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HOMOGENEOUS POLYNOMIALS WITH ISOMORPHIC MILNOR ALGEBRAS

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Abstract. We recall first Mather's Lemma providing effective necessary and sufficient conditions for a connected submanifold to be contained in an orbit. We show that two homogeneous polynomials having isomorphic Milnor algebras are right-equivalent.

Keywords: Milnor algebra, right-equivalence, homogeneous polynomial

MSC 2010: 14E05, 32S30, 14L30, 14J17, 16W22

1. Introduction

In this note we recall first Mather's Lemma 2.4 providing effective necessary and sufficient conditions for a connected submanifold to be contained in an orbit. In Theorem 3.2 we show that two homogeneous polynomials f and g having isomorphic Milnor algebras are right-equivalent.

This is similar to the celebrated theorem by Mather and Yau [4], saying that the isolated hypersurface singularities are determined by their Tjurina algebras. Our result applies only to homogeneous polynomials, but it is no longer necessary to impose the condition of having isolated singularities at the origin.

2. Preliminary results

We recall here some basic facts on semialgebraic sets, which are also called constructible sets, especially in the complex case. For a more complete introduction see [2, Chapter 1].

Definition 2.1. Let M be a smooth algebraic variety over K ($K = \mathbb{R}$ or $\mathbb{C}$ as usual).

(i) Complex Case. A subset $A \subset M$ is called semialgebraic if A belongs to the Boolean subalgebra generated by the Zariski closed subsets of M in the Boolean algebra $P(M)$ of all subsets of M.
(ii) Real Case. A subset $A \subset M$ is called semialgebraic if A belongs to Boolean subalgebra generated by the open sets $U_f = \{x \in U\colon\ f(x) > 0\}$ where $U \subset M$ is an algebraic open subset in M and $f\colon\ M \to \mathbb{R}$ is an algebraic function, in the Boolean algebra $P(M)$ of all subsets of M.

By definition, it follows that the class of semialgebraic subsets of M is closed under finite unions, finite intersections and complements. If $f\colon\ M \to N$ is an algebraic mapping between the smooth algebraic varieties M and N and if $B \subset N$ is semialgebraic, then clearly $f^{-1}(B)$ is semialgebraic in M. Conversely, we have the following basic result.

Theorem 2.2 (Tarski-Seidenberg-Chevalley). *If $A \subset M$ is semialgebraic, then $f(A) \subset N$ is also semialgebraic.*

Next consider the following useful result.

Proposition 2.3. *Let G be an algebraic group acting (algebraically) on a smooth algebraic variety M. Then the corresponding orbits are smooth semialgebraic subsets in M.*

Let $m\colon\ G \times M \to M$ be a smooth action. In order to decide whether two elements $x_0, x_1 \in M$ are G-transversal, we try to find a path (a homotopy) $P = \{x_t\colon\ t \in [0,1]\}$ such that P is entirely contained in a G-orbit. It turns out this naive approach works quite well and the next result gives effective necessary and sufficient conditions for a connected submanifold (in our case the path P) to be contained in an orbit.

Mather's Lemma 2.4 ([3]). *Let $m\colon\ G \times M \to M$ be a smooth action and $P \subset M$ a connected smooth submanifold. Then P is contained in a single G-orbit if and only if the following conditions are fulfilled:*
(a) $T_x(G.x) \supset T_xP$, *for any* $x \in P$.
(b) $\dim T_x(G.x)$ *is constant for* $x \in P$.

3. Main Theorem

For isolated hypersurface singularities, the following result was obtained by Mather and Yau, see [4].

Theorem 3.1. *Let $f, g\colon (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be two isolated hypersurface singularities having isomorphic Tjurina algebras $T(f) \simeq T(g)$. Then $f \overset{\mathcal{K}}{\sim} g$, where $\overset{\mathcal{K}}{\sim}$ denotes the contact equivalence.*

For arbitrary (i.e. not necessarily with isolated singularities) homogeneous polynomials we establish now the following result.

Theorem 3.2. *Let $f, g \in \mathbb{C}[x_1, \dots, x_n]_d = H^d(n, 1; \mathbb{C}) = H^d$ be two homogeneous polynomials of degree d such that $\mathcal{J}_f = \mathcal{J}_g$. Then $f \overset{\mathcal{R}}{\sim} g$, where $\overset{\mathcal{R}}{\sim}$ denotes the right equivalence.*

P r o o f. To prove this claim we choose an appropriate submanifold of $H^d(n, 1; \mathbb{C})$ containing f and g and then apply Mather's lemma to get the result. Here $f, g \in H^d(n, 1; \mathbb{C})$ are such that $\mathcal{J}_f = \mathcal{J}_g$. Set $f_t = (1-t)f + tg \in H^d(n, 1; \mathbb{C})$. Consider the $\mathcal{R}$-equivalence action on $H^d(n, 1; \mathbb{C})$ under the group $Gl(n, \mathbb{C})$. By eq. (4.16) [1, p. 35], we have

$$T_{f_t}(\mathrm{Gl}(n, \mathbb{C}).f_t) = \mathbb{C}\Big\langle x_j \frac{\partial f_t}{\partial x_i};\ i, j = 1, \dots, n\Big\rangle. \tag{3.1}$$

Now, note that the R.H.S of eq. (3.1) satisfies the relation

$$\mathbb{C}\Big\langle x_j \frac{\partial f_t}{\partial x_i};\ i, j = 1, \dots, n\Big\rangle \subset \mathcal{J}_{f_t} \cap H^d.$$

But $\mathcal{J}_{f_t} \cap H^d \subset \mathcal{J}_f \cap H^d$ since

$$\frac{\partial f_t}{\partial x_i} = (1-t)\frac{\partial f}{\partial x_i} + t\frac{\partial g}{\partial x_i} \in (1-t)\mathcal{J}_f + t\mathcal{J}_g = \mathcal{J}_f \quad (\text{because } \ \mathcal{J}_f = \mathcal{J}_g).$$

So, we have the inclusion of finite dimensional $\mathbb{C}$-vector spaces

$$T_{f_t}(\mathrm{Gl}(n, \mathbb{C}).f_t) = \mathbb{C}\Big\langle x_j \frac{\partial f_t}{\partial x_i};\ i, j = 1, \dots, n\Big\rangle \subset \mathcal{J}_f \cap H^d \tag{3.2}$$

with equality for $t = 0$ and $t = 1$.

Let us show that we have equality for all $t \in [0, 1]$ except finitely many values.

Clearly the dimension of the space $\mathcal{J}_f \cap H^d$ is at most n^2. Let us fix $\{e_1, \dots, e_m\}$ a basis of $\mathcal{J}_f \cap H^d$, where $m \leqslant n^2$. Consider the n^2 polynomials

$$\alpha_{ij}(t) = x_j \frac{\partial f_t}{\partial x_i} = x_j \Big[(1-t)\frac{\partial f}{\partial x_i} + t\frac{\partial g}{\partial x_i}\Big]$$

corresponding to the generators of the space (3.1). We can express each $\alpha_{ij}(t)$, $i, j = 1, \ldots, n$ in terms of the above mentioned fixed basis as

$$\alpha_{ij}(t) = \phi_{ij}^1(t)e_1 + \ldots + \phi_{ij}^m(t)e_m, \quad \forall\, i, j = 1, \ldots, n \tag{3.3}$$

where each $\phi_{ij}^k(t)$ is linear in t. Consider the matrix of transformation corresponding to the eqs. (3.3)

$$(\phi_{ij}^m(t))_{n^2\times m} = \begin{pmatrix} \phi_{11}^1(t) & \phi_{11}^2(t) & \ldots & \phi_{11}^m(t) \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{1n}^1(t) & \phi_{1n}^2(t) & \ldots & \phi_{1n}^m(t) \\ \phi_{21}^1(t) & \phi_{21}^2(t) & \ldots & \phi_{21}^m(t) \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{nn}^1(t) & \phi_{nn}^2(t) & \ldots & \phi_{nn}^m(t) \end{pmatrix}$$

having rank at most m. Note that the equality $\langle x_j \partial f_t/\partial x_i \rangle = \mathcal{J}_f \cap H^d$ holds for those values of t in $\mathbb{C}$ for which the rank of the above matrix is precisely m. As we know that the rank of the matrix of transformation is at most m, there must be $n^2 - m$ proportional rows. So, we have the $m \times m$ submatrix whose determinant is a polynomial of degree m in t and by the fundamental theorem of algebra it has at most m roots in $\mathbb{C}$, for which rank of the matrix of transformation will be less than m. Therefore, the above-mentioned equality does not hold for finitely many values, say $t_1, \ldots, t_p$ where $1 \leqslant p \leqslant m$.

It follows that the dimension of the space (3.1) is constant for all $t \in \mathbb{C}$ except finitely many values $\{t_1, \ldots, t_p\}$.

For an arbitrary smooth path

$$\alpha\colon\ \mathbb{C} \longrightarrow \mathbb{C} \setminus \{t_1, \ldots, t_p\}$$

with $\alpha(0) = 0$ and $\alpha(1) = 1$, we have the connected smooth submanifold

$$P = \{f_t = (1 - \alpha(t))f(x) + \alpha(t)g(x)\colon\ t \in \mathbb{C}\}$$

of H^d. By the above, it follows $\dim T_{f_t}((\mathrm{Gl}(n, \mathbb{C}).f_t))$ is constant for $f_t \in P$.

Now, to apply Mather's lemma, we need to show that the tangent space to the submanifold P is contained in that to the orbit $\mathrm{Gl}(n, \mathbb{C}).f_t$ for any $f_t \in P$. One clearly has

$$T_{f_t}P = \{\dot{f}_t = -\dot{\alpha}(t)f(x) + \dot{\alpha}(t)g(x)\ \forall\, t \in \mathbb{C}\}.$$

Therefore, by Euler's formula 7.6 [1, p. 101], we have

$$T_{f_t}P \subset T_{f_t}(\mathrm{Gl}(n, \mathbb{C}).f_t).$$

By Mather's lemma the submanifold P is contained in a single orbit. This implies that $f \overset{\mathcal{R}}{\sim} g$, as required. $\square$

Corollary 3.3. *Let $f, g \in H^d(n,1;\mathbb{C})$. If $M(f) \simeq M(g)$ (isomorphism of graded $\mathbb{C}$-algebras) then $f \overset{\mathcal{R}}{\sim} g$.*

P r o o f. We show firstly that an isomorphism of graded $\mathbb{C}$-algebras

$$\varphi\colon\ M(g) = \frac{\mathbb{C}[x_1,\ldots,x_n]}{\mathcal{J}_g} \overset{\simeq}{\longrightarrow} M(f) = \frac{\mathbb{C}[x_1,\ldots,x_n]}{\mathcal{J}_f}$$

is induced by a linear isomorphism $u\colon\ \mathbb{C}^n \longrightarrow \mathbb{C}^n$ such that $u^*(\mathcal{J}_g) = \mathcal{J}_f$. Consider the following commutative diagram.

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
\mathcal{J}_g \arrow[r, dotted, "u^*"] \arrow[d, "i"] & \mathcal{J}_f \arrow[d, "j"] \\
\mathbb{C}[x_1,\ldots,x_n] \arrow[r, dotted, "u^*"] \arrow[d, "p"] & \mathbb{C}[x_1,\ldots,x_n] \arrow[d, "q"] \\
M(g) \arrow[r, "\varphi", "\simeq"'] \arrow[d] & M(f) \arrow[d] \\
0 & 0
\end{tikzcd}$$

We note that the isomorphism φ is a degree preserving map and each of the Jacobian ideals $\mathcal{J}_f$ and $\mathcal{J}_g$ is generated by the homogeneous polynomials of degree $d-1$. The cases $d=1$ and $d=2$ are special, and we can treat them easily. Assume from now on that $d \geqslant 3$. Define the morphism $u^*\colon\ \mathbb{C}[x_1,\ldots,x_n] \to \mathbb{C}[x_1,\ldots,x_n]$ by

$$u^*(x_i) = L_i(x_1,\ldots,x_n) = a_{i1}x_1 + \ldots + a_{in}x_n, \quad a_{i1},\ldots,a_{in} \in \mathbb{C} \tag{3.4}$$

which is well defined by commutativity of the diagram below.

$$\begin{tikzcd}
x_i \arrow[r, mapsto, dotted, "u^*"] \arrow[d, mapsto, "p"] & L_i \arrow[d, mapsto, "q"] \\
\widehat{x_i} \arrow[r, mapsto, "\varphi", "\simeq"'] & \widehat{L_i}
\end{tikzcd}$$

Let us prove that u^* is an isomorphism. Consider the following commutative diagram at the level of degree $d = 1$.

$$\begin{tikzcd}
\mathbb{C}[x_1, \dots, x_n]_1 \arrow[r, dotted, "u_1^*"] \arrow[d, equal, "p"] & \mathbb{C}[x_1, \dots, x_n]_1 \arrow[d, equal, "q"] \\
(M(g))_1 \arrow[r, "\varphi_1", "\simeq"'] & (M(f))_1
\end{tikzcd}$$

Since here the Jacobian ideals $\mathcal{J}_f$ and $\mathcal{J}_g$ are generated by polynomials of degree $\geqslant 2$, therefore we have

$$(M(g))_1 = (M(f))_1 = \mathbb{C}[x_1, \dots, x_n]_1.$$

This implies that φ_1 and u_1^* are coincident. As φ is a given graded $\mathbb{C}$-algebra isomorphism, it follows that u_1^* is also an isomorphism. Hence u^* is an isomorphism.

Next we show that $u^*(\mathcal{J}_g) = \mathcal{J}_f$.

For every $G \in \mathcal{J}_g$, we have $u^*(G) \in \mathcal{J}_f$ by the commutative diagram below.

$$\begin{tikzcd}
G \arrow[r, mapsto, "u^*"] \arrow[d, mapsto, "p"] & F = u^*(G) \arrow[d, mapsto, "q"] \\
\widehat{0} \arrow[r, mapsto, "\varphi"] & \widehat{F} = \widehat{0}
\end{tikzcd}$$

This implies that $u^*(\mathcal{J}_g) \subset \mathcal{J}_f$. As u^* is an isomorphism, therefore it is invertible and by repeating the above argument for its inverse, we have $u^*(\mathcal{J}_g) \supset \mathcal{J}_f$.

Thus, u^* is an isomorphism with $u^*(\mathcal{J}_g) = \mathcal{J}_f$.

By eq. (3.4), the map $u\colon \mathbb{C}^n \to \mathbb{C}^n$ can be defined by

$$u(z_1, \dots, z_n) = (L_1(z_1, \dots, z_n), \dots, L_n(z_1, \dots, z_n))$$

where $L_i(z_1, \dots, z_n) = a_{i1}z_1 + \ldots + a_{in}z_n$.

Note that u is a linear isomorphism by Proposition 3.16 [1, p. 23].

In this way, we have shown that the isomorphism φ is induced by a linear isomorphism $u\colon \mathbb{C}^n \to \mathbb{C}^n$ such that $u^*(\mathcal{J}_g) = \mathcal{J}_f$.

Here $u^*(\mathcal{J}_g) = \langle g_1 \circ u, \dots, g_n \circ u \rangle = \mathcal{J}_{g \circ u}$, where $g_j = \partial g / \partial x_j$.

Therefore, $\mathcal{J}_{g \circ u} = \mathcal{J}_f \Rightarrow g \circ u \overset{\mathcal{R}}{\sim} f$. Now, $g \circ u \overset{\mathcal{R}}{\sim} g$ implies that $g \overset{\mathcal{R}}{\sim} f$. $\square$

Remark 3.4. The converse implication, namely

$$f \overset{\mathcal{R}}{\sim} g \Rightarrow M(f) \simeq M(g)$$

always holds (even for analytic germs f, g defining IHS), see [1, p. 90].

References

[1] *A. Dimca*: Topics on Real and Complex Singularities. Fried. Vieweg & Sohn, Braunschweig/Wiesbaden, 1987.

[2] *C. G. Gibson, K. Wirthmüller, A. A. Du Plessis, E. J. N. Looijenga*: Topological Stability of Smooth Mappings. Lect. Notes Math. Vol. 552. Springer, Berlin, 1976.

[3] *J. N. Mather*: Stability of C^∞-mappings. IV: Classification of stable germs by $\mathbb{R}$-algebras. Publ. Math., Inst. Hautes Étud. Sci. *37* (1969), 223–248.

[4] *J. N. Mather, S. S. T. Yau*: Classification of isolated hypersurface singularities by their moduli algebras. Invent. Math. *69* (1982), 243–251.

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