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GENERALIZED JORDAN DERIVATIONS ASSOCIATED WITH HOCHSCHILD 2-COCYCLES OF TRIANGULAR ALGEBRAS

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Abstract. In this paper, we investigate a new type of generalized derivations associated with Hochschild 2-cocycles which is introduced by A.Nakajima (Turk. J. Math. 30 (2006), 403–411). We show that if $\mathcal{U}$ is a triangular algebra, then every generalized Jordan derivation of above type from $\mathcal{U}$ into itself is a generalized derivation.

Keywords: generalized Jordan derivation, generalized derivation, Hochschild 2-cocycle, triangular algebra

MSC 2010: 47B47, 47L35

1. Introduction

Let $\mathcal{A}$ be an algebra and let $\mathcal{M}$ be an $\mathcal{A}$-bimodule. A linear (additive) mapping δ from $\mathcal{A}$ into $\mathcal{M}$ is said to be a linear (additive) *Jordan derivation* if $\delta(a^2) = \delta(a)a + a\delta(a)$ for all $a \in \mathcal{A}$. It is called a linear (additive) *derivation* if $\delta(ab) = \delta(a)b + a\delta(b)$ for all $a, b \in \mathcal{A}$. Each mapping of the form $a \to am - ma$, where $m \in \mathcal{M}$, will be called an inner derivation. Clearly, every derivation is Jordan derivation. The converse is false in general (see Benkovič [2]). Herstein [6] showed that each Jordan derivation from a 2-torsion free prime ring into itself is a derivation. Brešar [3] proved that Herstein's result is true for 2-torsion free semiprime rings. In [9], Zhang proved that every linear Jordan derivation on nest algebras is an inner derivation. In [7], Lu proved that every additive Jordan derivation on CSL algebras is an additive derivation.

Let $\mathcal{A}$ and $\mathcal{B}$ be unital algebra over a commutative ring $\mathcal{R}$, and $\mathcal{M}$ be a unital $(\mathcal{A}, \mathcal{B})$-bimodule, which is faithful as a left $\mathcal{A}$-module and also a right $\mathcal{B}$-module. The

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$\mathcal{R}$-algebra

$$\mathrm{Tri}(\mathcal{A},\mathcal{M},\mathcal{B}) = \left\{ \begin{pmatrix} a & m \\ 0 & b \end{pmatrix} : \ a \in \mathcal{A}, m \in \mathcal{M}, b \in \mathcal{B} \right\}$$

under the usual matrix operations will be called a *triangular algebra* [1]. It is clear that upper triangular matrix algebras and nontrivial nest algebras [4] are triangular algebras. Recently, Benkovič [2] showed that every Jordan derivation on an upper triangular matrix algebra into its bimodule is the sum of a derivation and an antiderivation. In [10], Zhang and Yu proved that every Jordan derivation of a triangular algebra is a derivation.

Recently, Nakajima introduced a new type of generalized derivation. Let $\mathcal{A}$ be an algebra and $\mathcal{M}$ be an $\mathcal{A}$-bimodule. Let $\alpha\colon \mathcal{A} \times \mathcal{A} \to \mathcal{M}$ be a bilinear (biadditive) mapping. α is called a *Hochschild* 2*-cocycle* if

$$x\alpha(y,z) - \alpha(xy,z) + \alpha(x,yz) - \alpha(x,y)z = 0. \tag{1}$$

A linear (additive) mapping $\delta\colon \mathcal{A} \to \mathcal{M}$ is called a linear(additive) *generalized derivation* if there is a 2-cocycle α such that

$$\delta(xy) = \delta(x)y + x\delta(y) + \alpha(x,y), \tag{2}$$

and δ is called a linear(additive) *generalized Jordan derivation* if

$$\delta(x^2) = \delta(x)x + x\delta(x) + \alpha(x,x). \tag{3}$$

We denote it by (δ,α). By the examples in [8], Nakajima showed that the *usual generalized derivations*, *left centralizers* and (σ,τ)*-derivations* are also generalized derivations in above sense.

In this paper, we generalize the result of [10] to generalized derivations of above type. We show that if $\mathcal{U}$ is a triangular algebra, then every additive generalized Jordan derivation from $\mathcal{U}$ into itself is an additive generalized derivation.

2. Main result

The following lemma, due to Nakajima [8], will be used repeatedly.

Lemma 2.1 [8, Lemma 2]**.** *Let $\mathcal{A}$ be an algebra and $\mathcal{M}$ be an $\mathcal{A}$-bimodule. If $(f,\alpha)\colon \mathcal{A} \to \mathcal{M}$ is a linear (additive) generalized Jordan derivations associated with Hochschild* 2*-cocycles α, then the following relations hold:*

(i) $f(xy+yx) = f(x)y + xf(y) + \alpha(x,y) + f(y)x + yf(x) + \alpha(y,x)$,
(ii) $f(xyx) = f(x)yx + xf(y)x + xyf(x) + x\alpha(y,x) + \alpha(x,yx)$,
(iii) $f(xyz + zyx) = f(x)yz + xf(y)z + xyf(z) + x\alpha(y,z) + \alpha(x,yz) + f(z)yx + zf(y)x + zyf(x) + z\alpha(y,x) + \alpha(z,yx)$.

Theorem 2.2. *Let $\mathcal{A},\mathcal{B}$ be unital algebras over a 2-torsion free commutative ring $\mathcal{R}$, and $\mathcal{M}$ be a unital $(\mathcal{A},\mathcal{B})$-bimodule that is faithful as left $\mathcal{A}$-module and also a right $\mathcal{B}$-module. Let $\mathcal{U} = \mathrm{Tri}(\mathcal{A},\mathcal{M},\mathcal{B})$ be the triangular algebra. If (δ,α) is an additive generalized Jordan derivation from $\mathcal{U}$ into $\mathcal{U}$, then (δ,α) is an additive generalized derivation.*

P r o o f. We write

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Since $\delta(P) = \delta(P^2) = \delta(P)P + P\delta(P) + \alpha(P,P)$, we have that

$$P\delta(P)P = -P\alpha(P,P)P, \quad Q\delta(P)Q = Q\alpha(P,P)Q.$$

So

$$\delta(P) = P\delta(P)Q + Q\alpha(P,P)Q - P\alpha(P,P)P.$$

For any $T \in \mathcal{U}$, by Lemma 2.1(i),

$$\begin{aligned} \delta(PTQ) &= \delta(PPTQ + PTQP) \\ &= \delta(P)PTQ + P\delta(PTQ) + \alpha(P,PTQ) \\ &\quad + \delta(PTQ)P + PTQ\delta(P) + \alpha(PTQ,P) \\ &= -\alpha(P,P)PTQ + P\delta(PTQ) + \alpha(P,PTQ) \\ &\quad + \delta(PTQ)P + PTQ\alpha(P,P) + \alpha(PTQ,P). \end{aligned}$$

Since

$$\begin{aligned} &PTQ\alpha(P,P) + \alpha(PTQ,P) - \alpha(PTQ,P)P = 0, \\ &P\alpha(P,PTQ) - \alpha(P,PTQ) + \alpha(P,PTQ) - \alpha(P,P)PTQ = 0, \end{aligned}$$

we have that

$$\delta(PTQ) = Q\alpha(P,PTQ) + P\delta(PTQ) + \delta(PTQ)P + \alpha(PTQ,P)P.$$

Thus

$$P\delta(PTQ)P = -P\alpha(PTQ,P)P, \quad Q\delta(PTQ)Q = Q\alpha(P,PTQ)Q.$$

So

$$\delta(PTQ) = P\delta(PTQ)Q + Q\alpha(P, PTQ)Q - P\alpha(PTQ, P)P.$$

By Lemma 2.1(ii),

$$\delta(PTP) = \delta(P)TP + P\delta(T)P + PT\delta(P) + P\alpha(T, P) + \alpha(P, TP).$$

So

$$Q\delta(PTP) = Q\alpha(P, PTP).$$

For any $S, T \in \mathcal{U}$,

$$\begin{aligned}
(4)\quad \delta(SPTQ) &= \delta(PSPPTQ + PTQPSP)\\
&= \delta(PSP)PTQ + PSP\delta(PTQ) + \alpha(PSP, PTQ) + \delta(PTQ)PSP\\
&\quad + PTQ\delta(PSP) + \alpha(PTQ, PSP)\\
&= (\delta(P)SP + P\delta(S)P + PS\delta(P) + P\alpha(S, P) + \alpha(P, SP))PTQ\\
&\quad + PSP\delta(PTQ) + \alpha(PSP, PTQ) + \delta(PTQ)PSP + PTQ\delta(PSP)\\
&\quad + \alpha(PTQ, PSP)\\
&= \delta(S)PTQ + S\delta(PTQ)\\
&\quad + (-\alpha(P, P)SPTQ + \alpha(P, SP)PTQ)\\
&\quad + (-S\alpha(P, P)PTQ + \alpha(S, P)PTQ - SQ\alpha(P, PTQ)Q + \alpha(SP, PTQ))\\
&\quad + (-\alpha(PTQ, P)PSP + PTQ\alpha(P, PTQ) + \alpha(PTQ, PSP)).
\end{aligned}$$

In the following, we reduce (4).

(a) Since

$$(P\alpha(P, SP) - \alpha(P, SP) + \alpha(P, PSP) - \alpha(P, P)SP)PTQ = 0,$$

it follows that

$$-\alpha(P, P)SPTQ + \alpha(P, SP)PTQ = 0.$$

(b) By

$$(SQ\alpha(P, PTQ) + \alpha(SQ, PTQ) - \alpha(SQ, P)PTQ)Q = 0$$

and

$$S\alpha(P, PTQ)P = SP\alpha(P, PTQ)P = 0,$$

it follows that

$$\begin{aligned}
&- S\alpha(P,P)PTQ + \alpha(S,P)PTQ - SQ\alpha(P,PTQ)Q + \alpha(SP,PTQ) \\
&\quad = -S\alpha(P,P)PTQ + \alpha(S,P)PTQ + \alpha(SQ,PTQ)Q - \alpha(SQ,P)PTQ \\
&\qquad + \alpha(SP,PTQ) \\
&\quad = -(S\alpha(P,P) - \alpha(SP,P))PTQ + \alpha(S,PTQ)Q - \alpha(SP,PTQ)Q \\
&\qquad + \alpha(SP,PTQ) \\
&\quad = \alpha(S,PTQ)Q + (S\alpha(P,PTQ) + \alpha(S,PTQ) - \alpha(S,P)PTQ)P \\
&\quad = \alpha(S,PTQ) + S\alpha(P,PTQ)P \\
&\quad = \alpha(S,PTQ).
\end{aligned}$$

(c) By

$$PTQ\alpha(P,PSP) - \alpha(PTQP,PSP) + \alpha(PTQ,PSP) - \alpha(PTQ,P)PSP = 0,$$

we have that

$$PTQ\alpha(P,PSP) + \alpha(PTQ,PSP) - \alpha(PTQ,P)PSP = 0.$$

From (a), (b), (c) and (4),

$$\delta(SPTQ) = \delta(S)PTQ + S\delta(PTQ) + \alpha(S,PTQ). \tag{5}$$

Similarly, we have that

$$\delta(PSQT) = \delta(PSQ)T + PSQ\delta(T) + \alpha(PSQ,T). \tag{6}$$

For any $A, B, T \in \mathcal{U}$, it follows from (5) and (6) that

$$\begin{aligned}
\delta(ABPTQ) &= \delta(AB)PTQ + AB\delta(PTQ) + \alpha(AB,PTQ), \\
\delta(ABPTQ) &= \delta(A)BPTQ + A\delta(BPTQ) + \alpha(A,BPTQ) \\
&= \delta(A)BPTQ + AB\delta(PTQ) + A\delta(B)PTQ \\
&\quad + A\alpha(B,PTQ) + \alpha(A,BPTQ).
\end{aligned}$$

So

$$\begin{aligned}
&\delta(AB)PTQ - \delta(A)BPTQ - A\delta(B)PTQ \\
&\qquad + \alpha(AB,PTQ) - \alpha(A,BPTQ) - A\alpha(B,PTQ) = 0.
\end{aligned}$$

Thus

$$(\delta(AB) - \delta(A)B - A\delta(B) - \alpha(A, B))PTQ = 0.$$

Since $P\mathcal{U}Q$ is faithful left $P\mathcal{U}P$-module, we have that

$$(\delta(AB) - \delta(A)B - A\delta(B) - \alpha(A, B))P = 0. \tag{7}$$

Similarly,

$$Q(\delta(AB) - \delta(A)B - A\delta(B) - \alpha(A, B)) = 0. \tag{8}$$

Define Δ by $\Delta(T) = \delta(T) - (T\delta(P) - \delta(P)T), T \in \mathcal{U}$. Thus (Δ, α) is also a generalized Jordan derivation. Since $\delta(P) = P\delta(P)Q + Q\alpha(P, P)Q - P\alpha(P, P)P$, we have that

$$\Delta(P) = \delta(P) - (P\delta(P) - \delta(P)P) = \delta(P) - P\delta(P)Q = Q\alpha(P, P)Q - P\alpha(P, P)P.$$

For any $T \in \mathcal{U}$, by Lemma 2.1(ii),

$$\begin{aligned}\Delta(TP) &= \Delta(PTP) \\ &= \Delta(P)TP + P\Delta(T)P + PT\Delta(P) + P\alpha(T, P) + \alpha(P, TP) \\ &= \Delta(T)P - \alpha(P, P)TP + PTQ\alpha(P, P)Q - T\alpha(P, P)P \\ &\quad + P\alpha(T, P) + \alpha(P, TP).\end{aligned}$$

So

$$\begin{aligned}\Delta(TP)Q &= PTQ\alpha(P, P)Q + P\alpha(T, P)Q + \alpha(P, TP)Q \\ &= (\alpha(PTQP, P) - \alpha(PTQ, P) + \alpha(PTQ, P)P)Q + \alpha(PT, P)Q \\ &= \alpha(PTP, P)Q = \alpha(TP, P)Q.\end{aligned} \tag{9}$$

Therefore, for any $A, B \in \mathcal{U}$,

$$\begin{aligned}&(\Delta(ABP) - \Delta(A)BP - A\Delta(BP) - \alpha(A, BP))Q \\ &\quad = \alpha(ABP, P)Q - A\alpha(BP, P)Q - \alpha(A, BP)Q \\ &\quad = -\alpha(A, BP)PQ = 0.\end{aligned} \tag{10}$$

By (7) and (10), we have that

$$\Delta(ABP) = \Delta(A)BP + A\Delta(BP) + \alpha(A, BP). \tag{11}$$

Since $\Delta(Q) = \Delta(Q^2) = \Delta(Q)Q + Q\Delta(Q) + \alpha(Q,Q)$, we have that $P\Delta(Q)P = P\alpha(Q,Q)P$ and $Q\Delta(Q)Q = -Q\alpha(Q,Q)Q$. Thus

$$\Delta(Q) = P\Delta(Q)Q + P\alpha(Q,Q)P - Q\alpha(Q,Q)Q.$$

By Lemma 2.1(ii),

$$\begin{aligned} P\Delta(QT) &= P\Delta(QTQ) \\ &= P(\Delta(Q)TQ + Q\Delta(T)Q + QT\Delta(Q) + Q\alpha(T,Q) + \alpha(Q,TQ)) \\ &= P\Delta(Q)TQ + P\alpha(Q,TQ). \end{aligned} \tag{12}$$

Therefore, for any $A, B \in \mathcal{U}$,

(13)

$$\begin{aligned} &P(\Delta(QAB) - \Delta(QA)B - QA\Delta(B) - \alpha(QA,B)) \\ &\quad = P\Delta(Q)ABQ + P\alpha(Q,ABQ) - P\Delta(Q)AQB - P\alpha(Q,AQ)B - P\alpha(QA,B) \\ &\quad = P\alpha(Q,Q)APBQ + P\alpha(Q,ABQ) - P\alpha(Q,AQ)B - P\alpha(QA,B) \\ &\quad = -P\alpha(Q,APBQ) + P\alpha(Q,ABQ) - P\alpha(Q,AQ)B - P\alpha(QA,B) \\ &\quad = P\alpha(Q,AQBQ) - P\alpha(Q,AQB) \\ &\quad = -P\alpha(Q,AQBP) = -P\alpha(Q,0) = 0. \end{aligned}$$

By (8) and (13), we have that

$$\Delta(QAB) = \Delta(QA)B + QA\Delta(B) + \alpha(QA,B). \tag{14}$$

Also, by (5) and (6),

$$\Delta(APBQ) = \Delta(A)PBQ + A\Delta(PBQ) + \alpha(A,PBQ), \tag{15}$$

$$\Delta(PAQB) = \Delta(PAQ)B + PAQ\Delta(B) + \alpha(PAQ,B). \tag{16}$$

Let $h(A,B) = \Delta(AB) - \Delta(A)B - A\Delta(B) - \alpha(A,B), A, B \in \mathcal{U}$. It follows from (11), (14), (15) and (16) that

$$h(A,BP) = h(QA,B) = h(A,PBQ) = h(PAQ,B) = 0.$$

Thus

$$h(A,PB) = h(AQ,B) = 0. \tag{17}$$

By (9), (12) and (17), we have that

$$h(A,B) = h(AP,QB) = \Delta(APQB) - \Delta(AP)QB - AP\Delta(QB) - \alpha(AP,QB)$$
$$= -\alpha(AP,P)QB - AP\Delta(Q)BQ - AP\alpha(Q,BQ) - \alpha(AP,QB).$$

Since

$$AP\alpha(Q,BQ) + \alpha(AP,QBQ) - \alpha(AP,Q)BQ = 0,$$

we have that

$$- AP\alpha(Q,BQ) - \alpha(AP,QBQ) - \alpha(AP,P)QB$$
$$= -\alpha(AP,Q)BQ - \alpha(AP,P)QBQ$$
$$= -AP\alpha(P,Q)BQ.$$

Thus

$$h(A,B) = h(AP,QB) = -AP\Delta(Q)BQ - AP\alpha(P,Q)BQ$$
$$= -A(P\Delta(Q) + P\alpha(P,Q))BQ.$$

Since $\Delta(I) = \delta(I) = -\alpha(I,I)$, we have that

$$\Delta(Q) = \Delta(I) - \Delta(P) = -\alpha(I,I) - Q\alpha(P,P)Q + P\alpha(P,P)P.$$

Thus $P\Delta(Q) = -P\alpha(I,I) + P\alpha(P,P)P$. Since $P\alpha(I,I) = PP\alpha(I,I) = P\alpha(P,I)$, it follows that

$$P\Delta(Q) = -P\alpha(P,I) + \alpha(P,P)P.$$

Thus

$$h(A,B) = -A(P\alpha(P,Q) - P\alpha(P,I) + \alpha(P,P)P)BQ$$
$$= A(P\alpha(P,P) - \alpha(P,P)P)BQ = 0.$$

Hence, (Δ,α) is a generalized derivation and (δ,α) is a generalized derivation. □

Let $\alpha = 0$, we can get the main result of Zhang [10].

Corollary 2.3 ([10, Theorem 2.1]). *Let $\mathcal{A},\mathcal{B}$ be unital algebras over a 2-torsion free commutative ring $\mathcal{R}$, and $\mathcal{M}$ be a unital $(\mathcal{A},\mathcal{B})$-bimodule that is faithful as left $\mathcal{A}$-module and also a right $\mathcal{B}$-module. Let $\mathcal{U} = \mathrm{Tri}(\mathcal{A},\mathcal{M},\mathcal{B})$ be the triangular algebra. Then every Jordan derivation from $\mathcal{U}$ into itself is a derivation.*

Remark 2.4. By Examples (1) in [8], we have that if δ is an additive generalized derivation, that is, there is an additive derivation $d\colon \mathcal{A} \to \mathcal{M}$ such that $\delta(xy) = \delta(x)y + xd(y)$, then the mapping $\alpha\colon \mathcal{A}\times\mathcal{A} \ni (x,y) \to x(d-\delta)(y) \in \mathcal{M}$ is biadditive and satisfies the 2-cocycle condition. Since $\delta(xy) = \delta(x)y + x\delta(y) + \alpha(x,y)$, it follows that a usual generalized derivation δ is a generalized derivation (δ,α).

By Theorem 2.2 and Remark 2.4, we can get the main result of Hou [5].

Corollary 2.5 ([5, Theorem 2.1]). *Let $\mathcal{L}$ be a nest on a Banach space X, and δ be an additive generalized Jordan derivation from* alg$\mathcal{L}$ *into itself. If there is a nontrivial element in $\mathcal{L}$ which is complemented in X, then δ is an additive generalized derivation.*

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