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Cong-hua Yan; Xiao-ke Wang

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INTUITIONISTIC I-FUZZY TOPOLOGICAL SPACES

Cong-hua Yan, Xiao-ke Wang, Nanjing

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Abstract. The main purpose of this paper is to introduce the concept of intuitionistic I-fuzzy quasi-coincident neighborhood systems of intuitiostic fuzzy points. The relation between the category of intuitionistic I-fuzzy topological spaces and the category of intuitionistic I-fuzzy quasi-coincident neighborhood spaces are studied. By using fuzzifying topology, the notion of generated intuitionistic I-fuzzy topology is proposed, and the connections among generated intuitionistic I-fuzzy topological spaces, fuzzifying topological spaces and I-fuzzy topological spaces are discussed. Finally, the properties of the operators $\mathrm{I}\omega$, ι are obtained.

Keywords: intuitionistic I-fuzzy topological space, intuitionistic fuzzy point, intuitionistic I-fuzzy quasi-coincident neighborhood space, fuzzifying topology, I-fuzzy topology

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1. Introduction

The idea of intuitionistic fuzzy sets was first proposed by Atanassov [1]. This is a generalization notion of fuzzy sets. From then on, this idea has been studied and applied in a variety areas. For example, D. Çoker [5] introduced the concept of intuitionistic fuzzy topological spaces. Park [16] defined the notion of intuitionistic fuzzy metric spaces. Xu and Yager [20] investigated the aggregation of intuitionistic fuzzy information, and developed some geometric aggregation operators. They also gave an application of these operators to multiple attribute decision making based on intuitionistic fuzzy sets, etc. Among of them, the research of the theory of intuitionistic fuzzy topology is similar to the the theory of fuzzy topology. In fact, the concept of intuitionistic fuzzy topological spaces given by Çoker [5] is originated from the fuzzy topology in the sense of Chang [4]. According to the standardized

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terminology in [10], it will be called intuitionistic I-topological spaces. Based on Çoker's work [5], many topological properties of intuitionistic I-topological spaces have been discussed ([8], [9], [13], [14]). On the other hand, Šostak [19] constructs a new notion of fuzzy topological spaces, and this new fuzzy topological structure has been accepted widely. Recently, Fang and Yue [11], [12] introduced the concept of I-fuzzy quasi-coincident neighborhood systems, and they proved the category of I-fuzzy topological spaces is isomorphic to the category of I-fuzzy quasi-coincident neighborhood spaces. Moreover, they used this notion to study the further properties of I-fuzzy topological spaces in [20].

Influenced by Šostak's work [19], Çoker [7] gave the notion of intuitionistic fuzzy topological spaces in the sense of Šostak. By the standardized terminology introduced in [10], we will call it intuitionistic I-fuzzy topological spaces in this paper. In [17], the authors studied the compactness in intuitionistic I-fuzzy topological spaces. The main purpose of this paper is to introduce the notion of intuitionistic I-fuzzy quasi-coincident neighborhood systems of intuitiostic fuzzy points. The relation between the category of intuitionistic I-fuzzy topological spaces and the category of intuitionistic I-fuzzy quasi-coincident neighborhood spaces are discussed. Then we construct the notion of generated intuitionistic I-fuzzy topology by using fuzzifying topologies, and the connections among generated intuitionistic I-fuzzy topological spaces, fuzzifying topological spaces and I-fuzzy topological spaces are studied. Finally, the properties of the operators $\mathrm{I}\omega$, ι are obtained.

Throughout this paper, $I = [0,1]$, $I_0 = (0,1]$, $I_1 = [0,1)$. Let X be a nonempty set; the family of all fuzzy sets and intuitionistic fuzzy sets on X are denoted by I^X and ζ^X, respectively. The notation $\mathrm{pt}(I^X)$ denotes the set of all fuzzy points on X. For all $\lambda \in I$, $\underline{\lambda}$ denotes the fuzzy set on X which takes the constant value λ. For each $A \in I^X$, the symbol $\underline{1} - A$ denotes the fuzzy set which value is $1 - A(x)$ for all $x \in X$. For each $A \subseteq X$, A^c denotes the complement of A with respect to X, and 1_A denotes the function $X \to I$, $1_A(x) = 1$ for all $x \in A$, otherwise its value is 0. For all $A \in \zeta^X$, let $A = \langle \mu_A, \gamma_A \rangle$. (For background on intuitionistic fuzzy sets, we refer to [1].)

2. Some preliminaries

Definition 2.1 ([21]). A fuzzifying topology on a set X is a function $\tau\colon 2^X \to I$, such that

(1) $\tau(\emptyset) = \tau(X) = 1$;

(2) $\forall A, B \subseteq X$, $\tau(A \wedge B) \geqslant \tau(A) \wedge \tau(B)$;

(3) $\forall\, A_t \subseteq X$, $t \in T$, $\tau\Big(\bigvee_{t\in T} A_t\Big) \geqslant \bigwedge_{t\in T} \tau(A_t)$.

The pair (X,τ) is called a fuzzifying topological space.

Definition 2.2 ([11], [12]). An I-fuzzy topology on a set X is a function $\eta\colon I^X \to I$ such that

(1) $\eta(\underline{0}) = \eta(\underline{1}) = 1$;

(2) $\forall A, B \in I^X$, $\eta(A \wedge B) \geqslant \eta(A) \wedge \eta(B)$;

(3) $\forall A_t \in I^X$, $t \in T$, $\eta\Big(\bigvee_{t\in T} A_t\Big) \geqslant \bigwedge_{t\in T} \eta(A_t)$.

If η is an I-fuzzy topology on X, then we say that (X,η) is an I-fuzzy topological space.

Lemma 2.3 ([22]). *Suppose that (X,τ) is a fuzzifying topological space, and for each $A \in I^X$, let $\omega(\tau)(A) = \bigwedge_{r\in I} \tau(\sigma_r(A))$, where $\sigma_r(A) = \{x\colon A(x) > r\}$. Then $\omega(\tau)$ is an I-fuzzy topology on X.*

Definition 2.4 ([9], [21], [22]). Let (X,τ_1), (Y,τ_2) be two fuzzifying topological spaces, and $f\colon X \to Y$ a mapping. Then f is called continuous if for all $A \subseteq Y, \tau_1(f^{\leftarrow}(A)) \geqslant \tau_2(A)$.

Definition 2.5 ([1], [2]). Let a, b be two real numbers in $[0,1]$ satisfying the inequality $a + b \leqslant 1$. Then the pair $\langle a, b\rangle$ is called an intuitionistic fuzzy pair.

Let $\langle a_1, b_1\rangle$, $\langle a_2, b_2\rangle$ be two intuitionistic fuzzy pairs, then we define

(1) $\langle a_1, b_1\rangle \leqslant \langle a_2, b_2\rangle$ if and only if $a_1 \leqslant a_2$ and $b_1 \geqslant b_2$;

(2) $\langle a_1, b_1\rangle = \langle a_2, b_2\rangle$ if and only if $a_1 = a_2$ and $b_1 = b_2$;

(3) if $\langle a_j, b_j\rangle_{j\in J}$ is a family of intuitionistic fuzzy pairs, then $\bigvee_{j\in J} \langle a_j, b_j\rangle = \Big\langle \bigvee_{j\in J} a_j, \bigwedge_{j\in J} b_j \Big\rangle$, and $\bigwedge_{j\in J} \langle a_j, b_j\rangle = \Big\langle \bigwedge_{j\in J} a_j, \bigvee_{j\in J} b_j \Big\rangle$;

(4) the complement of an intuitionistic fuzzy pair $\langle a, b\rangle$ is the intuitionistic fuzzy pair defined by $\overline{\langle a, b\rangle} = \langle b, a\rangle$.

In the following, for convenience, we will use the symbols $1^{\sim}$ and $0^{\sim}$ denote the intuitionistic fuzzy pairs $\langle 1, 0\rangle$ and $\langle 0, 1\rangle$. It is easy to find that the set of all intuitionistic fuzzy pairs with the above order forms a complete lattice, and $1^{\sim}$, $0^{\sim}$ are its top element and bottom element, respectively.

Definition 2.6 ([5]). Let X, Y be two nonempty sets and $f\colon X \to Y$ a function. If $B = \{\langle y, \mu_B(y), \gamma_B(y)\rangle\colon y \in Y\} \in \zeta^Y$, then the preimage of B under f, denoted by $f^{\leftarrow}(B)$, is the intuitionistic fuzzy set defined by

$$f^{\leftarrow}(B) = \{\langle x, f^{\leftarrow}(\mu_B)(x), f^{\leftarrow}(\gamma_B)(x)\rangle\colon x \in X\}.$$

Here $f^{\leftarrow}(\mu_B)(x) = \mu_B(f(x))$, $f^{\leftarrow}(\gamma_B)(x) = \gamma_B(f(x))$. (This notation is from [18].)

If $A = \{\langle x, \mu_A(x), \gamma_A(x)\rangle \colon x \in X\} \in \zeta^X$, then the image of A under f, denoted by $f^{\rightarrow}(A)$ is the intuitionistic fuzzy set defined by

$$f^{\rightarrow}(A) = \{\langle y, f^{\rightarrow}(\mu_A)(y), (\underline{1} - f^{\rightarrow}(\underline{1} - \gamma_A))(y)\rangle \colon y \in Y\},$$

where

$$f^{\rightarrow}(\mu_A)(y) = \begin{cases} \sup\limits_{x \in f^{\leftarrow}(y)} \mu_A(x), & \text{if } f^{\leftarrow}(y) \neq \emptyset, \\ 0, & \text{if } f^{\leftarrow}(y) = \emptyset, \end{cases}$$

$$\underline{1} - f^{\rightarrow}(\underline{1} - \gamma_A)(y) = \begin{cases} \inf\limits_{x \in f^{\leftarrow}(y)} \gamma_A(x), & \text{if } f^{\leftarrow}(y) \neq \emptyset, \\ 1, & \text{if } f^{\leftarrow}(y) = \emptyset. \end{cases}$$

Definition 2.7 ([3]). Let L be a complete lattice. L is called a completely distributive lattice (briefly CD lattice), if the following conditions hold:

(CD1) $\bigwedge\limits_{i \in \Gamma} \Big(\bigvee\limits_{j \in J_i} a_{i,j} \Big) = \bigvee\limits_{f \in \prod J_i} \Big(\bigwedge\limits_{i \in \Gamma} d_{i,f(i)} \Big)$;

(CD2) $\bigvee\limits_{i \in \Gamma} \Big(\bigwedge\limits_{j \in J_i} a_{i,j} \Big) = \bigwedge\limits_{f \in \prod J_i} \Big(\bigvee\limits_{i \in \Gamma} d_{i,f(i)} \Big)$,

where for each $i \in \Gamma$ and $j \in J_i$, $a_{i,j} \in L$, and $f \in \prod J_i$ means that f is a mapping $f \colon \Gamma \to \bigcup J_i$ such that $f(i) \in J_i$ for each $i \in I$.

3. Intuitionistic I-fuzzy topological spaces and intuitionistic I-fuzzy quasi-coincident neighborhood spaces

Definition 3.1 ([7]). Let X be a nonempty set, and $\delta \colon \zeta^X \to I \times I$ satisfy the following:

(1) $\delta(\langle \underline{0}, \underline{1}\rangle) = \delta(\langle \underline{1}, \underline{0}\rangle) = 1^{\sim}$;

(2) $\forall A, B \in \zeta^X$, $\delta(A \wedge B) \geqslant \delta(A) \bigwedge \delta(B)$;

(3) $\forall A_t \in \zeta^X$, $t \in T$, $\delta\Big(\bigvee\limits_{t \in T} A_t\Big) \geqslant \bigwedge\limits_{t \in T} \delta(A_t)$.

Then δ is called an intuitionistic I-fuzzy topology on X, and the pair (X, δ) is called an intuitionistic I-fuzzy topological space. For any $A \in \zeta^X$, we always suppose that $\delta(A) = \langle \mu_\delta(A), \gamma_\delta(A)\rangle$; the number $\mu_\delta(A)$ is called the openness degree of A, while $\gamma_\delta(A)$ is called the nonopenness degree of A. A fuzzy continuous mapping between two intuitionistic I-fuzzy topological spaces (ζ^X, δ_1) and (ζ^Y, δ_2) is a mapping $f \colon X \to Y$ such that $\delta_1(f^{\leftarrow}(A)) \geqslant \delta_2(A)$. The category of intuitionistic I-fuzzy topological spaces and fuzzy continuous mappings is denoted by II-**FTOP**.

Definition 3.2 ([6], [13], [14]). Let X be a nonempty set. An intuitionistic fuzzy point, denoted by $x_{(\alpha,\beta)}$, is an intuitionistic fuzzy set $A = \{\langle y, \mu_A(y), \gamma_A(y)\rangle \colon y \in X\}$, such that

$$\mu_A(y) = \begin{cases} \alpha & \text{if } y = x, \\ 0 & \text{if } y \neq x, \end{cases}$$

and

$$\gamma_A(y) = \begin{cases} \beta & \text{if } y = x, \\ 1 & \text{if } y \neq x. \end{cases}$$

Here $x \in X$ is a fixed point, and the constants $\alpha \in I_0$, $\beta \in I_1$ satisfy $\alpha + \beta \leqslant 1$. The set of all intuitionistic fuzzy points $x_{(\alpha,\beta)}$ is denoted by $\mathrm{pt}(\zeta^X)$.

Definition 3.3 ([14]). Let $x_{(\alpha,\beta)} \in \mathrm{pt}(\zeta^X)$ and $A, B \in \zeta^X$. We say that $x_{(\alpha,\beta)}$ quasi-coincides with A, or that $x_{(\alpha,\beta)}$ is quasi-coincident with A, denoted $x_{(\alpha,\beta)}\hat{q}A$, if $\mu_A(x) + \alpha > 1$ and $\gamma_A(x) + \beta < 1$. We say that A quasi-coincides with B at x, or that A is quasi-coincident with B at x, $A\hat{q}B$ at x, in short, if $\mu_A(x) + \mu_B(x) > 1$ and $\gamma_A(x) + \gamma_B(x) < 1$. We say that A quasi-coincides with B, or that A is quasi-coincident with B, if A is quasi-coincident with B at some point $x \in X$.

The relation"does not quasi-coincide with" or "is not quasi-coincident with" is denoted by $\neg\hat{q}$.

It is easily shown for $\forall\, x_{(\alpha,\beta)} \in \mathrm{pt}(\zeta^X)$ that $x_{(\alpha,\beta)}\hat{q}\langle\underline{1},\underline{0}\rangle$ and $x_{(\alpha,\beta)}\neg\hat{q}\langle\underline{0},\underline{1}\rangle$.

Definition 3.4. An intuitionistic I-fuzzy quasi-coincident neighborhood system of X is a set $Q = \{Q_{x_{(\alpha,\beta)}} \colon x_{(\alpha,\beta)} \in \mathrm{pt}(\zeta^X)\}$ of mappings $Q_{x_{(\alpha,\beta)}} \colon \zeta^X \to I \times I$ such that for all $U, V \in \zeta^X$

(1) $Q_{x_{(\alpha,\beta)}}(\langle\underline{1},\underline{0}\rangle) = 1^{\sim}$, $Q_{x_{(\alpha,\beta)}}(\langle\underline{0},\underline{1}\rangle) = 0^{\sim}$;
(2) $Q_{x_{(\alpha,\beta)}}(U) > 0 \Rightarrow x_{(\alpha,\beta)}\hat{q}U$;
(3) $Q_{x_{(\alpha,\beta)}}(U \wedge V) = Q_{x_{(\alpha,\beta)}}(U) \wedge Q_{x_{(\alpha,\beta)}}(V)$;
(4) $Q_{x_{(\alpha,\beta)}}(U) = \bigvee\limits_{x_{(\alpha,\beta)}\hat{q}V \leqslant U} \ \bigwedge\limits_{y_{(\varrho,\nu)}\hat{q}V} Q_{y_{(\varrho,\nu)}}(V)$.

A set X equipped with an intuitionistic I-fuzzy quasi-coincident neighborhood system $Q = \{Q_{x_{(\alpha,\beta)}} \colon x_{(\alpha,\beta)} \in \mathrm{pt}(\zeta^X)\}$, denoted by (X, Q), is called an intuitionistic I-fuzzy quasi-coincident neighborhood space. A fuzzy continuous mapping between two intuitionistic I-fuzzy quasi-coincident neighborhood spaces $(X, P) \to (Y, Q)$ is a map $f \colon X \to Y$ such that for all $x_{(\alpha,\beta)} \in \mathrm{pt}(\zeta^X)$, $U \in \zeta^Y$,

$$Q_{f^{\to}(x_{(\alpha,\beta)})}(U) \leqslant P_{x_{(\alpha,\beta)}}(f^{\leftarrow}(U)).$$

The category of all intuitionistic I-fuzzy quasi-coincident neighborhood spaces and their fuzzy continuous mappings is denoted by II-**FQN**.

Proposition 3.5. *Suppose that $\delta\colon \zeta^X \to I\times I$ is an intuitionistic I-fuzzy topology on X, and for all $x_{(\alpha,\beta)}\in \mathrm{pt}(\zeta^X)$, $U\in\zeta^X$, define*

$$Q^{\delta}_{x_{(\alpha,\beta)}}(U)=\begin{cases}\bigvee\limits_{x_{(\alpha,\beta)}\hat{q}V\leqslant U}\delta(V) & \text{if } x_{(\alpha,\beta)}\hat{q}U,\\ 0 & \text{if } x_{(\alpha,\beta)}\neg\hat{q}U.\end{cases}$$

Then the set $Q^{\delta}=\{Q^{\delta}_{x_{(\alpha,\beta)}}\colon x_{(\alpha,\beta)}\in\mathrm{pt}(\zeta^X)\}$ is an intuitionistic I-fuzzy quasi-coincident neighborhood system of X, called the intuitionistic I-fuzzy quasi-coincident neighborhood system determined by δ.

Proof. (1) Since for each $x_{(\alpha,\beta)}\in\mathrm{pt}(\zeta^X)$, $x_{(\alpha,\beta)}\hat{q}\langle\underline{1},\underline{0}\rangle$ and $x_{(\alpha,\beta)}\neg\hat{q}\langle\underline{0},\underline{1}\rangle$, then $Q^{\delta}_{x_{(\alpha,\beta)}}(\langle\underline{1},\underline{0}\rangle)\geqslant\delta(\langle\underline{1},\underline{0}\rangle)=1^{\sim}$, $Q^{\delta}_{x_{(\alpha,\beta)}}(\langle\underline{0},\underline{1}\rangle)=0^{\sim}$.

(2) Obvious.

(3) If $x_{(\alpha,\beta)}\neg\hat{q}(U\wedge V)$, we deduce that $x_{(\alpha,\beta)}\neg\hat{q}U$ or $x_{(\alpha,\beta)}\neg\hat{q}V$, hence

$$Q^{\delta}_{x_{(\alpha,\beta)}}(U\wedge V)=Q^{\delta}_{x_{(\alpha,\beta)}}(U)\wedge Q^{\delta}_{x_{(\alpha,\beta)}}(V)=0^{\sim}.$$

If $x_{(\alpha,\beta)}\hat{q}(U\wedge V)$, then $x_{(\alpha,\beta)}\hat{q}U$ and $x_{(\alpha,\beta)}\hat{q}V$. By the definition of $Q^{\delta}_{x_{(\alpha,\beta)}}$, for all $U,V\in\zeta^X$

$$U\leqslant V\Longrightarrow Q^{\delta}_{x_{(\alpha,\beta)}}(U)\leqslant Q^{\delta}_{x_{(\alpha,\beta)}}(V).$$

Thus, $Q^{\delta}_{x_{(\alpha,\beta)}}(U\wedge V)\leqslant Q^{\delta}_{x_{(\alpha,\beta)}}(U)\wedge Q^{\delta}_{x_{(\alpha,\beta)}}(V)$.

Suppose that $Q^{\delta}_{x_{(\alpha,\beta)}}(U)=\langle a_1,b_1\rangle$ and $Q^{\delta}_{x_{(\alpha,\beta)}}(V)=\langle a_2,b_2\rangle$. Then

$$Q^{\delta}_{x_{(\alpha,\beta)}}(U)\wedge Q^{\delta}_{x_{(\alpha,\beta)}}(V)=\langle a_1\wedge a_2,b_1\vee b_2\rangle.$$

For each $\varepsilon>0$, we have $(a_1\wedge a_2)-\varepsilon<a_1$, $(a_1\wedge a_2)-\varepsilon<a_2$, $(b_1\vee b_2)+\varepsilon>b_1$, $(b_1\vee b_2)+\varepsilon>b_2$. From the fact that $a_1=\bigvee\limits_{x_{(\alpha,\beta)}\hat{q}W\leqslant U}\mu_\delta(W)$, $a_2=\bigvee\limits_{x_{(\alpha,\beta)}\hat{q}W\leqslant V}\mu_\delta(W)$, there are $W_1,W_2\in\zeta^X$ with $x_{(\alpha,\beta)}\hat{q}W_1$, $x_{(\alpha,\beta)}\hat{q}W_2$ such that $(a_1\wedge a_2)-\varepsilon<\mu_\delta(W_1)$, $(a_1\wedge a_2)-\varepsilon<\mu_\delta(W_2)$. By Definition 3.1, we have $\mu_\delta(W_1\wedge W_2)\geqslant\mu_\delta(W_1)\wedge\mu_\delta(W_2)$, thus $(a_1\wedge a_2)-\varepsilon<\mu_\delta(W_1\wedge W_2)$. Using the same method, it follows that $(b_1\vee b_2)+\varepsilon>\gamma_\delta(W_3\wedge W_4)$. Put $W_5=W_1\wedge W_2\wedge W_3\wedge W_4$; clearly, $x_{(\alpha,\beta)}\hat{q}W_5\leqslant U\wedge V$ and $\mu_\delta(W_5)\geqslant\mu_\delta\bigl(W_1\wedge W_2\bigr)>(a_1\wedge a_2)-\varepsilon$, $\gamma_\delta(W_5)\leqslant\gamma_\delta(W_3\wedge W_4)<(b_1\vee b_2)+\varepsilon$. By the arbitrariness of ε, we have $a_1\wedge a_2\leqslant\mu_\delta(W_5)$, $b_1\vee b_2\geqslant\gamma_\delta(W_5)$. Thus

$$\begin{aligned}\langle a_1\wedge a_2,b_1\vee b_2\rangle&\leqslant\langle\mu_\delta(W_5),\gamma_\delta(W_5)\rangle=\delta(W_5)\\&\leqslant\bigvee_{x_{(\alpha,\beta)}\hat{q}W\leqslant U\wedge V}\delta(W)=Q^{\delta}_{x_{(\alpha,\beta)}}(U\wedge V).\end{aligned}$$

Hence $Q^{\delta}_{x_{(\alpha,\beta)}}\bigl(U\wedge V\bigr)=Q^{\delta}_{x_{(\alpha,\beta)}}(U)\wedge Q^{\delta}_{x_{(\alpha,\beta)}}(V)$.

(4) For any $V \in \zeta^X$ with $x_{(\alpha,\beta)}\hat{q}V \leqslant U$, we have $Q^{\delta}_{x_{(\alpha,\beta)}}(V) \geqslant \delta(V)$, so

$$\bigwedge_{y_{(\varrho,\nu)}\hat{q}V} Q^{\delta}_{x_{(\varrho,\nu)}}(V) \geqslant \delta(V).$$

Thus $\delta(V) \leqslant \bigwedge_{y_{(\varrho,\nu)}\hat{q}V} Q^{\delta}_{x_{(\varrho,\nu)}}(V) \leqslant Q^{\delta}_{x_{(\alpha,\beta)}}(V) \leqslant Q^{\delta}_{x_{(\alpha,\beta)}}(U)$, hence

$$Q^{\delta}_{x_{(\alpha,\beta)}}(U) = \bigvee_{x_{(\alpha,\beta)}\hat{q}V\leqslant U} \delta(V) \leqslant \bigvee_{x_{(\alpha,\beta)}\hat{q}V\leqslant U} \bigwedge_{y_{(\varrho,\nu)}\hat{q}V} Q^{\delta}_{x_{(\varrho,\nu)}}(V) \leqslant Q^{\delta}_{x_{(\alpha,\beta)}}(U).$$

Therefore $Q^{\delta}_{x_{(\alpha,\beta)}}(U) = \bigvee_{x_{(\alpha,\beta)}\hat{q}V\leqslant U} \bigwedge_{y_{(\varrho,\nu)}\hat{q}V} Q^{\delta}_{y_{(\varrho,\nu)}}(V)$. □

Lemma 3.6. *For each* $U \in \zeta^X$, $\delta(U) = \bigwedge_{x_{(\alpha,\beta)}\hat{q}U} Q^{\delta}_{x_{(\alpha,\beta)}}(U)$.

Proof. From the proof of Proposition 3.5 (4), $\delta(U) \leqslant \bigwedge_{x_{(\alpha,\beta)}\hat{q}U} Q^{\delta}_{x_{(\alpha,\beta)}}(U)$.

Thus the key of the proof is to prove that $\delta(U) \geqslant \bigwedge_{x_{(\alpha,\beta)}\hat{q}U} Q^{\delta}_{x_{(\alpha,\beta)}}(U)$.

In fact, by Definition 2.7 and from the fact that $I \times I$ is a CD lattice, we have

$$\begin{aligned}\bigwedge_{x_{(\alpha,\beta)}\hat{q}U} Q^{\delta}_{x_{(\alpha,\beta)}}(U) &= \bigwedge_{x_{(\alpha,\beta)}\hat{q}U} \bigvee_{x_{(\alpha,\beta)}\hat{q}V\leqslant U} \delta(V) \\ &= \bigvee_{f\in \prod\limits_{x_{(\alpha,\beta)}\hat{q}U} \mathcal{B}(x_{(\alpha,\beta)})} \bigwedge_{x_{(\alpha,\beta)}\hat{q}U} \delta(f(x_{(\alpha,\beta)})) \\ &\leqslant \bigvee_{f\in \prod\limits_{x_{(\alpha,\beta)}\hat{q}U} \mathcal{B}(x_{(\alpha,\beta)})} \delta\Big(\bigvee_{x_{(\alpha,\beta)}\hat{q}U} f(x_{(\alpha,\beta)})\Big) = \delta(U),\end{aligned}$$

where $\mathcal{B}(x_{(\alpha,\beta)}) = \{V \in \zeta^X\colon x_{(\alpha,\beta)}\hat{q}V \leqslant U\}$. The last equality is due to the fact that $\bigvee_{x_{(\alpha,\beta)}\hat{q}U} f(x_{(\alpha,\beta)}) = U$ for all $f \in \prod_{x_{(\alpha,\beta)}\hat{q}U} \mathcal{B}(x_{(\alpha,\beta)})$. □

From Lemma 3.6, we have

Proposition 3.7. *If* δ_1 *and* δ_2 *are two intuitionistic* I*-fuzzy topologies on* X *which determine the same intuitionistic* I*-fuzzy quasi-coincident neighborhood system of* X*, then* $\delta_1 = \delta_2$.

Theorem 3.8. *Suppose $f\colon (X,\delta_X)\to(Y,\delta_Y)$ is fuzzy continuous between intuitionistic I-fuzzy topological spaces. Then $f\colon (X,Q^{\delta_X})\to(Y,Q^{\delta_Y})$ is also fuzzy continuous with respect to the induced intuitionistic I-fuzzy quasi-coincident neighborhood systems.*

P r o o f. For each $U\in\zeta^Y$, suppose $f\colon (X,\delta_X)\to(Y,\delta_Y)$ is fuzzy continuous, then $\delta_X((f^{\leftarrow}(U))\geqslant\delta_Y(U)$. Notice that $x_{(\alpha,\beta)}\hat{q}f^{\leftarrow}(U)$ if and only if $f^{\rightarrow}(x_{(\alpha,\beta)})\hat{q}U$ for all $x_{(\alpha,\beta)}\in\mathrm{pt}(\zeta^X)$.

If $f^{\rightarrow}(x_{(\alpha,\beta)})\neg\hat{q}U$, then $Q^{\delta_Y}_{f^{\rightarrow}(x_{(\alpha,\beta)})}(U)=Q^{\delta_X}_{x_{(\alpha,\beta)}}(f^{\leftarrow}(U))=0$.

If $f^{\rightarrow}(x_{(\alpha,\beta)})\hat{q}U$, then

$$\begin{aligned}Q^{\delta_Y}_{f^{\rightarrow}(x_{(\alpha,\beta)})}(U)&=\bigvee_{f^{\rightarrow}(x_{(\alpha,\beta)})\hat{q}V\leqslant U}\delta_Y(V)\leqslant\bigvee_{x_{(\alpha,\beta)}\hat{q}f^{\leftarrow}(V)\leqslant f^{\leftarrow}(U)}\delta_Y(V)\\&\leqslant\bigvee_{x_{(\alpha,\beta)}\hat{q}f^{\leftarrow}(V)\leqslant f^{\leftarrow}(U)}\delta_X(f^{\leftarrow}(V))\leqslant\bigvee_{x_{(\alpha,\beta)}\hat{q}W\leqslant f^{\leftarrow}(U)}\delta_X(W)\\&=Q^{\delta_X}_{x_{(\alpha,\beta)}}(f^{\leftarrow}(U)).\end{aligned}$$

□

On the other hand, let $Q=\{Q_{x_{(\alpha,\beta)}}\colon x_{(\alpha,\beta)}\in\mathrm{pt}(\zeta^X)\}$ be an intuitionistic I-fuzzy quasi-coincident neighborhood system on X. Define the map $\delta^Q\colon\zeta^X\to I\times I$ as follows: for any $U\in\zeta^X$,

$$\delta^Q(U)=\bigwedge_{x_{(\alpha,\beta)}\hat{q}U}Q_{x_{(\alpha,\beta)}}(U).$$

Then we have

Theorem 3.9.

(i) *δ^Q defined as above is an intuitionistic I-fuzzy topology on X, and its quasi-coincident neighborhood system is just Q, called the intuitionistic I-fuzzy topology induced by $Q=\{Q_{x_{(\alpha,\beta)}}\colon x_{(\alpha,\beta)}\in\mathrm{pt}(\zeta^X)\}$.*

(ii) *Suppose that $P=\{P_{x_{(\alpha,\beta)}}\colon x_{(\alpha,\beta)}\in\mathrm{pt}(\zeta^X)\}$ has the same properties as Q, and they induce the same intuitionistic I-fuzzy topology on X. Then $P=Q$.*

P r o o f. (i) First we check δ^Q satisfies (1)–(3) of Definition 3.1.

(1) $\delta^Q(\langle\underline{1},\underline{0}\rangle)=\bigwedge_{x_{(\alpha,\beta)}\hat{q}\langle\underline{1},\underline{0}\rangle}Q_{x_{(\alpha,\beta)}}(\langle\underline{1},\underline{0}\rangle)=1^{\sim}$, and notice that for each $x_{(\alpha,\beta)}\in\mathrm{pt}(\zeta^X)$, $x_{(\alpha,\beta)}\neg\hat{q}\langle\underline{0},\underline{1}\rangle$, so $\delta^Q(\langle\underline{0},\underline{1}\rangle)=\bigwedge\emptyset=1^{\sim}$.

(2) For all $U, V \in \zeta^X$,

$$\delta^Q(U \wedge V) = \bigwedge_{x_{(\alpha,\beta)}\hat{q}(U\wedge V)} Q_{x_{(\alpha,\beta)}}(U \wedge V) = \bigwedge_{x_{(\alpha,\beta)}\hat{q}(U\wedge V)} [Q_{x_{(\alpha,\beta)}}(U) \wedge Q_{x_{(\alpha,\beta)}}(V)]$$
$$= \Big(\bigwedge_{x_{(\alpha,\beta)}\hat{q}(U\wedge V)} Q_{x_{(\alpha,\beta)}}(U)\Big) \wedge \Big(\bigwedge_{x_{(\alpha,\beta)}\hat{q}(U\wedge V)} Q_{x_{(\alpha,\beta)}}(V)\Big)$$
$$\geqslant \Big(\bigwedge_{x_{(\alpha,\beta)}\hat{q}U} Q_{x_{(\alpha,\beta)}}(U)\Big) \wedge \Big(\bigwedge_{x_{(\alpha,\beta)}\hat{q}V} Q_{x_{(\alpha,\beta)}}(V)\Big) = \delta^Q(U) \wedge \delta^Q(V).$$

(3) For all $U_j \in \zeta^X$, $j \in J$,

$$\delta^Q\Big(\bigvee_{j\in J} U_j\Big) = \bigwedge_{x_{(\alpha,\beta)}\hat{q}(\bigvee_{j\in J} U_j)} Q_{x_{(\alpha,\beta)}}\Big(\bigvee_{j\in J} U_j\Big) = \bigwedge_{j\in J} \bigwedge_{x_{(\alpha,\beta)}\hat{q}U_j} Q_{x_{(\alpha,\beta)}}\Big(\bigvee_{j\in J} U_j\Big)$$
$$\geqslant \bigwedge_{j\in J} \bigwedge_{x_{(\alpha,\beta)}\hat{q}(U_j)} (Q_{x_{(\alpha,\beta)}(U_j)} = \bigwedge_{j\in J} \delta^Q(U_j).$$

So δ^Q is an intuitionistic I-fuzzy topology on X.

(ii) Let $P = \{P_{x_{(\alpha,\beta)}} \colon x_{(\alpha,\beta)} \in \mathrm{pt}(\zeta^X)\}$ be the quasi-coincident neighborhood system determined by δ^Q. For any $x_{(\alpha,\beta)} \in \mathrm{pt}(\zeta^X)$, $U \in \zeta^X$, if $x_{(\alpha,\beta)}\hat{q}U$, we have $P_{x_{(\alpha,\beta)}}(U) = \bigvee_{x_{(\alpha,\beta)}\hat{q}V\leqslant U} \delta^Q(V) = \bigvee_{x_{(\alpha,\beta)}\hat{q}V\leqslant U} \bigwedge_{y_{(\varrho,\nu)}\hat{q}V} Q_{y_{(\varrho,\nu)}}(V) = Q_{x_{(\alpha,\beta)}}(U)$. Moreover, from the definition of $P_{x_{(\alpha,\beta)}}(U)$ we know that when $x_{(\alpha,\beta)}\neg\hat{q}U$, $P_{x_{(\alpha,\beta)}}(U) = 0^\sim$, thus $P_{x_{(\alpha,\beta)}}(U) = Q_{x_{(\alpha,\beta)}}(U)$. Therefore $Q = P$. □

Theorem 3.10. *Suppose that $P = \{P_{x_{(\alpha,\beta)}} \colon x_{(\alpha,\beta)} \in \mathrm{pt}(\zeta^X)\}$ and $Q = \{Q_{y_{(\alpha,\beta)}} \colon y_{(\alpha,\beta)} \in \mathrm{pt}(\zeta^Y)\}$ are intuitionistic I-fuzzy quasi-coincident neighborhood systems on X and Y, respectively. If $f \colon (X, P) \to (Y, Q)$ is fuzzy continuous, then $f \colon (X, \delta^P) \to (Y, \delta^Q)$ is also fuzzy continuous, where δ^P is the intuitionistic I-fuzzy topology on X induced by P and δ^Q is the intuitionistic I-fuzzy topology on Y induced by Q, respectively.*

Proof. Notice that for all $U \in \zeta^Y$, $x_{(\alpha,\beta)} \in \mathrm{pt}(\zeta^X)$, $\{y_{(\alpha,\beta)} \in \mathrm{pt}(\zeta^Y) \colon y_{(\alpha,\beta)}\hat{q}U\} \supseteq \{f(x)_{(\alpha,\beta)} \in \mathrm{pt}(\zeta^Y) \colon x_{(\alpha,\beta)} \in \mathrm{pt}(\zeta^X)$ and $f(x)_{(\alpha,\beta)}\hat{q}U\}$. So for each $U \in \zeta^Y$

$$\delta^Q(U) = \bigwedge_{y_{(\eta,\nu)}\hat{q}U} Q_{y_{(\eta,\nu)}}(U) \leqslant \bigwedge_{f(x)_{(\alpha,\beta)}\hat{q}U} Q_{f(x)_{(\alpha,\beta)}}(U)$$
$$\leqslant \bigwedge_{x_{(\alpha,\beta)}\hat{q}f^{\leftarrow}(U)} P_{x_{(\alpha,\beta)}}(f^{\leftarrow}(U)) = \delta^P(f^{\leftarrow}(U)).$$

This means that $f \colon (X, \delta^P) \to (Y, \delta^Q)$ is fuzzy continuous. □

By the above discussions, we easily obtain the following theorem.

Theorem 3.11. *The category* II-**FTOP** *is isomorphic to the category* II-**FQN**.

4. The generated intuitionistic I-fuzzy topology

Let (X,τ) be a fuzzifying topological space. By Lemma 2.3, $\omega(\tau)$ is an I-fuzzy topology on X. For each $A \in \zeta^X$, let $\mathrm{I}\omega(\tau)(A) = \langle \mu^\tau(A), \gamma^\tau(A)\rangle$, where $\mu^\tau(A) = \omega(\tau)(\mu_A) \wedge \omega(\tau)(\underline{1} - \gamma_A)$, $\gamma^\tau(A) = 1 - \mu^\tau(A)$. Then we have

Lemma 4.1. *Let (X,τ) be a fuzzifying topological space, then $\mathrm{I}\omega(\tau)$ is an intuitionistic I-fuzzy topology on X.*

Proof. (1) $\mu^\tau(\langle \underline{0}, \underline{1}\rangle) = \omega(\tau)(\underline{0}) \wedge \omega(\tau)(\underline{0}) = 1$, thus $\mathrm{I}\omega(\tau)(\langle \underline{0}, \underline{1}\rangle) = \langle 1, 0\rangle = 1^\sim$. Moreover $\mu^\tau(\langle \underline{1}, \underline{0}\rangle) = \omega(\tau)(\underline{1}) \wedge \omega(\tau)(\underline{1}) = 1$, and clearly, $\mathrm{I}\omega(\tau)\langle \underline{1}, \underline{0}\rangle = \langle 1, 0\rangle = 1^\sim$.

(2) For all $A, B \in \zeta^X$, $\mathrm{I}\omega(\tau)\big(A \wedge B\big) = \big\langle \mu^\tau\big(A \wedge B\big), \gamma^\tau\big(A \wedge B)\big\rangle$. Now

$$\begin{aligned}
\mu^\tau(A \wedge B) &= \mu^\tau(\langle \mu_A \wedge \mu_B, \gamma_A \vee \gamma_B\rangle)\\
&= \omega(\tau)(\mu_A \wedge \mu_B) \wedge \omega(\tau)(\underline{1} - (\gamma_A \vee \gamma_B))\\
&= \omega(\tau)(\mu_A \wedge \mu_B) \wedge \omega(\tau)((\underline{1} - \gamma_A) \wedge (\underline{1} - \gamma_B))\\
&\geqslant \omega(\tau)(\mu_A) \wedge \omega(\tau)(\mu_B) \wedge \omega(\tau)(\underline{1} - \gamma_A) \wedge \omega(\tau)(\underline{1} - \gamma_B)\\
&= \mu^\tau(A) \wedge \mu^\tau(B),\\
\gamma^\tau(A \wedge B) &= 1 - \mu^\tau(A \wedge B) \leqslant 1 - (\mu^\tau(A) \wedge \mu^\tau(B))\\
&= (1 - \mu^\tau(A)) \vee (1 - \mu^\tau(B))\\
&= \gamma^\tau(A) \vee \gamma^\tau(B).
\end{aligned}$$

Thus,

$$\begin{aligned}
&\mathrm{I}\omega(\tau)(A) \wedge \mathrm{I}\omega(\tau)(B)\\
&\qquad = \langle \mu^\tau(A), \gamma^\tau(A)\rangle \wedge \langle \mu^\tau(B), \gamma^\tau(B)\rangle\\
&\qquad = \langle \mu^\tau(A) \wedge \mu^\tau(B), \gamma^\tau(A) \vee \gamma^\tau(B)\rangle\\
&\qquad \leqslant \langle \mu^\tau(A \wedge B), \gamma^\tau(A \wedge B)\rangle\\
&\qquad = \mathrm{I}\omega(\tau)(A \wedge B).
\end{aligned}$$

(iii) For all $A_t \in \zeta^X$, $t \in T$, $\mathrm{I}\omega(\tau)\Big(\bigvee_{t\in T} A_t\Big) = \Big(\Big\langle \mu^\tau\Big(\bigvee_{t\in T} A_t\Big), \gamma^\tau\Big(\bigvee_{t\in T} A_t\Big)\Big\rangle$. In addition,

$$\begin{aligned}
\mu^\tau\Big(\bigvee_{t\in T} A_t\Big) &= \omega(\tau)\Big(\bigvee_{t\in T} \mu_{A_t}\Big) \wedge \omega(\tau)\Big(\underline{1} - \bigwedge_{t\in T} \gamma_{A_t}\Big) \\
&= \omega(\tau)\Big(\bigvee_{t\in T} \mu_{A_t}\Big) \wedge \omega(\tau)\Big(\bigvee_{t\in T} (\underline{1} - \gamma_{A_t})\Big) \\
&\geqslant \Big(\bigwedge_{t\in T} \omega(\tau)(\mu_{A_t})\Big) \bigwedge \Big(\bigwedge_{t\in T} \omega(\tau)(\underline{1} - \gamma_{A_t})\Big) \\
&= \bigwedge_{t\in T} (\omega(\tau)(\mu_{A_t}) \wedge \omega(\tau)(\underline{1} - \gamma_{A_t})) = \bigwedge_{t\in T} \mu^\tau(A_t), \\
\gamma^\tau\Big(\bigvee_{t\in T} A_t\Big) &= 1 - \mu^\tau\Big(\bigvee_{t\in T} A_t\Big) \leqslant 1 - \bigwedge_{t\in T} \mu^\tau(A_t) \\
&= \bigvee_{t\in T} (1 - \mu^\tau(A_t)) = \bigvee_{t\in T} \gamma^\tau(A_t).
\end{aligned}$$

Hence

$$\begin{aligned}
\bigwedge_{t\in T} \mathrm{I}\omega(\tau)(A_t) &= \bigwedge_{t\in T} \Big\langle \mu^\tau(A_t), \gamma^\tau(A_t)\Big\rangle = \Big\langle \bigwedge_{t\in T} \mu^\tau(A_t), \bigvee_{t\in T} \gamma^\tau(A_t)\Big\rangle \\
&\leqslant \Big\langle \mu^\tau\Big(\bigvee_{t\in T} A_t\Big), \gamma^\tau\Big(\bigvee_{t\in T} A_t\Big)\Big\rangle = \mathrm{I}\omega(\tau)\Big(\bigvee_{t\in T} A_t\Big).
\end{aligned}$$

Therefore, $\mathrm{I}\omega(\tau)$ is an intuitionistic I-fuzzy topology on X. □

Definition 4.2. By Lemma 4.1, we know $\mathrm{I}\omega(\tau)$ is an intuitionistic I-fuzzy topology on X. We say that $(\zeta^X, \mathrm{I}\omega(\tau))$ is the intuitionistic I-fuzzy topological space generated by fuzzifying topological space (X, τ).

Lemma 4.3. *Let (X, τ) be a fuzzifying topological space, then*
(1) $\forall\, A \subseteq X$, $\mu^\tau(\langle 1_A, 1_{A^c}\rangle) = \tau(A)$.
(2) $\forall\, A = \langle \underline{\alpha}, \underline{\beta}\rangle \in \zeta^X$, $\mathrm{I}\omega(\tau)(A) = 1^\sim$.

P r o o f. (1) For each $r \in I_0$, $A \subseteq X$, $\sigma_r(1_A) = A$, so

$$\begin{aligned}
\mu^\tau(\langle 1_A, 1_{A^c}\rangle) &= \omega(\tau)(1_A) \wedge \omega(\tau)(\underline{1} - 1_{A^c}) = \omega(\tau)(1_A) \\
&= \bigwedge_{r\in I} \tau(\sigma_r(1_A)) = \tau(A).
\end{aligned}$$

(2) For each constant map $\underline{\lambda} \in I^X$, $\lambda \in I$, we have

$$\omega(\tau)(\underline{\lambda}) = \bigwedge_{r\in I} \tau(\sigma_r(\underline{\lambda})) = \bigwedge_{r<\underline{\lambda}} \tau(\sigma_r(\underline{\lambda})) \wedge \bigwedge_{r\geqslant\lambda} \tau(\sigma_r(\underline{\lambda}))$$
$$= \tau(X) \wedge \tau(\emptyset) = 1.$$

Thus $\mu^\tau(\langle\underline{\alpha},\underline{\beta}\rangle) = \omega(\tau)(\underline{\alpha}) \wedge \omega(\tau)(\underline{1-\beta}) = 1$. Hence, $\mathrm{I}\omega(\tau)(\langle\underline{\alpha},\underline{\beta}\rangle) = \langle 1, 0\rangle = 1^\sim$.

□

Definition 4.4. Let $f\colon (\zeta^X, \delta_1) \to (\zeta^Y, \delta_2)$. If for each $A \in \zeta^Y$, $\delta_2(A) > 0$, implies the inequality $\delta_1(f^\leftarrow(A)) > 0$, then f is called a weak fuzzy continuous mapping from (ζ^X, δ_1) to (ζ^Y, δ_2).

Theorem 4.5. *Let (X,τ_1), (Y,τ_2) be two fuzzifying topological spaces. Then $f\colon (\zeta^X, \mathrm{I}\omega(\tau_1)) \to (\zeta^Y, \mathrm{I}\omega(\tau_2))$ is fuzzy continuous if and only if $f\colon (X,\tau_1) \to (Y,\tau_2)$ is continuous.*

Proof. *Necessity.* For each $B \subseteq Y$, by Lemma 4.3, we have

$$\tau_1(f^\leftarrow(B)) = \mu_1^\tau(\langle 1_{f^\leftarrow(B)}, 1_{(f^\leftarrow(B))^c}\rangle).$$

Since $f\colon (\zeta^X, \mathrm{I}\omega(\tau_1)) \to (\zeta^Y, \mathrm{I}\omega(\tau_2))$ is fuzzy continuous, we have

$$\mathrm{I}\omega(\tau_1)(\langle f^\leftarrow(1_B), f^\leftarrow(1_{B^c})\rangle) \geqslant \mathrm{I}\omega(\tau_2)(\langle 1_B, 1_{B^c}\rangle).$$

Thus

$$\mu^{\tau_1}(\langle 1_{f^\leftarrow(B)}, 1_{(f^\leftarrow(B))^c}\rangle) \geqslant \mu^{\tau_2}(\langle 1_B, 1_{B^c}\rangle) = \tau_2(B).$$

This implies that $\tau_1(f^\leftarrow(B)) \geqslant \tau_2(B)$. Thus $f\colon (X,\tau_1) \to (Y,\tau_2)$ is continuous.

Sufficiency. For all $B \in \zeta^Y$, $\mathrm{I}\omega(\tau_1)(f^\leftarrow(B)) = \langle \mu^{\tau_1}(f^\leftarrow(B)), \gamma^{\tau_1}(f^\leftarrow(B))\rangle$. Now

$$\mu^{\tau_1}(f^\leftarrow(B)) = \bigwedge_{r\in I} \tau_1(\sigma_r(f^\leftarrow(\mu_B))) \wedge \bigwedge_{r\in I} \tau_1(\sigma_r(\underline{1} - f^\leftarrow(\gamma_B)))$$
$$= \bigwedge_{r\in I} \tau_1(f^\leftarrow(\sigma_r(\mu_B))) \wedge \bigwedge_{r\in I} \tau_1(f^\leftarrow(\sigma_r(\underline{1} - \gamma_B))).$$

If $f\colon (X,\tau_1) \to (Y,\tau_2)$ is continuous, then

$$\forall r \in I,\ \tau_1(f^\leftarrow(\sigma_r(\mu_B))) \geqslant \tau_2(\sigma_r(\mu_B)),\ \ \tau_1(f^\leftarrow(\sigma_r(\underline{1} - \gamma_B))) \geqslant \tau_2(\sigma_r(\underline{1} - \gamma_B)).$$

Thus

$$\bigwedge_{r\in I} \tau_1(f^\leftarrow(\sigma_r(\mu_B))) \geqslant \bigwedge_{r\in I} \tau_2(\sigma_r(\mu_B)) = \omega(\tau_2)(\mu_B), \text{ and}$$
$$\bigwedge_{r\in I} \tau_1(f^\leftarrow(\sigma_r(\underline{1} - \gamma_B))) \geqslant \bigwedge_{r\in I} \tau_2(\sigma_r(\underline{1} - \gamma_B)) = \omega(\tau_2)(\underline{1} - \gamma_B).$$

Hence $\mu^{\tau_1}(f^{\leftarrow}(B)) \geqslant \omega(\tau_2)(\mu_B) \wedge \omega(\tau_2)(\underline{1} - \gamma_B) = \mu^{\tau_2}(B)$ and $\gamma^{\tau_1}(f^{\leftarrow}(B)) \leqslant \gamma^{\tau_2}(B)$. So $\mathrm{I}\omega(\tau_1)(< f^{\leftarrow}(B)) \geqslant \mathrm{I}\omega(\tau_2)(B)$.

This means that $f\colon (\zeta^X, \mathrm{I}\omega(\tau_1)) \to (\zeta^Y, \mathrm{I}\omega(\tau_2))$ is fuzzy continuous. $\square$

Theorem 4.6. *$f\colon (\zeta^X, \mathrm{I}\omega(\tau_1)) \to (\zeta^Y, \mathrm{I}\omega(\tau_2))$ is weak fuzzy continuous if and only if $f\colon (I^X, \omega(\tau_1)) \to (I^Y, \omega(\tau_2))$ is weak fuzzy continuous.*

P r o o f. *Necessity.* For each $A \in I^Y$ with $\omega(\tau_2)(A) > 0$, from the fact that

$$\mathrm{I}\omega(\tau_2)(\langle A, \underline{1} - A\rangle) = \langle \mu^{\tau_2}(\langle A, \underline{1} - A\rangle), \gamma^{\tau_2}(\langle A, \underline{1} - A\rangle)\rangle,$$

we have $\mu^{\tau_2}(\langle A, \underline{1} - A\rangle) = \omega(\tau_2)(A) > 0$. Thus $\mathrm{I}\omega(\tau_2)(\langle A, \underline{1} - A\rangle) > 0^{\sim}$. Since $f\colon (\zeta^X \mathrm{I}\omega(\tau_1)) \to (\zeta^Y, \mathrm{I}\omega(\tau_2))$ is weak fuzzy continuous, it follows that $\mathrm{I}\omega(\tau_1)(\langle f^{\leftarrow}(A), f^{\leftarrow}(\underline{1} - A)\rangle) > 0^{\sim}$. Thus

$$\omega(\tau_1)(f^{\leftarrow}(A)) = \mu^{\tau_1}(\langle f^{\leftarrow}(A), f^{\leftarrow}(\underline{1} - A)\rangle) > 0.$$

So $f\colon (I^X, \omega(\tau_1)) \to (I^Y, \omega(\tau_2))$ is weak fuzzy continuous.

Sufficiency. For each $\langle \mu_B, \gamma_B\rangle \in \zeta^Y$ with $\mathrm{I}\omega(\tau_2)(\langle \mu_B, \gamma_B\rangle) > 0^{\sim}$, we have

$$\mu^{\tau_2}(\langle \mu_B, \gamma_B\rangle) = \omega(\tau_2)(\mu_B) \wedge \omega(\tau_2)(\underline{1} - \gamma_B) > 0.$$

Thus $\omega(\tau_2)(\mu_B) > 0$, $\omega(\tau_2)(\underline{1} - \gamma_B) > 0$. Since $f\colon (I^X, \omega(\tau_1)) \to (I^Y, \omega(\tau_2))$ is weak fuzzy continuous, we have $\omega(\tau_1)(f^{\leftarrow}(\mu_B)) > 0$, $\omega(\tau_1)(f^{\leftarrow}(\underline{1} - \gamma_B)) = \omega(\tau_1)(\underline{1} - f^{\leftarrow}(\gamma_B)) > 0$. Hence

$$\mu^{\tau_1}(\langle f^{\leftarrow}(\mu_B), f^{\leftarrow}(\gamma_B)\rangle) = \omega(\tau_1)(f^{\leftarrow}(\mu_B)) \wedge \omega(\tau_1)(\underline{1} - f^{\leftarrow}(\gamma_B)) > 0.$$

Clearly, $\mathrm{I}\omega(\tau_1)(\langle f^{\leftarrow}(\mu_B), f^{\leftarrow}(\gamma_B)\rangle) > 0^{\sim}$, so $f\colon (\zeta^X, \mathrm{I}\omega(\tau_1)) \to (\zeta^Y, \mathrm{I}\omega(\tau_2))$ is weak fuzzy continuous. $\square$

Theorem 4.7. *Suppose (ζ^X, δ) is an intuitionistic I-fuzzy topological space. For each $A \subseteq X$, let $[\delta](A) = \mu_\delta(\langle 1_A, 1_{A^c}\rangle)$. Then $[\delta]$ is a fuzzifying topology on X.*

P r o o f. (i) $[\delta](\emptyset) = \mu_\delta(\langle 0_X, 1_X\rangle) = 1$, $[\delta](X) = \mu_\delta(\langle 1_X, 0_X\rangle) = 1$;

(ii) for all $A, B \subseteq X$,

$$\begin{aligned}[\delta](A \wedge B) &= \mu_\delta(\langle 1_{A\wedge B}, 1_{(A\wedge B)^c}\rangle) = \mu_\delta(\langle 1_A \wedge 1_B, 1_{A^c} \vee 1_{B^c}\rangle)\\ &= \mu_\delta(\langle 1_A, 1_{A^c}\rangle \wedge \langle 1_B, 1_{B^c}\rangle)\\ &\geqslant \mu_\delta(\langle 1_A, 1_{A^c}\rangle) \wedge \mu_\delta(\langle 1_B, 1_{B^c}\rangle) = [\delta](A) \wedge [\delta](B).\end{aligned}$$

(iii) For all $A_t \subseteq X$, $t \in T$,

$$[\delta]\Big(\bigvee_{t\in T} A_t\Big) = \mu_\delta\Big(\Big\langle 1_{\bigvee\limits_{t\in T} A_t}, 1_{(\bigvee\limits_{t\in T} A_t)^c}\Big\rangle\Big) = \mu_\delta\Big(\Big\langle \bigvee_{t\in T} 1_{A_t}, \bigwedge_{t\in T} 1_{A_t^c}\Big\rangle\Big)$$
$$= \mu_\delta\Big(\bigvee_{t\in T} \langle 1_{A_t}, 1_{A_t^c}\rangle\Big) \geqslant \bigwedge_{t\in T} \mu_\delta(\langle 1_{A_t}, 1_{A_t^c}\rangle) = \bigwedge_{t\in T} [\delta](A_t).$$

So $[\delta]$ is a fuzzifying topology on X. □

Theorem 4.8. *Suppose $f\colon (\zeta^X, \delta)$ is an intuitionistic I-fuzzy topological space, for each $A \in I^X$, let $\tilde{\delta}(A) = \mu_\delta(\langle A, \underline{1} - A\rangle)$. Then $\tilde{\delta}$ is an I-fuzzy topology on X.*

P r o o f. (i) $\tilde{\delta}(\underline{0}) = \mu_\delta(\langle \underline{0}, \underline{1}\rangle) = \mu_\delta(\langle \underline{1}, \underline{0}\rangle) = \tilde{\delta}(\underline{1}) = 1$;
(ii) for all $A, B \in I^X$,

$$\tilde{\delta}(A \wedge B) = \mu_\delta(\langle A \wedge B, \underline{1} - (A \wedge B)\rangle)$$
$$= \mu_\delta(\langle A, \underline{1} - A > \wedge \langle B, \underline{1} - B\rangle)$$
$$\geqslant \mu_\delta(\langle A, \underline{1} - A\rangle) \wedge \mu_\delta(\langle B, \underline{1} - B\rangle) = \tilde{\delta}(A) \wedge \tilde{\delta}(B);$$

(iii) for all $A_t \in I^X$, $t \in T$,

$$\tilde{\delta}\Big(\bigvee_{t\in T} A_t\Big) = \mu_\delta\Big(\Big\langle \bigvee_{t\in T} A_t, \underline{1} - \bigvee_{t\in T} A_t\Big\rangle\Big) = \mu_\delta\Big(\bigvee_{t\in T} \langle A_t, \underline{1} - A_t\rangle\Big)$$
$$\geqslant \bigwedge_{t\in T} \mu_\delta(\langle A_t, \underline{1} - A_t\rangle) = \bigwedge_{t\in T} \tilde{\delta}(A_t).$$

So $\tilde{\delta}$ is an I-fuzzy topology on X. □

Theorem 4.9. *Suppose that (ζ^X, δ) is an intuitionistic I-fuzzy topological space. For each $r \in I_0$, $A \subseteq X$, let*
$\iota_1(\delta)^r(A) = \bigvee\{\mu_\delta(B)\colon\ \sigma_r(\mu_B) = A,\ B \in \zeta^X\}$,
$\iota_2(\delta)^r(A) = \bigvee\{\mu_\delta(B)\colon\ \sigma_r(\underline{1} - \gamma_B) = A,\ B \in \zeta^X\}$.
Then both $\iota_1(\delta)^r$ and $\iota_2(\delta)^r$ are fuzzifying topologies on X.

P r o o f. We only need to prove that $\iota_1(\delta)^r$ is a fuzzifying topology on X. The proof for $\iota_2(\delta)^r$ is similar to $\iota_1(\delta)^r$.

(i) $\iota_1(\delta)^r(\emptyset) \geqslant \mu_\delta(\langle \underline{0}, \underline{1}\rangle) = 1$, so $\iota_1(\delta)^r(\emptyset) = 1$, $\iota_1(\delta)^r(X) \geqslant \mu_\delta(\langle \underline{1}, \underline{0}\rangle) = 1$, so $\iota_1(\delta)^r(X) = 1$.

(ii) For all $A, B \subseteq X$, and any $\varepsilon > 0$, there exist $A_1, B_1 \in \zeta^X$ such that $\sigma_r(\mu_{A_1}) = A$, $\sigma_r(\mu_{B_1}) = B$ and $\iota_1(\delta)^r(A) - \varepsilon < \mu_\delta(A_1)$, $\iota_1(\delta)^r(B) - \varepsilon < \mu_\delta(B_1)$.

From $\sigma_r(\mu_{A_1} \wedge \mu_{B_1}) = A \wedge B$, we have

$$\begin{aligned}\iota_1(\delta)^r(A \wedge B) &\geqslant \mu_\delta(A_1 \wedge B_1) \geqslant \mu_\delta(A_1) \wedge \mu_\delta(B_1)\\ &> (\iota_1(\delta)^r(A) - \varepsilon) \wedge (\iota_1(\delta)^r(B) - \varepsilon) = \iota_1(\delta)^r(A) \wedge (\iota_1(\delta)^r(B) - \varepsilon.\end{aligned}$$

By the arbitrariness of ε, we obtain $\iota_1(\delta)^r(A \wedge B) \geqslant \iota_1(\delta)^r(A) \wedge (\iota_1(\delta)^r(B)$.

(iii) For all $A_t \subseteq X$, $t \in T$, and $\varepsilon > 0$, there exists $B_t \in \zeta^X$ such that $\sigma_r(\mu_{B_t}) = A_t$ and $\iota_1(\delta)^r(A_t) - \varepsilon \leqslant \mu_\delta(B_t)$. Since $\sigma_r\Big(\bigvee\limits_{t \in T} \mu_{B_t}\Big) = \bigvee\limits_{t \in T} \sigma_r(\mu_{B_t})$, then

$$\begin{aligned}\iota_1(\delta)^r\Big(\bigvee_{t \in T} A_t\Big) &\geqslant \mu_\delta\Big(\bigvee_{t \in T} B_t\Big) \geqslant \bigwedge_{t \in T} \mu_\delta(B_t)\\ &\geqslant \bigwedge_{t \in T} (\iota_1(\delta)^r(A_t) - \varepsilon) = \bigwedge_{t \in T} \iota_1(\delta)^r(A_t) - \varepsilon.\end{aligned}$$

By the arbitrariness of ε again, we obtain $\iota_1(\delta)^r\Big(\bigvee\limits_{t \in T} A_t\Big) \geqslant \bigwedge\limits_{t \in T} \iota_1(\delta)^r(A_t)$.

Therefore $\iota_1(\delta)^r$ is a fuzzifying topology on X. □

Let $\iota_1(\delta) = \bigwedge\limits_{r \in I_0} \iota_1(\delta)^r$, $\iota_2(\delta) = \bigwedge\limits_{r \in I_0} \iota_2(\delta)^r$, then both $\iota_1(\delta)$ and $\iota_2(\delta)$ are fuzzifying topology on X.

Theorem 4.10. *Suppose that $f\colon (\zeta^X, \delta_1) \to (\zeta^Y, \delta_2)$ is fuzzy continuous, then $f\colon (X, \iota_i(\delta_1)) \to (Y, \iota_i(\delta_2))$ $(i = 1, 2)$ is continuous.*

Proof. (1) For all $A \subseteq Y$,

$\iota_1(\delta_1)(f^{\leftarrow}(A)) = \bigwedge\limits_{r \in I_0} \bigvee\{\delta_1(B)\colon \sigma_r(\mu_B) = f^{\leftarrow}(A),\ B \in \zeta^X\}$,

$\iota_1(\delta_2)(A) = \bigwedge\limits_{r \in I_0} \bigvee\{\delta_2(B)\colon \sigma_r(\mu_B) = A,\ B \in \zeta^Y\}$.

For each $B \in \zeta^Y$, if $\sigma_r(\mu_B) = A$, then $\sigma_r(f^{\leftarrow}(\mu_B)) = f^{\leftarrow}(A)$. Since $f\colon (\zeta^X, \delta_1) \to (\zeta^Y, \delta_2)$ is fuzzy continuous, we have $\delta_1(f^{\leftarrow}(B)) \geqslant \delta_2(B)$ for every $B \in \zeta^Y$. So

$$\begin{aligned}\iota_1(\delta_1)(f^{\leftarrow}(A)) &\geqslant \bigwedge_{r \in I_0} \bigvee\{\delta_1(f^{\leftarrow}(B))\colon \sigma_r(\mu_B) = A,\ B \in \zeta^Y\}\\ &\geqslant \bigwedge_{r \in I_0} \bigvee\{\delta_2(B)\colon \sigma_r(\mu_B) = A,\ B \in \zeta^Y\} = \iota_1(\delta_2)(A).\end{aligned}$$

This means $f\colon (X, \iota_1(\delta_1)) \to (Y, \iota_1(\delta_2))$ is continuous.

As for the continuity of $f\colon (X, \iota_2(\delta_1)) \to (Y, \iota_2(\delta_2))$, the proof is similar to (1). □

Theorem 4.11. *Let (X,τ) be a fuzzifying topological space and (X,δ) an intuitionistic I-fuzzy topological space. Then*
(1) $[\mathrm{I}\omega(\tau)] = \tau$;
(2) $\forall\, A \subseteq X$, $\mu^{[\delta]}(\langle 1_A, 1_{A^c}\rangle) = \mu_\delta(\langle 1_A, 1_{A^c}\rangle)$;
(3) $[\delta] \leqslant \iota_1(\delta)$, $[\delta] \leqslant \iota_2(\delta)$.

Proof. (1) For all $A \subseteq X$, $[\mathrm{I}\omega(\tau)](A) = \mu_{\mathrm{I}\omega(\tau)}(\langle 1_A, 1_{A^c}\rangle) = \tau(A)$ (by Lemma 4.3);

(2) for all $A \subseteq X$,

$$\begin{aligned}\mu^{[\delta]}(\langle 1_A, 1_{A^c}\rangle) &= \omega([\delta])(1_A) \wedge \omega([\delta])(\underline{1} - 1_{A^c}) \\ &= \omega([\delta])(1_A) = [\delta](1_A) = \mu_\delta(\langle 1_A, 1_{A^c}\rangle).\end{aligned}$$

(3) for all $A \subseteq X$, $[\delta](A) = \mu_\delta(\langle 1_A, 1_{A^c}\rangle)$.
$\forall\, r \in I_0$, $\sigma_r(1_A) = A$, thus $\iota_1(\delta)^r(A) \geqslant \mu_\delta(\langle 1_A, 1_{A^c}\rangle)$, so

$$\iota_1(\delta)(A) = \bigwedge_{r\in I_0} \iota_1(\delta)^r(A) \geqslant \mu_\delta(\langle 1_A, 1_{A^c}\rangle) = [\delta](A).$$

Moreover, $\sigma_r(\underline{1} - 1_{A^c}) = A$, thus $\iota_2(\delta)^r(A) \geqslant \mu_\delta(\langle 1_A, 1_{A^c}\rangle)$, so $\iota_2(\delta)(A) \geqslant [\delta](A)$. $\square$

Theorem 4.12. *Suppose that $f\colon X \to Y$, and (X,τ) is a fuzzifying topological space on X. Then $\tau/f = \tau \circ f^{\leftarrow}$ is a fuzzifying topology on Y, and $\mathrm{I}\omega(\tau/f) = \mathrm{I}\omega(\tau)/f$.*

Proof. For the first part of the theorem see [4, Theorem 1.5].
For all $A \in \zeta^Y$, $\mathrm{I}\omega(\tau/f)(A) = \langle \mu^{(\tau/f)}(A), \gamma^{(\tau/f)}(A)\rangle$. Now

$$\begin{aligned}\mu^{(\tau/f)}(A) &= \omega(\tau/f)(\mu_A) \wedge \omega(\tau/f)(\underline{1} - \gamma_A) \\ &= \bigwedge_{r\in I} (\tau/f)(\sigma_r(\mu_A)) \wedge \bigwedge_{r\in I} (\tau/f)(\sigma_r(\underline{1} - \gamma_A)) \\ &= \bigwedge_{r\in I} \tau(f^{\leftarrow}(\sigma_r(\mu_A))) \wedge \bigwedge_{r\in I} \tau(f^{\leftarrow}(\sigma_r(\underline{1} - \gamma_A))) \\ &= \bigwedge_{r\in I} \tau(\sigma_r(f^{\leftarrow}(\mu_A))) \wedge \bigwedge_{r\in I} \tau(\sigma_r f^{\leftarrow}((\underline{1} - \gamma_A))) \\ &= \omega(\tau)(f^{\leftarrow}(\mu_A)) \wedge \omega(\tau)(\underline{1} - f^{\leftarrow}(\gamma_A)) = \mu^{\tau}(f^{\leftarrow}(A)).\end{aligned}$$

Thus

$$\mathrm{I}\omega(\tau/f)(A) = \langle \mu^{(\tau/f)}(A), \gamma^{(\tau/f)}(A)\rangle = \langle \mu^{\tau}(f^{\leftarrow}(A)), \gamma^{\tau}(f^{\leftarrow}(A))\rangle = \mathrm{I}\omega(\tau)/f(A).$$

This completes the proof. $\square$

Theorem 4.13. *Suppose that $f\colon X \to Y$ is a bijection and (Y,τ) a fuzzifying topological space. Then $f^{\leftarrow}(\tau) = \tau \circ f$ is a fuzzifying topology on X, and $f^{\leftarrow}(\mathrm{I}\omega(\tau)) = \mathrm{I}\omega(f^{\leftarrow}(\tau))$.*

P r o o f. For the first part of the theorem see [4, Theorem 1.6]. Next,

$$A \in \zeta^X, \quad f^{\leftarrow}(\mathrm{I}\omega(\tau))(A) = \mathrm{I}\omega(\tau)(f^{\rightarrow}(A)) = \langle \mu^{\tau}(f^{\rightarrow}(A)), \gamma^{\tau}(f^{\rightarrow}(A))\rangle.$$

Notice that, for each $A = \{\langle x, \mu_A(x), \gamma_A(x)\rangle\colon\ x \in X\} \in \zeta^X$,

$$\begin{aligned}
f^{\rightarrow}(A) &= \{\langle y, f^{\rightarrow}(\mu_A)(y), \underline{1} - f^{\rightarrow}(\underline{1} - \gamma_A)(y)\rangle\colon\ y \in Y\}, \quad f \text{ is bijection},\\
\mu^{\tau}(f^{\rightarrow}(A)) &= \omega(\tau)(f^{\rightarrow}(\mu_A)) \wedge \omega(\tau)(f^{\rightarrow}(\underline{1} - \gamma_A))\\
&= \bigwedge_{r\in I} \tau(\sigma_r(f^{\rightarrow}(\mu_A))) \wedge \bigwedge_{r\in I} \tau(\sigma_r(f^{\rightarrow}(\underline{1} - \gamma_A)))\\
&= \bigwedge_{r\in I} \tau(f^{\rightarrow}(\sigma_r(\mu_A))) \wedge \bigwedge_{r\in I} \tau(f^{\rightarrow}(\sigma_r(\underline{1} - \gamma_A)))\\
&= \bigwedge_{r\in I} f^{\leftarrow}(\tau(\sigma_r(\mu_A))) \wedge \bigwedge_{r\in I} f^{\leftarrow}(\tau(\sigma_r(\underline{1} - \gamma_A)))\\
&= \bigwedge_{r\in I} \omega(f^{\leftarrow}(\tau))(\mu_A) \wedge \bigwedge_{r\in I} \omega(f^{\leftarrow}(\tau))(\underline{1} - \gamma_A) = \mu^{f^{\leftarrow}(\tau)}(A).
\end{aligned}$$

Obviously, $\gamma^{\tau}(f^{\rightarrow}(A)) = \gamma^{f^{\leftarrow}(\tau)}(A)$. Hence

$$\begin{aligned}
f^{\leftarrow}(\mathrm{I}\omega(\tau))(A) &= \langle \mu^{\tau}(f^{\rightarrow}(A)), \gamma^{\tau}(f^{\rightarrow}(A))\rangle\\
&= \langle \mu^{f^{\leftarrow}(\tau)}(A), \gamma^{f^{\leftarrow}(\tau)}(A)\rangle = \mathrm{I}\omega(f^{\leftarrow}(\tau))(A).
\end{aligned}$$

Therefore $f^{\leftarrow}(\mathrm{I}\omega(\tau)) = \mathrm{I}\omega(f^{\leftarrow}(\tau))$. □

Theorem 4.14. *Suppose that $f\colon X \to Y$ is a mapping, and (ζ^X, δ) is an intuitionistic I-fuzzy topological space. Then $\delta/f = \delta \circ f^{\leftarrow}$ is an intuitionistic I-fuzzy topology on Y, and $[\delta/f] = [\delta]/f$.*

P r o o f. (i)

$$\begin{aligned}
(\delta/f)(\langle \underline{0}, \underline{1}\rangle) &= \delta \circ f^{\leftarrow}(\langle \underline{0}, \underline{1}\rangle)\\
&= \delta(\langle f^{\leftarrow}(\underline{0}), f^{\leftarrow}(\underline{1})\rangle) = \delta(\langle \underline{0}, \underline{1}\rangle) = 1^{\sim},\\
(\delta/f)(\langle \underline{1}, \underline{0}\rangle) &= \delta \circ f^{\leftarrow}(\langle \underline{1}, \underline{0}\rangle)\\
&= \delta(\langle f^{\leftarrow}(\underline{1}), f^{\leftarrow}(\underline{0})\rangle) = \delta(\langle \underline{1}, \underline{0}\rangle) = 1^{\sim};
\end{aligned}$$

(ii) for all $A, B \in \zeta^Y$, $A = \langle \mu_A, \gamma_A \rangle$, $B = \langle \mu_B, \gamma_B \rangle$,

$$\begin{aligned}(\delta/f)(A \wedge B) &= \delta \circ f^{\leftarrow}(A \wedge B) = \delta(f^{\leftarrow}(A) \wedge f^{\leftarrow}(B)) \\ &\geqslant \delta(f^{\leftarrow}(A)) \wedge \delta(f^{\leftarrow}(B)) = (\delta/f)(A) \wedge (\delta/f)(B);\end{aligned}$$

(iii) for all $A_t \in \zeta^Y$, $t \in T$, $A_t = \langle \mu_{A_t}, \gamma_{A_t} \rangle$,

$$\begin{aligned}(\delta/f)\Big(\bigvee_{t\in T} A_t\Big) &= \delta \circ f^{\leftarrow}\Big(\bigvee_{t\in T} A_t\Big) = \delta\Big(\bigvee_{t\in T} f^{\leftarrow}(A_t)\Big) \\ &\geqslant \bigwedge_{t\in T} \delta(f^{\leftarrow}(A_t) \geqslant \bigwedge_{t\in T} (\delta/f)(A_t).\end{aligned}$$

So δ/f is an intuitionistic I-fuzzy topology on Y.

For each $A \subseteq Y$

$$\begin{aligned}[\delta/f](A) &= (\delta/f)(\langle 1_A, 1_{A^c} \rangle)) = \delta \circ f^{\leftarrow}(\langle 1_A, 1_{A^c} \rangle) \\ &= \delta(\langle f^{\leftarrow}(1_A), f^{\leftarrow}(1_{A^c}) \rangle) = \delta(\langle 1_{f^{\leftarrow}(A)}, 1_{(f^{\leftarrow}(A)^c} \rangle) \\ &= [\delta](f^{\leftarrow}(A)) = [\delta]/f(A).\end{aligned}$$

Hence $[\delta/f] = [\delta]/f$. □

Theorem 4.15. *Suppose that $f\colon \zeta^X \to \zeta^Y$ is a bijection and (ζ^Y, δ) is an intuitionistic I-fuzzy topological space, then $f^{\leftarrow}(\delta) = \delta \circ f$ is an intuitionistic I-fuzzy topological space on X, and $f^{\leftarrow}([\delta]) = [f^{\leftarrow}(\delta)]$.*

P r o o f. Notice that f is bijection, and by the definition of $f^{\rightarrow}(A)$ $(A \in \zeta^X)$, we have $f^{\rightarrow}(\langle \underline{0}, \underline{1} \rangle) = \langle \underline{0}, \underline{1} \rangle$, $f^{\rightarrow}(\langle \underline{1}, \underline{0} \rangle) = \langle \underline{1}, \underline{0} \rangle$.

(i) $f^{\leftarrow}(\delta)(\langle \underline{0}, \underline{1} \rangle) = \delta(\langle \underline{0}, \underline{1} \rangle) = 1^{\sim}$, $f^{\leftarrow}(\delta)(\langle \underline{1}, \underline{0} \rangle) = \delta(\langle \underline{1}, \underline{0} \rangle) = 1^{\sim}$;

(ii) for all $A, B \in \zeta^X$, since f is bijection, the following relation holds

$$\begin{aligned}f^{\leftarrow}(\delta)(A \wedge B) &= \delta(f^{\rightarrow}(A) \wedge f^{\rightarrow}(B)) \geqslant \delta(f^{\rightarrow}(A)) \wedge \delta(f^{\rightarrow}(B)) \\ &= f^{\leftarrow}(\delta)(A) \wedge f^{\leftarrow}(\delta)(B);\end{aligned}$$

(iii) for all $A_t \in \zeta^X$, $t \in T$,

$$\begin{aligned}f^{\leftarrow}(\delta)\Big(\bigvee_{t\in T} A_t\Big) &= \delta\Big(f^{\rightarrow}\Big(\bigvee_{t\in T} A_t\Big)\Big) = \delta\Big(\bigvee_{t\in T} f^{\rightarrow}(A_t)\Big) \\ &\geqslant \bigwedge_{t\in T} \delta(f^{\rightarrow}(A_t)) = \bigwedge_{t\in T} f^{\leftarrow}(\delta)(A_t).\end{aligned}$$

Therefore $f^{\leftarrow}(\delta)$ is an intuitionistic I-fuzzy topological space on X.

For each $A \subseteq X$, $f^{\leftarrow}([\delta])(A) = [\delta](f^{\rightarrow}(A)) = \mu_\delta(\langle 1_{f^{\rightarrow}(A)}, 1_{(f^{\rightarrow}(A))^c}\rangle)$,

$$[f^{\leftarrow}(\delta)](A) = \mu_{f^{\leftarrow}(\delta)}(\langle 1_A, 1_{A^c}\rangle) = \mu_\delta(\langle f^{\rightarrow}(1_A), \underline{1} - f^{\rightarrow}(1_A)\rangle).$$

Since f is bijection, so $1_{f^{\rightarrow}(A)} = f^{\rightarrow}(1_A)$, $1_{(f^{\rightarrow}(A))^c} = \underline{1} - f^{\rightarrow}(1_A)$.
Hence $f^{\leftarrow}([\delta])(A) = [f^{\leftarrow}(\delta)](A)$, thus $f^{\leftarrow}([\delta]) = [f^{\leftarrow}(\delta)]$. □

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Authors' address: Cong-hua Yan, Xiao-ke Wang, Institute of Mathematics, School of Math., Nanjing Normal University, Nanjing Jiangsu 210046, P. R. China, e-mail: chyan@njnu.edu.cn, xiaokewang@sohu.com.