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GLOBAL STRUCTURE OF POSITIVE SOLUTIONS FOR SUPERLINEAR $2m$th-BOUNDARY VALUE PROBLEMS

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Abstract. We consider boundary value problems for nonlinear $2m$th-order eigenvalue problem

$$\begin{aligned}(-1)^m u^{(2m)}(t) &= \lambda a(t) f(u(t)), \qquad 0 < t < 1,\\ u^{(2i)}(0) = u^{(2i)}(1) &= 0, \qquad i = 0, 1, 2, \ldots, m-1.\end{aligned}$$

where $a \in C([0,1],[0,\infty))$ and $a(t_0) > 0$ for some $t_0 \in [0,1]$, $f \in C([0,\infty),[0,\infty))$ and $f(s) > 0$ for $s > 0$, and $f_0 = \infty$, where $f_0 = \lim\limits_{s\to 0^+} f(s)/s$. We investigate the global structure of positive solutions by using Rabinowitz's global bifurcation theorem.

Keywords: multiplicity results, Lidstone boundary value problem, eigenvalues, bifurcation methods, positive solutions

MSC 2010: 34B10, 34G20

1. Introduction

We are interested in the study of the global structure of positive solutions of the problem

$$(-1)^m u^{(2m)}(t) = \lambda a(t) f(u(t)), \qquad t \in (0,1), \tag{1.1}$$

$$u^{(2i)}(0) = u^{(2i)}(1) = 0, \qquad i = 0, 1, 2, \ldots, m-1, \tag{1.2}$$

where λ is a positive parameter, $a\colon [0,1] \to [0,\infty)$ and $f\colon [0,\infty) \to [0,\infty)$ is continuous. We note that when $m = 2$, (1.1), (1.2) may describe the deformations of an elastic beam whose both ends are simply supported, see Gupta [1].

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In the past twenty years, the existence of solutions, especially the existence of positive solutions, of (1.1), (1.2) and its general cases, has been extensively studied by using the Leray-Schauder degree and the fixed point theorem in cones, see Agarwal [1], Agarwal and Wong [2], Aftabizadeh [3], Yang [4], Del Pino and Manásevich [5], Ma and Wang [6], Ma, Zhang and Fu [7], Bai and Wang [8], Bai and Ge [9], Yao [10], Y. Li [11], F. Li *et al.* [12] and references therein. Also, the global structure of positive solution set (and nodal solutions set) are investigated by several authors, see for example, the interesting contributions [13]–[15] by Bari and Rynne.

Very recently Ma [16]–[18] studied the global bifurcation phenomena of nodal solutions of (1.1), (1.2) when $m = 2$ and $f_0 \in (0,\infty)$, where $f_0 = \lim\limits_{s\to 0^+} f(s)/s$. However, relatively little is known about the global structure of solutions in the case that $f_0 = \infty$, and no global results are found in the available literature when $f_0 = \infty = f_\infty$. The probable reason is that the global bifurcation techniques cannot be applied directly in this case.

In the present work we obtain complete description of the global structure of positive solutions of (1.1), (1.2) under the assumptions

(A1) $a\colon [0,1] \to [0,\infty)$ is continuous and $a(t) \not\equiv 0$ on $[0,1]$;

(A2) $f\colon [0,\infty) \to [0,\infty)$ is continuous and $f(s) > 0$ for $s > 0$;

(A3) $f_0 = \infty$, where $f_0 = \lim\limits_{s\to 0^+} f(s)/s$;

(A4) $f_\infty \in [0,\infty]$, where $f_\infty = \lim\limits_{s\to +\infty} f(s)/s$.

Let Y be the Banach space $C[0,1]$ with the norm

$$\|u\|_0 = \max\{|u(t)| \mid t \in [0,1]\}.$$

Let E denote the Banach space defined by

$$E = \{u \in C^{2m-1}[0,1];\ u^{(2i)}(0) = u^{(2i)}(1) = 0,\ i = 0,1,2,\dots,m-1\}$$

equipped with the norm

$$\|u\| = \max\{\|u\|_0, \|u'\|_0, \dots, \|u^{2m-1}\|_0\}.$$

Define an operator $L\colon (E \cap C^{2m}[0,1]) \to Y$ by

$$Lu = (-1)^m u^{(2m)}, \qquad u \in E \cap C^{2m}[0,1].$$

To state our main results, we need the spectrum theory for the linear eigenvalue problem

$$Lu = \lambda a(t) u, \quad t \in (0,1), \tag{1.3}$$

$$u^{(2i)}(0) = u^{(2i)}(1) = 0, \quad i = 0,1,2,\dots,m-1. \tag{1.4}$$

Lemma 1.0 ([19]). *Let* (A1) *hold. Then the eigenvalues of* (1.3), (1.4) $\{\lambda_k\}_{k\in\mathbb{N}} \subset (0,\infty)$ *satisfy*

(i) $0 < \lambda_1 < \lambda_2 < \ldots < \lambda_k < \lambda_{k+1} < \ldots, \lim_{k\to\infty} \lambda_k = \infty$;

(ii) *for each* $k \in \mathbb{N}$, *the algebraic multiplicity of* λ_k *is equal to* 1;

(iii) *for each* $k \in \mathbb{N}$, *if* $v \in \ker(L - \lambda_k I) \setminus \{0\}$, *then* v *has exactly* $k-1$ *simple zeros in* $(0,1)$.

Let M be a subset of E. A *component* of M is a maximal connected subset of M, i.e. a connected subset of M which is not contained in any other connected subset of M.

The main results of this paper are the following theorems.

Theorem 1.1. *Let* (A1)–(A3) *hold.*

(a) *If* $f_\infty = 0$, *then there exists a component* ζ *of* Σ *with* $(0,0) \in \zeta$ *and*

$$\mathrm{Proj}_{\mathbb{R}}\zeta = [0,\infty).$$

(b) *If* $f_\infty \in (0,\infty)$, *then there exists a component* ζ *of* Σ *with*

$$(0,0) \in \zeta, \quad \mathrm{Proj}_{\mathbb{R}}\zeta \subseteq \Big[0, \frac{\lambda_1}{f_\infty}\Big).$$

(c) *If* $f_\infty = 0$, *then there exists a component* ζ *of* Σ *with* $(0,0) \in \zeta$, $\mathrm{Proj}_{\mathbb{R}}\zeta$ *is a bounded closed interval, and* ζ *approaches* $(0,\infty)$ *as* $\|u\| \to \infty$.

Theorem 1.2. *Let* (A1)–(A3) *hold.*

(a) *If* $f_\infty = 0$, *then* (1.1), (1.2) *has at least one positive solution for* $\lambda \in (0,\infty)$.

(b) *If* $f_\infty \in (0,\infty)$, *then* (1.1), (1.2) *has at least one positive solution for* $\lambda \in (0, \lambda_1/f_\infty)$.

(c) *If* $f_\infty = 0$, *then there exists* $\lambda_* > 0$ *such that* (1.1), (1.2) *has at least two positive solutions for* $\lambda \in (0,\lambda_*)$.

We will develop a bifurcation approach to treat the case $f_0 = \infty$. Crucial to this approach is to construct a sequence of functions $\{f^{[n]}\}$ which is asymptotically linear at 0 and satisfies

$$f^{[n]} \to f, \quad (f^{[n]})_0 \to \infty.$$

By means of the corresponding auxiliary equations, we obtain a sequence of unbounded components $\{C_+^{[n]}\}$ via Rabinnowitz's global bifurcation theorem [11], and this enables us to find an unbounded component $\mathcal{C}$ satisfying

$$(0,0) \in \mathcal{C} \subset \limsup C_+^{[n]}.$$

We now conclude this introduction by outlining the rest of the paper: In Section 2, we introduce some notation and prove some preliminary results. Finally, in Section 3, we prove our main results by global bifurcation techniques.

2. Some preliminaries

In this section we introduce some notation and preliminary results which will be used in the proofs of our main results.

Definition 2.1 [20]. Let X be a Banach space and $\{C_n;\ n=1,2,\ldots\}$ a family of subsets of X. Then the *limit superior* $\mathcal{D}$ of $\{C_n\}$ is defined by

$$\mathcal{D} := \limsup_{n\to\infty} C_n = \{x\in X;\ \exists\{n_i\}\subset\mathbb{N} \text{ and } x_{n_i}\in C_{n_i} \text{ such that } x_{n_i}\to x\}.$$

Lemma 2.1 [20]. *Suppose that Y is a compact metric space, A and B are non-intersecting closed subsets of Y, and no component of Y intersects both A and B. Then there exist two disjoint compact subsets Y_A and Y_B such that $Y=Y_A\cup Y_B$, $A\subset Y_A$, $B\subset Y_B$.*

Lemma 2.2. *Let X be a Banach space and let $\{C_n\}$ be a family of closed connected subsets of X. Assume that*
(i) *there exist $z_n\in C_n$, $n=1,2,\ldots$, and $z^*\in X$ such that $z_n\to z^*$;*
(ii) $\lim\limits_{n\to\infty} r_n=\infty$, *where $r_n=\sup\{\|x\|;\ x\in C_n\}$;*
(iii) *for every $R>0$, $\Big(\bigcup\limits_{n=1}^{\infty} C_n\Big)\cap B_R$ is a relatively compact set of X, where*

$$B_R=\{x\in X;\ \|x\|\leqslant R\}.$$

Then $\mathcal{D}\ (=\limsup\limits_{n\to\infty} C_n)$ contains an unbounded component $\mathcal{C}$ with $z^*\in\mathcal{C}$.

Proof. By the definition of $\mathcal{D}$, $z^*\in\mathcal{D}$. Suppose on the contrary that the component $\mathcal{C}$ in $\mathcal{D}$ which contains z^* is bounded. Note that $\mathcal{D}$ is closed in X. It follows that $\mathcal{C}$ is a closed subset of $\mathcal{D}$, and consequently $\mathcal{C}$ is a closed subset of X. It is easy to see that $\mathcal{C}$ is a compact subset of X by (iii). Take $\delta>0$, and let U_1 be a δ-neighborhood of $\mathcal{C}$ in X.

We discuss two cases.

Case 1. $\partial U_1\cap\mathcal{D}\neq\emptyset$.

In this case, we have from (iii) that $\overline{U}_1\cap\mathcal{D}$ is a compact metric space. Obviously, C and $\partial U_1\cap\mathcal{D}$ are two disjoint closed subsets of X. Because of the maximal connectedness of $\mathcal{C}$, there exists no component $\mathcal{C}^*$ of $\mathcal{D}\cap\overline{U}_1$ such that $\mathcal{C}^*\cap\mathcal{C}\neq\emptyset$,

$\mathcal{C}^* \cap (\partial U_1 \cap \mathcal{D}) \neq \emptyset$. By Lemma 2.1, there exist two disjoint compact sets X_A and X_B of $D \cap \overline{U}_1$ such that $D \cap \overline{U}_1 = X_A \cup X_B$, $\mathcal{C} \subset X_A$, $\partial U_1 \cap \mathcal{D} \subset X_B$. Evidently, $d(X_A, X_B) > 0$. Let $\delta_1 = \frac{1}{3} d(X_A, X_B)$, and let U_2 be the $\frac{1}{3}\delta_1$-neighborhood of X_A. Set $U = U_1 \cap U_2$. Then

$$\mathcal{C} \subset U, \quad \partial U \cap \mathcal{D} = \emptyset. \tag{2.1}$$

Case 2. $\partial U_1 \cap \mathcal{D} = \emptyset$.

In this case, we take $U = U_1$. It is obvious that (2.1) holds.

Since $z_n \to z^*$, we may assume that $\{z_n\} \subset U$. By (ii) and the connectedness of C_n, there exists $n_0 > 0$ such that for all n: $n \geqslant n_0 \Rightarrow C_n \cap \partial U \neq \emptyset$. Take $y_n \in C_n \cap \partial U$, then $\{y_n;\ n \geqslant n_0\}$ is a relative compact subset of X, so there exist $y^* \in \partial U$ and a subsequence $\{y_{n_k}\}$ of $\{y_n;\ n \geqslant n_0\}$ such that $y_{n_k} \to y^*$. Obviously, $y^* \in \mathcal{D}$. Therefore, $y^* \in \partial U \cap \mathcal{D}$. However, this contradicts (2.1). The proof is completed. □

Now, let σ be a constant with $0 < \sigma < \frac{1}{2}$. Denote the cone K in Y by

$$K = \{u \in Y;\ u(t) \geqslant 0 \text{ on } (0,1), \text{ and } \min_{\sigma \leqslant t \leqslant 1-\sigma} u(t) \geqslant \Gamma \|u\|_0\}, \tag{2.2}$$

where $\Gamma = \sigma^m [\int_\sigma^{1-\sigma} G(\tau,\tau)\,\mathrm{d}\tau]^{m-1}$. For $r > 0$, let

$$\Omega_r = \{u \in K;\ \|u\|_0 < r\}.$$

Define a map $T_\lambda\colon K \to Y$ by

$$T_\lambda u(t) = \lambda \int_0^1 G_m(t,s) a(s) f(u(s))\,\mathrm{d}s, \quad t \in [0,1], \tag{2.3}$$

where $G_m(t,s)$ can be expressed by the recurrence

$$G_m(t,s) = \int_0^1 G(t,\tau) G_{m-1}(\tau,s)\,\mathrm{d}\tau, \quad m \geqslant 2 \tag{2.4}$$

and

$$G_1(t,s) = \begin{cases} (1-t)s, & 0 \leqslant s \leqslant t \leqslant 1, \\ t(1-s), & 0 \leqslant t \leqslant s \leqslant 1, \end{cases} \tag{2.5}$$

see [8] for the details. It is easy to verify that the following lemma holds.

Lemma 2.3 ([21]). (i) *For any* $(t,s)\in[0,1]\times[0,1]$ *we have*

$$G_m(t,s)\leqslant\left[\int_0^1 G(\tau,\tau)\,\mathrm{d}\tau\right]^{m-1}G(s,s)=\frac{1}{6^{m-1}}G(s,s),\quad\forall\, m\in\mathbb{N}. \tag{2.6}$$

(ii) *For any* $(t,s)\in[\sigma,1-\sigma]\times[0,1]$ *we have*

$$G_m(t,s)\geqslant\Gamma G(s,s),\quad\forall\, m\in\mathbb{N}. \tag{2.7}$$

Lemma 2.4. *Assume that* (A1), (A2) *hold. Then* $T_r\colon K\to K$ *is completely continuous.*

Lemma 2.5. *Let* (A1), (A2) *hold. If* $u\in\partial\Omega_r,\ r>0$, *then*

$$\|T_\lambda u\|_0\leqslant\frac{\lambda\widehat{M_r}}{6^{m-1}}\int_0^1 G(s,s)a(s)\,\mathrm{d}s.$$

where $\widehat{M_r}=1+\max\limits_{0\leqslant s\leqslant r}\{f(s)\}$.

P r o o f. Since $f(u(t))\leqslant\widehat{M_r}$ for $t\in[0,1]$, it follows that

$$\|T_\lambda u\|_0\leqslant\frac{\lambda}{6^{m-1}}\int_0^1 G(s,s)a(s)f(u(s))\,\mathrm{d}s\leqslant\frac{\lambda\widehat{M_r}}{6^{m-1}}\int_0^1 G(s,s)a(s)\,\mathrm{d}s.$$

□

Lemma 2.6. *Let* (A1), (A2) *hold. Assume that* $\{(\mu_k,y_k)\}\subset(0,+\infty)\times K$ *is a sequence of positive solutions of* (1.1), (1.2). *Assume that* $|\mu_k|\leqslant C_0$ *for some constant* $C_0>0$, *and*

$$\lim_{k\to\infty}\|y_k\|=\infty.$$

Then

$$\lim_{k\to\infty}\|y_k\|_0=\infty.$$

P r o o f. From (1.1), (1.2), we have

$$\begin{aligned} y_k^{(2m-1)}(t)&=\int_\eta^t y_k^{(2m)}(s)\,\mathrm{d}s=\int_\eta^t \mu_k a(s)f(y_k(s))\,\mathrm{d}s\\ &\leqslant C_0\int_\eta^t a(s)f(y_k(s))\,\mathrm{d}s,\quad\forall\, t\in(0,1)\end{aligned} \tag{2.8}$$

for some $\eta\in(0,1)$ such that $y_k^{(2m-1)}(\eta)=0$. (2.8) implies that $\{\|y_k^{2m-1}\|_0\}$ is bounded whenever $\{\|y_k\|_0\}$ is bounded. This together with the boundary condition (1.2) implies that $\{\|y_k^{(j)}\|_0\}$ $(j=1,2,\ldots,2m-2)$ is bounded if $\{\|y_k\|_0\}$ is bounded.

□

3. Proof of the main results

Define $f^{[n]}(s)\colon [0,\infty) \to [0,\infty)$ by

$$f^{[n]}(s) = \begin{cases} f(s), & s > (\frac{1}{n}, \infty), \\ nf\left(\frac{1}{n}\right) s, & s \in \left[0, \frac{1}{n}\right]. \end{cases} \tag{3.1}$$

Then $f^{[n]} \in C([0,\infty),[0,\infty))$ with

$$f^{[n]}(s) > 0, \quad \forall\, s \in (0,\infty), \qquad \text{and } (f^{[n]})_0 = nf\left(\frac{1}{n}\right).$$

By (A3), it follows that

$$\lim_{n\to\infty} (f^{[n]})_0 = \infty.$$

To apply the Global Bifurcation Theorem, we extend f by an odd function $g\colon \mathbb{R} \to \mathbb{R}$ such that

$$g(s) = \begin{cases} f(s), & s \geqslant 0; \\ -f(-s), & s < 0. \end{cases} \tag{3.2}$$

Similarly we may extend $f^{[n]}$ to an odd function $g^{[n]}\colon \mathbb{R} \to \mathbb{R}$ for each $n \in \mathbb{N}$.

Now let us consider an auxiliary family of equations

$$(-1)^m u^{(2m)}(t) = \lambda a(t) g^{[n]}(u), \quad t \in (0,1), \tag{3.3}$$

$$u^{(2i)}(0) = u^{(2i)}(1) = 0, \qquad i = 0,1,2,\dots,m-1. \tag{3.4}$$

Let $\zeta \in C(R)$ be such that

$$g^{[n]}(u) = (g^{[n]})_0 u + \zeta(u) = nf\left(\frac{1}{n}\right)u + \zeta(u). \tag{3.5}$$

Note that

$$\lim_{|s|\to 0} \frac{\zeta(s)}{s} = 0. \tag{3.6}$$

Let us consider

$$Lu - \lambda a(t)(g^{[n]})_0 u = \lambda a(t)\zeta(u) \tag{3.7}$$

as a bifurcation problem for the trivial solution $u \equiv 0$.

Equation (3.7) can be converted to the equivalent equation

$$
\begin{aligned}
u(t) &= \int_0^1 G_m(t,s)[\lambda a(s)(g^{[n]})_0 u(s) + \lambda a(s)\zeta(u(s))]\,\mathrm{d}s \\
&:= (\lambda L^{-1}[a(\cdot)(g^{[n]})_0 u(\cdot)](t) + \lambda L^{-1}[a(\cdot)\zeta(u(\cdot))])(t).
\end{aligned} \tag{3.8}
$$

Further, we note that $\|L^{-1}[a(\cdot)\zeta(u(\cdot)]\| = o(\|u\|)$ for u near 0 in E.

The results of Rabinowitz [22] for (3.7) can be stated as follows: For each integer $n \geqslant 1$, $\nu \in \{+, -\}$, there exists a continuum $C_\nu^{[n]}$ of solutions of (3.7) joining $\big(\lambda_1/(g^{[n]})_0, 0\big)$ to infinity in νK. Moreover, $C_\nu^{[n]} \setminus \{(\lambda_1/(g^{[n]})_0, 0)\} \subset \nu\,(\operatorname{int} K)$.

P r o o f of Theorem 1.1. Let us verify that $\{C_+^{[n]}\}$ satisfies all of the conditions of Lemma 2.2.

Since

$$
\operatorname*{limit}_{n\to\infty} \frac{\lambda_1}{(g^{[n]})_0} = \operatorname*{limit}_{n\to\infty} \frac{\lambda_1}{n\,f(1/n)} = 0,
$$

Condition (i) in Lemma 2.4 is satisfied with $z^* = (0,0)$. Obviously

$$
r_n = \sup\{|\mu| + \|y\|_0\,;\ |\ (\mu, y) \in C_+^{[n]}\} = \infty,
$$

and accordingly, (ii) holds. (iii) can be deduced directly from the Arzela-Ascoli Theorem and the definition of $g^{[n]}$. Therefore, the limit superior of $\{C_+^{[n]}\}$, $\mathcal{D}$, contains an unbounded connected component $\mathcal{C}$ with $(0,0) \in \mathcal{C}$.

(a) $f_\infty = 0$.

In this case, we show that $\operatorname{Proj}_{\mathbb{R}} \mathcal{C} = [0, \infty)$.

Assume on the contrary that

$$
\sup\{\lambda;\ (\lambda, y) \in \mathcal{C}\} < \infty, \tag{3.9}
$$

then there exists a sequence $\{(\mu_k, y_k)\} \subset \mathcal{C}$ such that

$$
\lim_{k\to\infty} \|y_k\| = \infty, \quad |\mu_k| \leqslant C_0, \tag{3.10}
$$

for some positive constant C_0 independent of k. From Lemma 2.8, we have that

$$
\lim_{k\to\infty} \|y_k\|_0 = \infty, \tag{3.11}
$$

This together with the fact

$$
\min_{\sigma\leqslant t\leqslant 1-\sigma} y_k(t) \geqslant \Gamma \|y_k\|_0, \quad \forall\ 0 < \sigma < \min\{t_0, 1 - t_0\} \tag{3.12}
$$

implies that

$$\lim_{n\to\infty} y_k(t) = \infty \quad \text{uniformly for } t\in[\sigma, 1-\sigma]. \tag{3.13}$$

Since $(\mu_k, y_k) \in \mathcal{C}$, we have

$$(-1)^m y_k^{(2m)}(t) = \mu_k a(t) g^{[n]}(y_k(t)), \quad t\in(0,1), \tag{3.14}$$

$$y_k^{(2i)}(0) = y_k^{(2i)}(1) = 0, \quad i = 0,1,2,\dots,m-1. \tag{3.15}$$

Set $v_k(t) = y_k(t)/\|y_k\|_0$. Then

$$\|v_k\|_0 = 1. \tag{3.16}$$

Now, choosing a subsequence and relabelling if necessary, it follows that there exists $(\mu_*, v_*) \in [0, C_0] \times E$ with

$$\|v_*\|_0 = 1 \tag{3.17}$$

such that

$$\lim_{k\to\infty} (\mu_k, v_k) = (\mu_*, v_*) \quad \text{in } \mathbb{R}\times E. \tag{3.18}$$

Moreover, using (3.13), (3.14), (3.15) and the assumption $f_\infty = 0$, it follows that

$$(-1)^m v_*^{(2m)}(t) = \mu_* a(t)\cdot 0, \quad t\in(0,1), \tag{3.19}$$

$$v_*^{(2i)}(0) = v_*^{(2i)}(1) = 0, \quad i = 0,1,2,\dots,m-1, \tag{3.20}$$

and consequently, $v_*(t) \equiv 0$ for $t\in[0,1]$. This contradicts (3.17). Therefore

$$\sup\{\lambda; (\lambda, y)\in\mathcal{C}\} = \infty.$$

(b) $f_\infty \in (0,\infty)$.

In this case, we show that $\mathrm{Proj}_{\mathbb{R}}\,\mathcal{C} \subseteq [0, \lambda_1/f_\infty)$.

Let us rewrite (1.1), (1.2) to the form

$$(-1)^m u^{(2m)}(t) - \lambda a(t) g_\infty u - \lambda a(t)\xi(u(t)) = 0, \quad t\in(0,1), \tag{3.21}$$

$$u^{(2i)}(0) = u^{(2i)}(1) = 0, \quad i = 0,1,2,\dots,m-1, \tag{3.22}$$

where $\xi(s) = g(s) - g_\infty s$. Obviously

$$\lim_{|s|\to\infty} \frac{\xi(s)}{s} = 0. \tag{3.23}$$

Now by the same method as that used to prove [21, Theorem 1.1], we may prove that $\mathcal{C}$ joins $(0,0)$ with $(0,\lambda_1/f_\infty)$.

(c) $f_\infty=\infty$.

In this case, we show that $\mathcal{C}$ joins $(0,0)$ with $(0,\infty)$.

Let $\{(\mu_k,y_k)\}\subset\mathcal{C}$ be such that

$$|\mu_k|+\|y_k\|\to\infty,\qquad k\to\infty. \tag{3.24}$$

Then

$$(-1)^m y_k^{(2m)}(t)=\mu_k a(t)g(y_k(t)),\quad t\in(0,1), \tag{3.25}$$

$$y_k^{(2i)}(0)=y_k^{(2i)}(1)=0,\qquad i=0,1,2,\dots,m-1. \tag{3.26}$$

If $\{\|y_k\|\}$ is bounded, say, $\|y_k\|\leqslant M_1$ for some M_1 independent of k, then we may assume that

$$\lim_{k\to\infty}\mu_k=\infty.$$

Note that

$$\frac{g(y_k(t))}{y_k(t)}\geqslant\inf\left\{\frac{g(s)}{s};\ 0<s\leqslant M_1\right\}>0$$

and

$$(-1)^m y_k^{(2m)}(t)=\mu_k a(t)\frac{g(y_k(t))}{y_k(t)}y_k(t),\quad t\in(0,1). \tag{3.27}$$

The proof of Lemma 4 in [19] (see also the remarks in the final paragraph on p. 43 of [19]) shows that y_k must change its sign on $(0,1)$ if k is large enough. This is a contradiction. Hence $\{\|y_k\|\}$ is unbounded.

Now, taking $\{(\mu_k,y_k)\}\subset\mathcal{C}$ such that

$$\|y_k\|\to\infty\qquad\text{as } k\to\infty, \tag{3.28}$$

we show that $\lim\limits_{k\to\infty}\mu_k=0$.

Suppose on the contrary that, choosing a subsequence and relabelling if necessary, $\mu_k\geqslant a_0$ for some constant $a_0>0$. Then we have from (3.28)

$$\|y_k\|_0\to\infty\qquad\text{as } k\to\infty. \tag{3.29}$$

This together with (3.13) and condition (A1) implies that there exist constants α_1, β_1 with $\sigma<\alpha_1<\beta_1<1-\sigma$,

$$a(t)>0,\quad \lim_{k\to\infty}\mu_k\frac{g(y_k(t))}{y_k(t)}=\infty,\qquad\forall\, t\in[\alpha_1,\beta_1] \tag{3.30}$$

for every fixed constant σ with $0<\sigma<\min\{t_0,1-t_0\}$.

Thus, we have from (3.27) that y_k must change its sign on $[\sigma,1-\sigma]$ if k is large enough. This is a contradiction. Therefore $\lim\limits_{k\to\infty}\mu_k=0$. $\square$

P r o o f of Theorem 1.2. (a) and (b) are immediate consequences of Theorem 1.1 (a) and (b), respectively.

To prove (c), we rewrite (1.1), (1.2) to

$$u = \lambda \int_0^1 G_m(t,s)a(s)f(u(s))\,\mathrm{d}s =: T_\lambda u(t). \tag{3.31}$$

By Lemma 2.5, for every $r > 0$ and $u \in \partial\Omega_r$ we have

$$\|T_\lambda u\|_0 \leqslant \frac{\lambda \widehat{M}_r}{6^{m-1}} \int_0^1 G(s,s)a(s)\,\mathrm{d}s,$$

where $\widehat{M}_r = 1 + \max\limits_{0\leqslant s\leqslant r}\{f(s)\}$.

Let $\lambda_r > 0$ be such that

$$\frac{\lambda_r \widehat{M}_r}{6^{m-1}} \int_0^1 G(s,s)a(s)\,\mathrm{d}s = r.$$

Then for $\lambda \in (0,\lambda_r)$ and $u \in \partial\Omega_r$

$$\|T_\lambda u\|_0 < \|u\|_0.$$

This means that

$$\Sigma \cap \{(\lambda,u) \in (0,\infty)\times K;\ 0<\lambda<\lambda_r,\ u\in K\colon\ \|u\|_0 = r\} = \emptyset. \tag{3.32}$$

By Lemma 2.6 and Theorem 1.1, it follows that $\mathcal{C}$ is also an unbounded component joining $(0,0)$ and $(0,\infty)$ in $[0,\infty)\times Y$. Thus, (3.32) implies that for $\lambda \in (0,\lambda_r)$, problem (1.1), (1.2) has at least two positive solutions. $\square$

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