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ON DEFORMATIONS OF SPHERICAL ISOMETRIC FOLDINGS

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Abstract. The behavior of special classes of isometric foldings of the Riemannian sphere S^2 under the action of angular conformal deformations is considered. It is shown that within these classes any isometric folding is continuously deformable into the *standard* spherical isometric folding f_s defined by $f_s(x,y,z) = (x,y,|z|)$.

Keywords: isometric foldings, edge-to-edge spherical tilings, homotopy

MSC 2010: 52C20, 57Q55, 55P10, 52B05

1. Introduction

Suppose that a plane sheet of paper is crumpled gently in the hands, and then crushed flat against a desk top. The effect is to criss-cross the sheet with a pattern of creases.

It was S. A. Robertson [7], who in 1977 first observed that the patterns of creases so formed obey certain simples rules, namely:

(i) all the creases are composed of straight line segments;

(ii) if p is the end-point of such a segment then the total number of crease-segments that end at p is even. Moreover, the sum of alternated angles between creases at p is equal to π.

Replacing both the sheet of paper and the desk-top by the Euclidean plane $\mathbb{R}^2$ equipped with its standard Riemannian structure, the physical crumpling-crushing process can then be modelled mathematically by a map $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ that sends each piecewise-straight path in $\mathbb{R}^2$ to a piecewise-straight path in $\mathbb{R}^2$ of the same length.

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More generally, we can think about maps, called *isometric foldings* of Riemannian manifolds, that send finite piecewise geodesics to finite piecewise geodesics of the same length.

For any two smooth Riemannian manifolds, M and N, we denote by $\mathcal{F}(M,N)$ the set of all isometrics foldings from M into N. We may conclude that:

(i) $\mathcal{F}(M) = \mathcal{F}(M,M)$ is a semigroup with identity element id_M and contains the isometry group $\mathcal{I}(M)$ as a subsemigroup;

(ii) for all $x, y \in M$, $d_N(f(x), f(y)) \leqslant d_M(x,y)$, where d_M and d_N are, respectively, the metrics on M and N induced by their Riemannian structure. Consequently, any isometric folding is a continuous map;

(iii) any differentiable isometric folding is an isometry.

The map $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by $f(x,y) = (x, |y|)$ is an isometric folding of the real plane equipped with its standard structure, which is not differentiable at any point of the straight line $y = 0$.

A point $x \in M$ where an isometric folding $f\colon M \to N$ fails to be differentiable is called a *singularity* of f. The set of all singularities of f is denoted by Σf. An isometric folding f is called *nontrivial* if $\Sigma f \neq \emptyset$.

A general description of Σf for any $f \in \mathcal{F}(M,N)$ was given by Robertson in [7]. When M and N are complete Riemannian 2-manifolds this description can be stated as follows: for each $x \in \Sigma f$ the singularities of f near x form the image of an even number of geodesic rays emanating from x and making alternated angles α_1, β_1, α_2, β_2, …, α_n, β_n, where

$$\sum_{j=1}^{n} \alpha_j = \sum_{j=1}^{n} \beta_j = \pi. \tag{1.1}$$

In other words, the singularity set of an isometric folding on surfaces can be seen as an embedded graph of even valency satisfying the angle folding relation (1.1). Figure 1 shows a singularity set near a vertex of valency six.

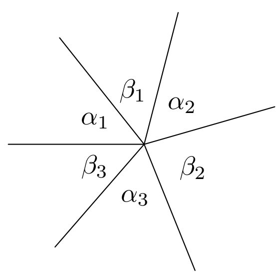

Fig. 1. The angle folding relation (with $n = 3$).

Our interest is focused on the set of isometric foldings of the Riemannian sphere $\mathcal{F}(S^2)$. The compactness of the sphere ensures that the singularity set of any spherical isometric folding (as an embedded graph of S^2) is connected with finitely many regions.

The compactness of S^2 allows us to conclude that any $f \in \mathcal{F}(S^2)$ is a proper map, [6]. S. Robertson established that the Hopf degree of any nontrivial spherical isometric folding is 0.

2. The compact-open topology on $\mathcal{F}(S^2)$

By a *spherical folding tiling* we mean an edge-to-edge finite polygonal-tiling τ of S^2 whose underlying graph is of the type described in (1.1). We shall denote by $\mathcal{T}(S^2)$ the set of all folding tilings of S^2, identifying the singularity sets of nontrivial foldings with spherical folding tilings.

Classification of spherical folding tilings with a specified fixed type of prototiles was obtained in [2], [3], [4] and [5].

Definition 2.1. Two folding tilings, τ_1 and τ_2, of S^2, are said to be congruent if there exists an isometry k of S^2 such that $k(\tau_1) = \tau_2$.

Proposition 2.1. *Let f and g be isometric foldings of S^2. Then*

(i) $\Sigma f = \Sigma g$ *iff there exists* $a \in \mathrm{Iso}\,(S^2)$ *such that* $g = a \circ f$;

(ii) Σf *and* Σg *are congruent iff there exist* $a, b \in \mathrm{Iso}(S^2)$ *such that* $g = a \circ f \circ b$.

Proof. (i) If $g = a \circ f$ for some $a \in \mathcal{I}(S^2)$, then $\Sigma g = \Sigma f \cup f^{-1}(\Sigma a) = \Sigma f$, since $\Sigma a = \emptyset$.

Conversely, suppose that $\Sigma f = \Sigma g = \tau$. If F is a face of τ, there are $i, j \in \mathcal{I}(S^2)$ such that $(i \circ f)|_F = (j \circ g)|_F = \mathrm{id}_F$. In [1] it was shown that necessarily $i \circ f = j \circ g$, and so $g = a \circ f$, where $a = j^{-1} \circ i$.

(ii) Suppose that $g = a \circ f \circ b$ for some $a, b \in \mathcal{I}(S^2)$. Then $\Sigma g = \Sigma\,(a \circ (f \circ b)) = \Sigma(f \circ b) = b^{-1}(\Sigma f)$, and so Σf and Σg are congruent.

On the other hand, if Σf and Σg are congruent, then $\Sigma g = k(\Sigma f)$ for some $k \in \mathrm{Iso}\,(S^2)$, and so $\Sigma g = \Sigma(f \circ k^{-1})$, since $k(\Sigma f) = \Sigma(f \circ k^{-1})$. Now, using the case (i) one gets $g = a \circ f \circ k^{-1}$ for some $a \in \mathrm{Iso}\,(S^2)$. □

Let us consider the *compact-open topology* on $\mathcal{F}(S^2)$, i.e., the topology generated by sets of the form

$$B(K, U) = \left\{ f \in \mathcal{F}(S^2) \colon\ f(K) \subset U \right\},$$

where K is compact in S^2 and U is open in S^2.

Definition 2.2. Let f, $g \in \mathcal{F}(S^2)$. We say that f is *deformable* into g iff there exists a map, (homotopy) $H\colon [0,1] \times S^2 \to S^2$ such that

(i) H is continuous;

(ii) for each $t \in [0,1]$, H_t defined by $H_t(x) = H(t,x)$, $x \in S^2$ is an isometric folding;

(iii) $H(0,x) = f(x)$ and $H(1,x) = g(x)$, $\forall x \in S^2$.

As we are considering the compact open topology, f is deformable into g iff they belong to the same path connected component, i.e., there is a continuous map $\gamma\colon [0,1] \to \mathcal{F}(S^2)$ such that $\gamma(0) = f$ and $\gamma(1) = g$.

The relation of deformation is obviously an equivalence relation on $\mathcal{F}(S^2)$.

Definition 2.3. An isometric folding $f \in \mathcal{F}(S^2)$ is *simple* if Σf is a great circle of S^2. The (simple) *standard folding*, denoted by f_s, is defined by

$$f_s(x,y,z) = (x,y,|z|),\ \forall (x,y,z) \in S^2.$$

In [1] it was established that any nontrivial isometric folding with Hopf degree zero in the Euclidean plane is deformable into the standard planar folding $f\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, defined by $f(x,y) = (x,|y|)$ and was conjectured that (Breda-Roberton conjecture)

(2.1) *any nontrivial spherical isometric folding is deformable into* f_s.

In other words, all spherical isometric foldings with zero Hopf degree belong to the same path connected component of $\mathcal{F}(S^2)$. It should be pointed out that a proof given for the sphere cannot be a simple adaptation of the one used for the plane, since here the dilatations played a crucial role. However, some contributions to the above conjecture will be given.

3. Deformation of simple foldings

Here we show that the sub-semigroup of $\mathcal{F}(S^2)$ generated by spherical simple foldings verifies the Breda-Roberton conjecture.

Theorem 3.1. *Let f, $g \in \mathcal{F}(S^2)$. If f is deformable into f_0 and g is deformable into g_0 for some f_0 and $g_0 \in \mathcal{F}(S^2)$, then $f \circ g$ is deformable into $f_0 \circ g_0$.*

Proof. Suppose that $H\colon [0,1] \times S^2 \to S^2$ is a deformation of f into f_0. The map $\mathscr{H} = H \circ g\colon [0,1] \times S^2 \to S^2$ defined by $\mathscr{H}(t,x) = H(t,g(x))$ is a deformation of $f \circ g$ into $f_0 \circ g$. In fact, $\mathscr{H}$ is continuous; for each $t \in [0,1]$, $\mathscr{H}_t$ defined by $\mathscr{H}_t(x) = \mathscr{H}(t,x) = (H_t \circ g)(x)$ is a spherical isometric folding; and $\mathscr{H}(0,x) = H(0,g(x)) = f(g(x))$ and $\mathscr{H}(1,x) = H(1,g(x)) = f_0(g(x))$.

Let now $H'\colon [0,1]\times S^2 \to S^2$ be a deformation of g into g_0. Then $\mathscr{H}' = f_0 \circ H'\colon [0,1]\times S^2 \to S^2$ defined by $\mathscr{H}'(t,x) = f_0(H'(t,x))$ is a deformation of $f_0\circ g$ into $f_0\circ g_0$. In fact, $\mathscr{H}'$ is continuous; for each $t \in [0,1]$, $\mathscr{H}'_t$ defined by $\mathscr{H}'_t(x) = \mathscr{H}'(t,x) = (f_0 \circ H'_t)(x)$ is a spherical isometric folding; and $\mathscr{H}'(0,x) = f_0(H'(0,x)) = f_0(g(x))$ and $\mathscr{H}'(1,x) = f_0(H'(1,x)) = f_0(g_0(x))$. It follows that $f\circ g$ is deformable into $f_0 \circ g_0$. □

Since $f_s \circ f_s = f_s$, immediately

Corollary 3.1. *If $f_1,\dots,f_n$ are isometric foldings of S^2 deformable into f_s, then $f = f_1 \circ \ldots \circ f_n$ is deformable into f_s.*

Next, we exhibit a homotopy joining f_s to $\overline{f}_s = \varrho^{xy}\circ f_s$, where ϱ^{xy} is the reflection on the coordinate plane xOy. Observe that $\overline{f}_s(x,y,z) = (x,y,-|z|)$ for all $(x,y,z) \in S^2$.

Lemma 3.1. *Let Π_t, $t \in [0,1]$ be the plane containing the yy axis and making an angle $\frac{1}{2}\pi t$ with the plane xOy (as being Π_0). Further, let ϱ_t be the spherical reflection on Π_t. Then the map $H\colon [0,1]\times S^2 \to S^2$ such that*

$$H(t,x) = (\varrho_t \circ f_s \circ \varrho_t)\,(x)$$

is a deformation of $\overline{f}_s$ into f_s.

P r o o f. Clearly H is continuous; for each $t \in [0,1]$, H_t defined by $H_t(x) = H(t,x)$, $x \in S^2$, is a spherical isometric folding joining $H_0 = \overline{f}_s$ and $H_1 = f_s$. □

Lemma 3.2. *Let i be an orientation preserving (reversing) isometry of S^2. Then there exists a continuous map $\Gamma\colon [0,1]\times S^2 \to S^2$ such that for each $t\in[0,1]$, $\Gamma_t = \Gamma(t,*)$ is an orientation preserving (reversing) isometry, $\Gamma_0 = i$ and $\Gamma_1 = \mathrm{id}_{S^2}$ ($\Gamma_1 = -\mathrm{id}_{S^2}$).*

P r o o f. We shall show that any rotation can be joined to id_{S^2} and any reflection can be joined to $-\mathrm{id}_{S^2}$. In fact, if R^L_θ is the rotation of S^2 through an angle θ around a line L, then the map $H(t,x) = R^L_{(1-t)\theta}(x)$, $t\in[0,1]$, $x \in S^2$, is a deformation of R^L_θ into id_{S^2}.

Let now ϱ denote a reflection of S^2. The axial symmetry $-\mathrm{id}_{S^2}$ can be written as $-\mathrm{id}_{S^2} = R^z_\pi \circ \varrho^{xy} = \varrho^{xy}\circ R^z_\pi$, where ϱ^{xy} is the reflection on the plane xOy and R^z_π is the rotation of S^2 through the angle π around the zz axis. On the other hand, the map $\varrho\circ R^z_\pi \circ \varrho^{xy}$ is a rotation of S^2, say $R'_{\theta'}$, through an angle θ' around a line L', for some θ' and L'. It follows that $H'(t,x) = (\varrho\circ R'_{t\theta'})(x)$, $t\in[0,1]$, $x\in S^2$, is a deformation of $H'(0,x) = \varrho(x)$ into $H'(1,x) = (\varrho\circ R'_{\theta'})(x) = (\varrho\circ\varrho\circ R^z_\pi\circ\varrho^{xy})(x) = (R^z_\pi\circ\varrho^{xy})(x) = -x$.

Since any spherical isometry is either a rotation, a reflection or a glide-reflection (the product of a reflection in a line L with a rotation which maps L onto itself) the result follows. $\square$

Lemma 3.3. *The isometric foldings*

$$f_s(-x), \quad -f_s(x) \quad \text{and} \quad -f_s(-x)$$

are all deformable into f_s.

Proof. Let R^z_α be the rotation of S^2 through an angle α around the zz axis. Then $f_s(-x) = R^z_\pi \circ f_s$ and so $H_t = R^z_{t\pi} \circ f_s$, $t \in [0,1]$ joins $f_s(-x)$ and f_s.

On the other hand, $-f_s(x) = R^z_\pi \circ \varrho^{xy} \circ f_s$, where $R^z_\pi \circ \varrho^{xy}$ is the axial symmetry. It follows that $L_t = R^z_{t\pi} \circ \varrho^{xy} \circ f_s$, $t \in [0,1]$ joins $-f_s(x)$ and $-f_s(-x)$.

By Lemma 3.1, the folding $\overline{f}_s = -f_s(-x)$ is deformable into f_s. The result follows. $\square$

Theorem 3.2. *Let f be a simple isometric folding of S^2. Then f is deformable into f_s.*

Proof. By Proposition 2.1, $f = a \circ f_s \circ b$ for some $a, b \in \operatorname{Iso}(S^2)$. By Lemma 3.2 and Theorem 3.1 f is deformable into $\pm f_s(\pm x)$. Using Lemma 3.3 we can join any one of these foldings to f_s. $\square$

Corollary 3.2. *If $f \in \mathscr{F}(S^2)$ is of the form $f = f_1 \circ \ldots \circ f_n$, where f_i $(i = 1, \ldots, n)$ is a simple isometric folding, then f is deformable into f_s.*

Proof. Use Theorem 3.1 and Theorem 3.2. $\square$

Theorem 3.3. *Let f, $g \in \mathscr{F}(S^2)$ be such that f is deformable into f_s. If Σf and Σg are congruent then g is deformable into f_s.*

Proof. Let a, $b \in \operatorname{Iso}(S^2)$ be such that $g = a \circ f \circ b$. By Theorem 3.1 and Lemma 3.2, g is deformable into $\pm f_s(\pm x)$. By Lemma 3.3 the result follows. $\square$

The problem stated in (2.1) is now partially solved, since it is verified in the interesting class of all isometric foldings that are compositions (of a finite number) of simple foldings or any isometric folding whose singularity set is congruent to such a folding.

Figure 2 shows a dihedral f-tiling τ (obtained in [4]) with prototiles a spherical rhombus with distinct pairs of opposite angles $\frac{2}{3}\pi$ and $\frac{2}{5}\pi$, and a scalene spherical triangle of angles $\frac{1}{2}\pi$, $\frac{1}{3}\pi$ and $\frac{1}{5}\pi$. We observe that τ is identified with the set of singularities of a spherical isometric folding obtained by composition of simple foldings.

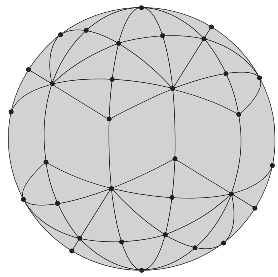

Fig. 2. A spherical isometric folding composition of simple foldings.

4. Perfect foldings and their deformations

Now we focus our attention on isometric foldings $f\colon S^2 \to S^2$ such that f has no singularities in the interior of its image.

Definition 4.1. A nontrivial isometric folding $f\colon S^2 \to S^2$ is said to be perfect if $\Sigma f = f^{-1}(\partial \mathrm{Im}\, f)$ (or $\Sigma f \cap f^{-1}(\widehat{\overset{\circ}{\mathrm{Im}}\, f}) = \emptyset$).

Here we describe, up to an isomorphism, the class of all perfect foldings $f\colon S^2 \to S^2$. General properties of $\tau = \Sigma f$ are also given.

A tiling τ will be called monohedral if every tile of τ is congruent to one fixed set T called the prototile of τ.

Lemma 4.1. *If τ is a monohedral polygonal spherical tiling with even vertex valency then the prototile of τ must be a triangle (or a spherical moon).*

Proof. Let P be the prototile of τ. We may suppose that P is an n-sided spherical polygon, where $n \geqslant 3$. We shall denote by V, E, F and V_r ($r \geqslant 2$), respectively, the number of vertices, edges, faces and vertices of valency $2r$ of τ. Then

$$\begin{cases} nF = 2E, \\ \sum\limits_{r=2}^{L} 2rV_r = 2E \quad \text{for some } L \geqslant 2, \\ V = \sum\limits_{r=2}^{L} V_r. \end{cases}$$

Taking into account the Euler's relation $F - E + V = 2$, one gets

$$\sum_{r=2}^{L} \Bigl(\frac{2r}{n} - r + 1\Bigr) V_r = 2.$$

As $f(r) = 2r/n - r + 1 \leqslant 0$ for $r \geqslant 3$, hence necessarily $f(2) > 0$ and consequently $n = 3$. □

Theorem 4.1. *Let f be a perfect folding of S^2, $\tau = \Sigma f$ and $F = \operatorname{Im} f$. Then:*

(i) *τ is a monohedral f-tiling with prototile F;*

(ii) *if e is an edge of τ, then the great circle containing e is contained in τ;*

(iii) *F is either a spherical moon with internal angle π/k, $k \geqslant 1$, or a spherical triangle with internal angles (up to an order) $(\frac{1}{2}\pi, \frac{1}{2}\pi, \pi/k)$, $k \geqslant 2$ or $(\frac{1}{2}\pi, \frac{1}{3}\pi, \pi/k)$, $k \in \{3, 4, 5\}$;*

(iv) *f is a composition of simple foldings.*

Proof. The description of the singularity set of f implies that each face of $\tau = \Sigma f$ is an n-sided convex polygon.

(i) Without loss of generality we may suppose that $f|_F = \mathrm{id}_F$ ($F = \operatorname{Im} f$) and so F is a face of τ. Let e and s be, respectively, an edge of F and a spherical segment contained in F. If s' is the reflection of s on the great circle containing e then $s' \in f^{-1}(s)$, and as $\Sigma f = f^{-1}(\partial \operatorname{Im} f)$, if s is an edge of F then s' is an edge of τ. Consequently, each face of τ adjacent to F is congruent to F (by reflection on an edge of F). Repeating this argument for any other face, we conclude that all faces of τ are congruent to F. And so τ is a monohedral f-tiling with prototile F. In fact, any face of τ is obtained from F by successive reflections on its edges.

(ii) Let v be a vertex of τ. Then v is of even valency and by the previous case all the angles surrounding v are congruent. Now, if e is an edge of τ incident to v, then the great circle containing e is contained in τ. In particular, the antipode $-v$ is also a vertex of τ congruent to v.

(iii) By Lemma 4.1, F must be a spherical triangle or a spherical moon. If F is a triangle with angles, say, α, β and γ ($\alpha \geqslant \beta \geqslant \gamma$), then $\alpha = \pi/k$, $\beta = \pi/l$ and $\alpha = \pi/m$ for some $k, l, m \geqslant 2$. Taking in account that $\pi < \alpha + \beta + \gamma < \frac{3}{2}\pi$, the unique possible combinations are those refereed above.

(iv) If f is a perfect isometric folding of S^2, then $\tau = \Sigma f$ is illustrated, up to an isomorphism, in Figure 3.

In the first case the prototile is a spherical moon of angle π/k for some $k \geqslant 1$, and f can be seen as a composition of k simple foldings. In the other cases the prototile is a triangle, and each tiling is obtained by reflecting a tiled spherical moon on its edges. Choosing now a triangle sharing a vertex with a spherical moon, then it can be reflected successively on its edges forming the whole spherical moon. □

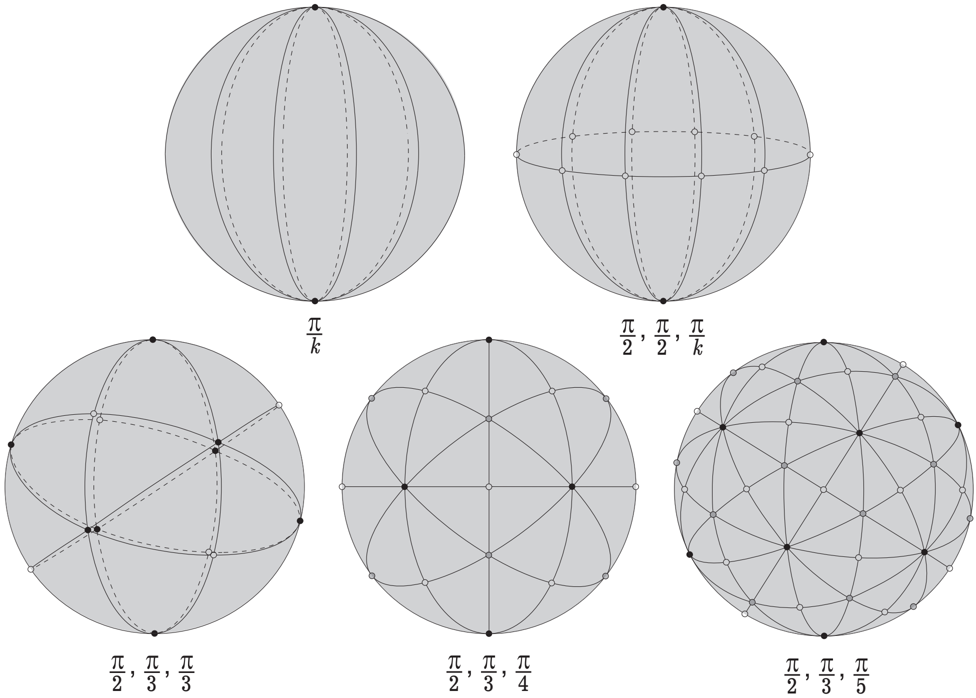

Fig. 3. Perfect spherical foldings.

Corollary 4.1. *Any perfect folding of the sphere is deformable into* f_s.

This result can be generalized to other Riemannian 2-manifolds.

4.1. Spherical foldings over perfect foldings. We extend the family of isometric foldings deformable into the standard folding, proceeding as follows:

Definition 4.2. Let F be a convex spherical polygon and let $g\colon F \to S^2$ be a continuous map such that $g|_{\mathring{F}}$ is an isometric folding. We say that g is deformable into id_F if there exists a continuous map $H\colon [0,1] \times F \to S^2$ such that for each $t \in [0,1]$, H_t is an isometric folding of $\mathring{F}$ into S^2 and $H(0,x) = g(x)$ and $H(1,x) = x$, $\forall x \in F$.

Theorem 4.2. *Let f be a perfect isometric folding of S^2. If $g\colon F = \operatorname{Im} f \to S^2$ is a continuous map such that $g|_{\mathring{F}}$ is an isometric folding deformable into id_F, then $g \circ f$ is an isometric folding of S^2 deformable into f_s.*

P r o o f. Suppose that $H\colon [0,1] \times F \to S^2$ is a deformation of g into id_F. Then the map $\mathscr{H} = H \circ f\colon [0,1] \times S^2 \to S^2$ defined by $\mathscr{H}(t,x) = H(t, f(x))$ is a deformation of $g \circ f$ into f. In fact,

$$\mathscr{H}(0,x) = H(0,f(x)) = g(f(x)) \text{ and } \mathscr{H}(1,x) = H(1,f(x)) = f(x) \in F,\ x \in S^2.$$

By Corollary 4.1, f is a composition of simple foldings, therefore f is deformable into f_s. And so, $g \circ f$ is deformable into f_s. $\square$

Remark. Let f and g be isometric foldings satisfying the conditions of Theorem 4.2, and let $\mathscr{H}$ be the deformation of $g \circ f$ into f described before. By Theorem 4.1 any face F' of $\tau = \Sigma f$ is obtained from $F = \operatorname{Im} f$ by successive reflections on its edges. In other words, there are spherical reflections $\varrho_1, \ldots, \varrho_k$ such that

$$F' = \overbrace{\varrho_k \circ \ldots \circ \varrho_1}^{\varrho_{F'}}(F),$$

where ϱ_1 is a spherical reflection in an edge of F and ϱ_i $(i = 2, \ldots, k)$ is a reflection in an edge of $(\varrho_{i-1} \circ \ldots \circ \varrho_1)(F)$. In Figure 4, we have taken $k = 5$. One has $\Sigma(g \circ f) = \Sigma f \cup f^{-1}(\Sigma g)$. Now, if $\alpha\colon [0,1] \to \mathcal{T}(S^2)$ is defined by $\alpha(t) = \Sigma \mathscr{H}_t$, then

$$f^{-1}(\Sigma g) = \bigcup_{F' \text{ face of } \Sigma f} \varrho_{F'}(\alpha(t) \cap F) \quad \text{and} \quad \varrho_{F'}(\alpha(t) \cap F) = \alpha(t) \cap F',$$

where $\varrho_{F'} = \varrho_k \circ \ldots \circ \varrho_1$. In particular, for $t = 0$ one has

$$\varrho_{F'}(\alpha(0) \cap F) = \varrho_{F'}\big((\Sigma(g \circ f)) \cap F\big) = (\Sigma(g \circ f)) \cap F'$$

and, for $t = 1$, one has

$$\varrho_{F'}(\alpha(1) \cap F) = \varrho_{F'}(\Sigma f \cap F) = \varrho_{F'}(\partial F) = \partial F' = \Sigma f \cap F'.$$

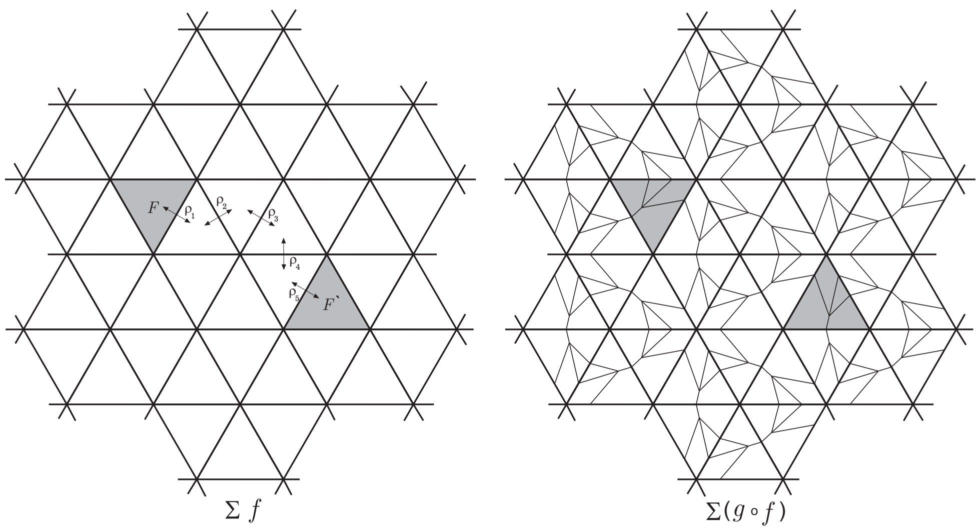

Fig. 4. F-tilings Σf and $\Sigma(g \circ f)$.

It is not difficult to find isometric foldings which are not over perfect ones deformable in the standard folding. In Figure 5 we provide one such example. A very interesting question, for future work, is to find how far from the set of non trivial spherical foldings the set of spherical foldings over prefect ones is.

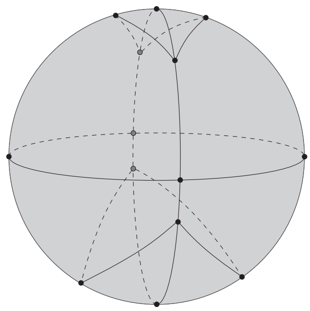

Fig. 5. The singular set of a spherical folding not over a perfect one.

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