n by n matrix $\mathcal{A} = [\alpha_{ij}]$ with entries in $\{+, -, 0\}$ is a *sign pattern* (*matrix*). If $\mathcal{A}$ has no zero entries, then $\mathcal{A}$ is a *full* sign pattern. If $\alpha_{ij} = \alpha_{ji}$ for all $i \neq j$, then the sign pattern $\mathcal{A}$ is *symmetric*. The set $Q(\mathcal{A})$ of all n by n real matrices $A = [a_{ij}]$ with $\operatorname{sign}(a_{ij}) = \alpha_{ij}$ for all i, j is the *sign pattern class* of $\mathcal{A}$. When $\mathcal{A}$ is symmetric, we define the *sign symmetric class*

$$Q_{\mathrm{SYM}}(\mathcal{A}) = \{A \in \mathbb{R}^{n\times n} \mid A \in Q(\mathcal{A}) \text{ and } A^T = A\}.$$

The *inertia* of an n by n real symmetric matrix A, denoted by $\operatorname{In}(A)$, is the ordered integer triple (p, q, r), where $p + q + r = n$, and p (resp. q and r) is the number of positive (resp. negative and zero) eigenvalues. Note that the rank of A is equal to $p + q$. For a symmetric sign pattern $\mathcal{A}$, the *inertia of* $\mathcal{A}$ is

$$\operatorname{In}(\mathcal{A}) = \{\operatorname{In}(A) \mid A \in Q_{\mathrm{SYM}}(\mathcal{A})\}.$$

This work was inspired by the American Institute of Mathematics (AIM) Workshop, “Spectra of families of matrices described by graphs, digraphs, and sign patterns”, and In-Jae Kim thanks AIM and NSF for their support.

The inertias of symmetric sign patterns have been studied in the literature; for example, see [2], [3], [5], [6].

By [1, Lemma 5.1], there are no symmetric sign patterns that allow all the possible inertias. Thus, it is natural to consider maximal elements in a partially ordered set (with set inclusion) of inertias of symmetric sign patterns. If the inertia of an n by n symmetric sign pattern is a maximal element in the partially ordered set of inertias of symmetric sign patterns of order n, then it is called *maximal*.

In this note we study maximal inertias of symmetric sign patterns, and find symmetric sign patterns having these maximal inertias by a rank-one perturbation.

2. Results

Note that the zero sign pattern of order n is the only n by n symmetric sign pattern with inertia $(0,0,n)$. In other words, if an n by n symmetric sign pattern $\mathcal{A}$ allows inertia $(0,0,n)$, then $(0,0,n)$ is the only inertia realized by a symmetric matrix in $Q_{\mathrm{SYM}}(\mathcal{A})$, which implies that $M_0 = \{(0,0,n)\}$ is a maximal inertia for symmetric sign patterns of order n. Thus the largest possible value of r for an inertia (p,q,r) of a nonzero real symmetric matrix of order n is $n-1$.

Let $\mathbb{Z}_+$ be the set of nonnegative integers. In the following it is shown that the sets $M_1 = \{(p,q,r) \in \mathbb{Z}_+^3 \mid p \geqslant 1,\ p+q+r=n\}$ and $M_2 = \{(p,q,r) \in \mathbb{Z}_+^3 \mid q \geqslant 1,\ p+q+r=n\}$ are the only maximal inertias for nonzero symmetric sign patterns of order n, and that the symmetric sign patterns with all $+$'s and all $-$'s are, up to some congruence, the only symmetric sign patterns having the inertias M_1 and M_2, respectively. By a rank-one perturbation, we first show that an element of the inertia $\mathrm{In}(\mathcal{A})$ of a full symmetric sign pattern $\mathcal{A}$ guarantees the existence of some other elements in $\mathrm{In}(\mathcal{A})$.

Lemma 1. *Let $\mathcal{A}$ be a full symmetric sign pattern of order n. If a nonnegative integer triple (p,q,r) with $p+q+r=n$ and $1 \leqslant r \leqslant n-1$ is in $\mathrm{In}(\mathcal{A})$, then $(p+1,q,r-1)$ and $(p,q+1,r-1)$ are also in $\mathrm{In}(\mathcal{A})$.*

P r o o f. Let A be a symmetric matrix in $Q_{\mathrm{SYM}}(\mathcal{A})$ with $i(A)=(p,q,r)$ satisfying $1 \leqslant r \leqslant n-1$, and $\mathbf{x}$ be a nonzero nullvector of A. Let $B = A + \varepsilon \mathbf{x}\mathbf{x}^T$ for $\varepsilon \in \mathbb{R}$. For sufficiently small ε, $B \in Q_{\mathrm{SYM}}(\mathcal{A})$. Since there exists a set of n orthogonal eigenvectors for the symmetric matrix A, including the nullvector $\mathbf{x}$, it follows that

$$\mathrm{In}(B) = \begin{cases} (p+1,q,r-1) & \text{if } \varepsilon > 0, \\ (p,q+1,r-1) & \text{if } \varepsilon < 0. \end{cases}$$

□

By using Lemma 1 repeatedly, we get the following result.

Theorem 2. *Let $\mathcal{A}$ be a full symmetric sign pattern of order n. If a nonnegative integer triple (p,q,r) with $1 \leqslant r \leqslant n-1$ is in In($\mathcal{A}$), then each inertia (p',q',r'), with $p' \in \{p,\ldots,p+r\}$, $q' \in \{q,\ldots,q+r\}$, $r' \in \{0,\ldots,r\}$ and $p'+q'+r'=n$, is in* $\operatorname{In}(\mathcal{A})$.

A diagonal sign pattern $\mathcal{D}$ without zero main diagonal entries is a *signature pattern*. For two symmetric sign patterns $\mathcal{A}$ and $\mathcal{B}$, if $\mathcal{A} = \mathcal{D}\mathcal{B}\mathcal{D}^T$, then we say $\mathcal{A}$ and $\mathcal{B}$ are *signature congruent*. Note that signature congruent symmetric sign patterns have the same inertia (see [4, Theorem 4.5.8]). Let $\mathcal{J}_n$ be the n by n full symmetric sign pattern with all entries $+$.

Theorem 3. *The sets $M_0 = \{(0,0,n)\}$, $M_1 = \{(p,q,r) \in \mathbb{Z}_+^3 \mid p \geqslant 1,\, p+q+r = n\}$ and $M_2 = \{(p,q,r) \in \mathbb{Z}_+^3 \mid q \geqslant 1,\, p+q+r = n\}$ are the only maximal inertias of symmetric sign patterns of order n. Furthermore, the n by n symmetric sign pattern $\mathcal{A}$ with the inertia M_0 is the zero sign pattern of order n, and an n by n symmetric sign pattern $\mathcal{A}$ having the inertia M_1 (resp. M_2) is signature congruent to $\mathcal{J}_n$ (resp. $-\mathcal{J}_n$).*

P r o o f. First, we show that $\mathcal{J}_n$ (resp. $-\mathcal{J}_n$) has the inertia M_1 (resp. M_2). By the Perron-Frobenius Theory of nonnegative matrices (see, for example, [4, Theorem 8.4.4]), it follows that $\operatorname{In}(\mathcal{J}_n) \subseteq M_1$ (resp. $\operatorname{In}(-\mathcal{J}_n) \subseteq M_2$). Note that the matrix J_n (resp. $-J_n$) with all entries one (resp. negative one) is in $Q_{\mathrm{SYM}}(\mathcal{J}_n)$ (resp. $Q_{\mathrm{SYM}}(-\mathcal{J}_n)$). Since $\operatorname{In}(J_n) = (1,0,n-1)$ (resp. $\operatorname{In}(-J_n) = (0,1,n-1)$), by Theorem 2, $M_1 \subseteq \operatorname{In}(\mathcal{J}_n)$ (resp. $M_2 \subseteq \operatorname{In}(-\mathcal{J}_n)$). Hence, $\operatorname{In}(\mathcal{J}_n) = M_1$ (resp. $\operatorname{In}(-\mathcal{J}_n) = M_2$).

Recall that M_0 is maximal. We now claim that M_1 and M_2 are maximal, and these inertias together with M_0 are the only maximal inertias for symmetric sign patterns of order n. Let M be a non-empty subset of $M_1 \cup M_2$ such that $M \not\subseteq M_1$ and $M \not\subseteq M_2$. Then M must have $(k,0,n-k)$ and $(0,\ell,n-\ell)$ for some positive integers k,ℓ $(\leqslant n)$. Suppose that there is an n by n symmetric sign pattern $\mathcal{A}$ with $\operatorname{In}(\mathcal{A}) = M$. Let A be a symmetric matrix in $Q_{\mathrm{SYM}}(\mathcal{A})$ with $\operatorname{In}(A) = (k,0,n-k)$. Then A is a nonzero positive semidefinite matrix. This implies that the main diagonal entries of $\mathcal{A}$ are $+$ or 0, and at least one main diagonal entry is $+$. However, the existence of a symmetric matrix $B \in Q_{\mathrm{SYM}}(\mathcal{A})$ with $\operatorname{In}(B) = (0,\ell,n-\ell)$, which is negative semidefinite, implies that the main diagonal entries of $\mathcal{A}$ are nonpositive, which gives a contradiction. Hence, the claim holds.

Next, suppose that the inertia of an n by n symmetric sign pattern $\mathcal{A}$ is equal to $M_1 = \{(p,q,r) \in \mathbb{Z}_+^3 \mid p \geqslant 1,\, p+q+r = n\}$. Since $(1,0,n-1) \in \operatorname{In}(\mathcal{A})$, there exists a symmetric matrix A in $Q_{\mathrm{SYM}}(\mathcal{A})$ such that A has a positive entry on the main diagonal and the rank of A is 1. By permuting rows and columns of A in the same way, we can place a positive entry on the $(1,1)$-position of the resulting matrix A'. By a signature congruence, it can be shown that there exists a matrix $\widehat{A}$ with all nonnegative entries in the first column. Since $\widehat{A}$ is symmetric, every entry in the first row of $\widehat{A}$ is also nonnegative. Note that the rank of $\widehat{A}$ is 1, and hence

$$\widehat{A} = \mathbf{u}\mathbf{u}^T,$$

where each entry of the nonzero n by 1 vector $\mathbf{u}$ is nonnegative. Since the sign pattern $\mathcal{A}$ also allows inertia $(n,0,0)$, $\mathcal{A}$ allows rank n. This implies that the vector $\mathbf{u}$ cannot have any zero entries. Thus, we conclude that $\mathcal{A}$ is signature congruent to $\mathcal{J}_n$.

Similarly, the case when $\mathrm{In}(\mathcal{A}) = M_2$ can be proved. □

The inertia of $\mathcal{J}_n$ can also be found in [5, Proposition 4.1], and a similar argument to the argument used to show that M_1 and M_2 are maximal can be found in [5, p. 228]. The use of a rank-one perturbation to get the results in Lemma 1 and Theorem 2 is new and very effective to find the inertias of $\mathcal{J}_n$ and $-\mathcal{J}_n$. We also have shown that the sign pattern $\mathcal{J}_n$ (resp. $-\mathcal{J}_n$) is, up to signature congruence, the only symmetric sign pattern having the inertia M_1 (resp. M_2).

Acknowledgement. The authors would like to thank a referee for his/her helpful comments which greatly improved the presentation of this paper.

References

[1] *M. S. Cavers, K. N. Vander Meulen*: Spectrally and inertially arbitrary sign patterns. Linear Algebra Appl. *394* (2005), 53–72.

[2] *C. M. da Fonseca*: On the inertia sets of some symmetric sign patterns. Czechoslovak Math. J. *56* (2006), 875–883.

[3] *Y. Gao, Y. Shao*: The inertia set of nonnegative symmetric sign pattern with zero diagonal. Czechoslovak Math. J. *53* (2003), 925–934.

[4] *R. A. Horn, C. R. Johnson*: Matrix Analysis. Cambridge Univeristy Press, Cambridge, 1985.

[5] *F. J. Hall, Z. Li*: Inertia sets of symmetric sign pattern matrices. Numer. Math., J. Chin. Univ. (English Ser.) *10* (2001), 226–240.

[6] *F. J. Hall, Z. Li, and Di Wang*: Symmetric sign pattern matrices that require unique inertia. Linear Algebra Appl. *338* (2001), 153–169.

Authors' address: In-Jae Kim (corresponding author), Ch. Waters, Department of Mathematics and Statistics, Minnesota State University, Mankato, MN 56001, U.S.A, e-mail: `in-jae.kim@mnsu.edu`; `charles.waters@mnsu.edu`.