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CONVERGENCE CONDITIONS FOR SECANT-TYPE METHODS

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Abstract. We provide new sufficient convergence conditions for the convergence of the secant-type methods to a locally unique solution of a nonlinear equation in a Banach space. Our new idea uses recurrent functions, and Lipschitz-type and center-Lipschitz-type instead of just Lipschitz-type conditions on the divided difference of the operator involved. It turns out that this way our error bounds are more precise than earlier ones and under our convergence hypotheses we can cover cases where earlier conditions are violated. Numerical examples are also provided.

Keywords: secant method, Banach space, majorizing sequence, divided difference, Fréchet-derivative

MSC 2010: 65H10, 65B05, 65G99, 65N30, 49M15

1. Introduction

In this study we are concerned with the problem of approximating a locally unique solution $x^\star$ of the equation

$$F(x) = 0, \tag{1.1}$$

where F is a Fréchet-differentiable operator defined on a convex subset $\mathcal{D}$ of a Banach space $\mathcal{X}$ with values in a Banach space $\mathcal{Y}$.

A large number of problems in applied mathematics and also in engineering are solved by finding the solutions of certain equations. For example, dynamic systems are mathematically modeled by difference or differential equations, and their solutions usually represent the states of the systems. For the sake of simplicity, assume that a time-invariant system is driven by the equation $\dot{x} = Q(x)$ for a suitable operator Q, where x is the state. Then the equilibrium states are determined by solving equation (1.1). Similar equations are used in the case of discrete systems. The unknowns of engineering equations can be functions (difference, differential, and integral equations), vectors (systems of linear or nonlinear algebraic equations), or real or complex numbers (single algebraic equations with single unknowns). Except in special cases, the solution methods most commonly used are iterative-when starting from one or several initial approximations a sequence is constructed that converges to a solution of the equation. Iteration methods are also applied for solving optimization problems. In such cases, the iteration sequences converge to an optimal solution of the problem at hand. Since all of these methods have the same recursive structure, they can be introduced and discussed in a general framework.

We consider the Secant method in the form

$$x_{n+1} = x_n - \delta F(x_{n-1}, x_n)^{-1} F(x_n) \quad (n \geqslant 0), \quad (x_{-1}, x_0 \in \mathcal{D}) \tag{1.2}$$

where $\delta F(x,y) \in \mathcal{L}(\mathcal{X},\mathcal{Y})$ $(x, y \in \mathcal{D})$ is a consistent approximation of the Fréchet-derivative of F [5], [13]. Bosarge and Falb [7], Dennis [9], Potra [16], Argyros [1]–[5], Hernández et al. [10] and others [11], [15], [18] have provided sufficient convergence conditions for the Secant method based on Lipschitz-type conditions on δF (see also relevant results in [6]–[9], [12], [14], [16], [17], [19]).

The conditions usually associated with the semilocal convergence of the Secant method (1.2) are:

- F is a nonlinear operator defined on a convex subset $\mathcal{D}$ of a Banach space $\mathcal{X}$ with values in a Banach space $\mathcal{Y}$;
- x_{-1} and x_0 are two points belonging to the interior $\mathcal{D}^0$ of $\mathcal{D}$ and satisfying the inequality
$$\|x_0 - x_{-1}\| \leqslant c;$$
- F is Fréchet-differentiable on $\mathcal{D}^0$, and there exists an operator $\delta F\colon \mathcal{D}^0 \times \mathcal{D}^0 \to \mathcal{L}(\mathcal{X},\mathcal{Y})$ such that:

the linear operator $A = \delta F(x_{-1}, x_0)$ is invertible, its inverse A^{-1} is bounded, and

$$\|A^{-1}F(x_0)\| \leqslant \eta;$$
$$\|A^{-1}[\delta F(x,y) - F'(z)]\| \leqslant l(\|x - z\| + \|y - z\|)$$

for all $x, y, z \in \mathcal{D}$;

$$\overline{U}(x_0, r) = \{x \in \mathcal{X}\colon\ \|x - x_0\| \leqslant r\} \subseteq \mathcal{D}^0$$

for some $r > 0$ depending on l, c, and η; and

$$lc + 2\sqrt{l\eta} \leqslant 1. \tag{1.3}$$

The sufficient convergence condition (1.3) is easily violated. Indeed, for solving equation (1.1), let $l = 1$, $\eta = .18$, and $c = .185$. Then (1.3) does not hold, since $lc + 2\sqrt{l\eta} = 1.033528137$. Hence, there is no guarantee that equation (1.1) under the information (l, c, η) has a solution that can be found using the Secant method (1.2). In this study we are motived by optimization considerations, and the above observation.

Here using Lipschitz-type and center-Lipschitz-type conditions, we provide a semilocal convergence analysis for (1.2). It turns out that our error bounds are more precise and our convergence conditions hold in cases where the corresponding hypotheses in earlier references mentioned above are violated. Newton's method is also examined as a special case. Numerical examples are also provided.

2. Semilocal convergence analysis of the Secant method

We need the following result on majorizing sequences for the Secant method (1.2).

Lemma 2.1. *Let $l_0 > 0$, $l > 0$, $c \geqslant 0$ and $\eta \in [0, c]$ be given parameters. Assume*

$$(l + 2l_0)\eta + l_0 c < 1 \tag{2.1}$$

for

$$\delta_0 = \frac{l(c+\eta)}{1 - l_0(c+\eta)}, \qquad s_\infty = \frac{1 - l_0(c+2\eta)}{1 - l_0 c},$$

δ the unique positive zero of equation

$$f(t) = l_0 t^3 + (l_0 + l)t^2 - l = 0 \quad \text{in } (0, 1); \tag{2.2}$$

$$\delta_0 \leqslant \delta \leqslant s_\infty. \tag{2.3}$$

Then the scalar sequence $\{t_n\}$ $(n \geqslant -1)$ given by

$$t_{-1} = 0,\ t_0 = c, t_1 = c + \eta,\ t_{n+2} = t_{n+1} + \frac{l(t_{n+1} - t_{n-1})(t_{n+1} - t_n)}{1 - l_0(t_{n+1} - t_0 + t_n)} \tag{2.4}$$

is non-decreasing, bounded above by

$$t^{\star\star} = \frac{\eta}{1 - \delta} + c, \tag{2.5}$$

and converges to its unique least upper bound $t^\star$ such that

$$0 \leqslant t^\star \leqslant t^{\star\star}. \tag{2.6}$$

Moreover, the following estimates hold for all $n \geqslant 0$*:*

$$0 \leqslant t_{n+2} - t_{n+1} \leqslant \delta(t_{n+1} - t_n) \leqslant \delta^{n+1}\eta. \tag{2.7}$$

P r o o f. Using (2.2), we obtain $f(0) = -l < 0$, $f(l) = 2l_0 > 0$, and $f'(t) = 3l_0t^2 + 2(l_0 + l)t > 0$ $(t > 0)$. The existence of δ_1 follows from the intermediate value theorem on $(0, 1)$ and the fact that f crosses the positive x-axis only once. Similarly, we show the existence of s_1 using (2.1).

We shall show using induction on $k \geqslant 0$ that

$$0 \leqslant t_{k+2} - t_{k+1} \leqslant \delta(t_{k+1} - t_k). \tag{2.8}$$

Using (2.4) for $k = 0$, we must show

$$0 \leqslant \frac{l(t_1 - t_{-1})}{1 - l_0t_1} \leqslant \delta$$

or

$$0 \leqslant \frac{l(c+\eta)}{1 - l_0(c+\eta)} \leqslant \delta,$$

which is true by virtue of (2.1) and the choice of $\delta \geqslant \delta_0$.

Let us assume that (2.8) holds for $k \leqslant n + 1$.

It then follows from the induction hypotheses

$$\begin{aligned} t_{k+2} &\leqslant t_{k+1} + \delta(t_{k+1} - t_k) \\ &\leqslant t_k + \delta(t_k - t_{k-1}) + \delta(t_{k+1} - t_k) \\ &\leqslant t_1 + \delta(t_1 - t_0) + \ldots + \delta(t_{k+1} - t_k) \\ &\leqslant c + \eta + \delta\eta + \ldots + \delta^{k+1}\eta \\ &= c + \frac{1 - \delta^{k+2}}{1 - \delta}\eta < \frac{\eta}{1 - \delta} + c = t^{\star\star}. \end{aligned} \tag{2.9}$$

Moreover, we have

$$\begin{aligned} &l(t_{k+2} - t_k) + \delta l_0(t_{k+2} - t_0 + t_{k+1}) \\ &\leqslant l((t_{k+2} - t_{k+1}) + (t_{k+1} - t_k)) + \delta l_0\Big(\frac{1 - \delta^{k+2}}{1 - \delta} + \frac{1 - \delta^{k+1}}{1 - \delta}\Big)\eta + \delta l_0 c \\ &\leqslant l(\delta^k + \delta^{k+1})\eta + \frac{\delta l_0}{1 - \delta}(2 - \delta^{k+1} - \delta^{k+2})\eta + \delta l_0 c. \end{aligned} \tag{2.10}$$

To show (2.8), using (2.4) we obtain

$$l(\delta^k + \delta^{k+1})\eta + \frac{\delta l_0}{1 - \delta}(2 - \delta^{k+1} - \delta^{k+2})\eta + \delta l_0 c \leqslant \delta \tag{2.11}$$

or

$$l(\delta^{k-1}+\delta^k)\eta + l_0((1+\delta+\ldots+\delta^k)+(1+\delta+\ldots+\delta^{k+1}))\eta + l_0c - 1 \leqslant 0. \tag{2.12}$$

In view of (2.12), we are motivated to define (for $\delta = s$) for $k \geqslant 1$ functions

$$f_k(s) = l(s^{k-1}+s^k)\eta + l_0(2(1+s+\ldots+s^k)+s^{k+1})\eta + l_0c - 1. \tag{2.13}$$

We need the relationship between two consecutive functions f_k. Using (2.13), we obtain

$$\begin{aligned} f_{k+1}(s) &= l(s^k+s^{k+1})\eta + l_0(2(1+s+\ldots+s^{k+1})+s^{k+2})\eta + l_0c - 1 \\ &= l(s^{k-1}+s^k)\eta + l(s^k+s^{k+1})\eta - l(s^{k-1}+s^k)\eta \\ &\quad + l_0(2(1+s+\ldots+s^k)+s^{k+1})\eta + l_0(2s^{k+1}+s^{k+2})\eta \\ &\quad - l_0 s^{k+1}\eta + l_0c - 1 \\ &= f_k(s) + l(s^{k+1}-s^{k-1})\eta + l_0(s^{k+1}+s^{k+2})\eta \\ &= f(s)s^{k-1}\eta + f_k(s). \end{aligned} \tag{2.14}$$

We shall show

$$f_k(\delta) \leqslant 0 \quad (k \geqslant 1). \tag{2.15}$$

But we have by (2.14)

$$f_k(\delta) = f_{k-1}(\delta) = \ldots = f_1(\delta). \tag{2.16}$$

Define the function f_∞ on $[0,1)$:

$$f_\infty(s) = \lim_{k \longrightarrow \infty} f_k(s). \tag{2.17}$$

Then, we can show instead of (2.15)

$$f_\infty(\delta) = \frac{2l_0\eta}{1-\delta} + l_0c - 1 \leqslant 0,$$

which is true by (2.3) and (2.13). Hence, we showed that the sequence $\{t_n\}$ $(n \geqslant -1)$ is non-decreasing and bounded above by $t^{\star\star}$, so that (2.7) holds. It follows that there exists $t^\star \in [0, t^{\star\star}]$ such that $\lim_{n \longrightarrow \infty} t_n = t^\star$.

This completes the proof of Lemma 2.1. □

We shall study the Secant method (1.2) for triplets (F, x_{-1}, x_0) belonging to the class $\mathcal{C}(l, l_0, \eta, c, \delta)$ defined as follows:

Definition 2.2. Let l, l_0, η, c, δ be non-negative parameters satisfying the hypotheses of Lemma 2.1.

We say that a triplet (F, x_{-1}, x_0) belongs to the class $\mathcal{C}(l, l_0, \eta, c, \delta)$ if

(c_1) F is a nonlinear operator defined on a convex subset $\mathcal{D}$ of a Banach space $\mathcal{X}$ with values in a Banach space $\mathcal{Y}$;

(c_2) x_{-1} and x_0 are two points belonging to the interior $\mathcal{D}^0$ of $\mathcal{D}$ and satisfying the inequality

$$\|x_0 - x_{-1}\| \leqslant c;$$

(c_3) F is Fréchet-differentiable on $\mathcal{D}^0$, and there exists an operator $\delta F\colon\ \mathcal{D}^0 \times \mathcal{D}^0 \to \mathcal{L}(\mathcal{X}, \mathcal{Y})$ such that

the linear operator $A = \delta F(x_{-1}, x_0)$ is invertible, its inverse A^{-1} is bounded and

$$\|A^{-1}F(x_0)\| \leqslant \eta;$$
$$\|A^{-1}[\delta F(x, y) - F'(z)]\| \leqslant l(\|x - z\| + \|y - z\|);$$
$$\|A^{-1}[\delta F(x, y) - F'(x_0)]\| \leqslant l_0(\|x - x_0\| + \|y - x_0\|)$$

for all $x, y, z \in \mathcal{D}$;

(c_4) the set $\mathcal{D}_c = \{x \in \mathcal{D};\ F \text{ is continuous at } x\}$ contains the closed ball $\overline{U}(x_0, t^\star) = \{x \in X \mid \|x - x_0\| \leqslant t^\star\}$ where $t^\star$ is given in Lemma 2.1.

We present the following semilocal convergence theorem for the Secant method (1.2).

Theorem 2.3. *If $(F, x_{-1}, x_0) \in \mathcal{C}(l, l_0, \eta, c, \delta)$, then the sequence $\{x_n\}$ $(n \geqslant -1)$ generated by the Secant method* (1.2) *is well defined, remains in $\overline{U}(x_0, t^\star)$ for all $n \geqslant 0$ and converges to a unique solution $x^\star \in \overline{U}(x_0, t^\star)$ of the equation $F(x) = 0$.*

Moreover, the following estimates hold for all $n \geqslant 0$:

$$\|x_{n+2} - x_{n+1}\| \leqslant t_{n+2} - t_{n+1}, \tag{2.18}$$

and

$$\|x_n - x^\star\| \leqslant t^\star - t_n \tag{2.19}$$

where the sequence $\{t_n\}$ $(n \geqslant 0)$ is given by (2.4).

Furthermore, if there exists $R \geqslant t^\star - t_0$ such that

$$l_0(c + \eta + R) < 1, \tag{2.20}$$

$$U(x_0, R) \subseteq \mathcal{D}, \tag{2.21}$$

and

$$\delta_0 \leqslant \delta \leqslant \min\{\delta_1, s_\infty\}, \tag{2.22}$$

where

$$\delta_1 = \frac{1 - l_0(c + \eta + R)}{1 - l_0(c + R)},$$

then the solution $x^\star$ is unique in $U(x_0, R)$.

Proof. We first show that the operator $L = \delta F(x_k, x_{k+1})$ is invertible for $x_k, x_{k+1} \in \overline{U}(x_0, t^\star)$. It follows from (2.4), (2.5), (c_2) and (c_3) that

$$\begin{aligned} \|I - A^{-1}L\| = \|A^{-1}(L - A)\| &\leqslant \|A^{-1}(L - F'(x_0))\| + \|A^{-1}(F'(x_0) - A)\| \\ &\leqslant l_0(\|x_k - x_0\| + \|x_{k+1} - x_0\| + \|x_0 - x_{-1}\|) \\ &\leqslant l_0(t_k - t_0 + t_{k+1} - t_0 + c) \\ &\leqslant l_0(t^\star - t_0 + t^\star - t_0 + c) \\ &\leqslant l_0\Big(2\Big(\frac{\eta}{1-\delta} + c\Big) - c\Big) \leqslant 1 \end{aligned} \tag{2.23}$$

since $\delta \leqslant s_\infty$.

According to the Banach Lemma on invertible operators [5], [13], and (2.23), L is invertible and

$$\|L^{-1}A\| \leqslant (1 - l_0(\|x_k - x_0\| + \|x_{k+1} - x_0\| + c))^{-1}. \tag{2.24}$$

The second condition in (c_3) implies the Lipschitz condition for F' :

$$\|A^{-1}(F'(u) - F'(v))\| \leqslant 2l\|u - v\|, \quad u, v \in \mathcal{D}^0. \tag{2.25}$$

By the identity

$$F(x) - F(y) = \int_0^1 F'(y + t(x - y))\,\mathrm{d}t(x - y) \tag{2.26}$$

we get

$$\|A_0^{-1}[F(x) - F(y) - F'(u)(x - y)]\| \leqslant l(\|x - u\| + \|y - u\|)\|x - y\| \tag{2.27}$$

and

$$\|A_0^{-1}[F(x) - F(y) - \delta F(u, v)(x - y)]\| \leqslant l(\|x - v\| + \|y - v\| + \|u - v\|)\|x - y\| \tag{2.28}$$

for all $x, y, u, v \in \mathcal{D}^0$. By a continuity argument (2.25)–(2.28) remain valid if x and/or y belong to $\mathcal{D}_c$.

We first show (2.18). If (2.18) holds for all $n \leqslant k$ and if $\{x_n\}$ $(n \geqslant 0)$ is well defined for $n = 0, 1, 2, \ldots, k$ then

$$\|x_0 - x_n\| \leqslant t_n - t_0 < t^\star - t_0, \quad n \leqslant k. \tag{2.29}$$

That is, (1.2) is well defined for $n = k+1$. For $n = -1$ and $n = 0$, (2.18) reduces to $\|x_{-1} - x_0\| \leqslant c$ and $\|x_0 - x_1\| \leqslant \eta$. Suppose (2.18) holds for $n = -1, 0, 1, \ldots, k$ $(k \geqslant 0)$. Using (2.24), (2.28) and

$$F(x_{k+1}) = F(x_{k+1}) - F(x_k) - \delta F(x_{k-1}, x_k)(x_{k+1} - x_k) \tag{2.30}$$

we obtain in turn

$$\begin{aligned} \|x_{k+2} - x_{k+1}\| &= \|\delta F(x_k, x_{k+1})^{-1} F(x_{k+1})\| \\ &\leqslant \|\delta F(x_k, x_{k+1})^{-1} A\| \|A^{-1} F(x_{k+1})\| \\ &\leqslant \frac{l(\|x_{k+1} - x_k\| + \|x_k - x_{k-1}\|)}{1 - l_0(\|x_{k+1} - x_0\| + \|x_k - x_0\| + c)} \|x_{k+1} - x_k\| \\ &\leqslant \frac{l(t_{k+1} - t_k + t_k - t_{k-1})}{1 - l_0(t_{k+1} - t_0 + t_k - t_0 + t_0 - t_{-1})} (t_{k+1} - t_k) \\ &= t_{k+2} - t_{k+1}. \end{aligned} \tag{2.31}$$

The induction for (2.18) is completed. It follows from (2.18) and Lemma 2.1 that the sequence $\{x_n\}$ $(n \geqslant -1)$ is Cauchy in the Banach space $\mathcal{X}$, and as such it converges to some $x^\star \in \overline{U}(x_0, t^\star)$ (since $\overline{U}(x_0, t^\star)$ is a closed set). By letting $k \to \infty$ in (2.31) we obtain $F(x^\star) = 0$.

The estimate (2.19) follows from (2.18) by using the standard majoration techniques [1], [5], [13].

We shall first show uniqueness in $\overline{U}(x_0, t^\star)$. Let $y^\star \in \overline{U}(x_0, t^\star)$ be a solution of equation (1.1).

Set

$$\mathcal{M} = \int_0^1 F'(y^\star + t(y^\star - x^\star))\,\mathrm{d}t.$$

It then follows by (c_3) that

$$\begin{aligned} \|A^{-1}(A - \mathcal{M})\| &= l_0(\|y^\star - x_0\| + \|x^\star - x_0\| + \|x_0 - x_{-1}\|) \\ &\leqslant l_0((t^\star - t_0) + (t^\star - t_0) + t_0) \\ &< l_0\Big(2\Big(\frac{\eta}{1-\delta} + c\Big) - c\Big) = l_0\Big(\frac{2\eta}{1-\delta} + c\Big) \leqslant 1, \end{aligned} \tag{2.32}$$

since $\delta \leqslant s_\infty$.

It follows from (2.32) and the Banach lemma on invertible operators that $\mathcal{M}^{-1}$ exists on $\overline{U}(x_0, t^\star)$.

Using the identity

$$F(x^\star) - F(y^\star) = \mathcal{M}(x^\star - y^\star) \tag{2.33}$$

we deduce $x^\star = y^\star$. Finally, we shall show uniqueness in $U(x_0, R)$. As in (2.32), we arrive at

$$\|A^{-1}(A - \mathcal{M})\| < l_0\Big(\frac{\eta}{1-\delta} + c + R\Big) < 1$$

by (2.20), and (2.22).

That completes the proof of Theorem 2.3. □

Remark 2.4.

(a) The root δ_1 of the function f has an unatractive closed form found using Mathematica and given by

$$\delta_1 = d_0 + \frac{d_1}{3l_0 d_2} + d_4,$$

where

$$d_0 = -\frac{l_0 + l}{3l_0}, \quad d_1 = 2^{1/3}(l_0 + l)^2,$$
$$d_2 = \big(-2(l_0 + l)^3 + 27l_0^2 l + 3\sqrt{3d_3}\big)^{1/3},$$
$$d_3 = -4l_0^2(l_0 + l)^3 l + 27l_0^4 l^2,$$

and

$$d_4 = \frac{d_2}{32^{1/3} l_0}.$$

(b) Returning to the example given in the introduction, say $l_0 = 0.1$, we obtain $\delta_0 = 0.378827193$, $s_\infty = 0.948747087$, $\delta = 0.913742123$, whereas (2.1) holds, since $0.2405 < 1$. That is, our results apply, whereas the ones using (1.3) cannot.

Remark 2.5. Let us define the majoring sequence $\{w_n\}$ used in [4], [5] (under condition (1.3)):

$$w_{-1} = 0,\ w_0 = c,\ w_1 = c + \eta,\ w_{n+2} = w_{n+1} + \frac{l(w_{n+1} - w_{n-1})(w_{n+1} - t_n)}{1 - l(w_{n+1} - w_0 + w_n)}. \tag{2.34}$$

Note that in general

$$l_0 \leqslant l \tag{2.35}$$

holds, and l/l_0 can be arbitrarily large [3], [5]. In the case $l_0 = l$ we have $t_n = w_n$ $(n \geqslant -1)$. Otherwise:

$$t_n < w_n \quad (n \geqslant 2), \tag{2.36}$$

$$t_{n+1} - t_n \leqslant w_{n+1} - w_n \qquad (n \geqslant 2), \tag{2.37}$$

$$0 \leqslant t^\star - t_n \leqslant w^\star - w_n \quad (n \geqslant 0), \quad w^\star = \lim_{n \longrightarrow \infty} w_n. \tag{2.38}$$

Note also that strict inequality holds in (2.36) for $n \geqslant 1$ if $l_0 < l$.

The proof of (2.36)–(2.38) can be found in [5]. Note that the only difference in the proofs is that the conditions of Lemma 2.1 are used here instead of the ones in [4]. However, this makes no difference between the proofs.

3. Special cases and applications

We shall consider Newton's method

$$x_{n+1} = x_n - F'(x_n)^{-1}F(x_n) \quad (n \geqslant 0), \quad (x_0 \in \mathcal{D}) \tag{3.1}$$

as a special case of the Secant method (1.2).

We need a result similar to Lemma 2.1 for Newton's method (3.1).

Lemma 3.1. *Let $l_0 > 0$, $l > 0$, $\eta > 0$ be given parameters.*
Assume

$$2h_0 = b\eta \leqslant 1, \tag{3.2}$$

where

$$b = \frac{l + 4l_0 + \sqrt{l^2 + 8l_0 l}}{4}.$$

Then the scalar sequence $\{t_n\}$ $(n \geqslant 0)$ given by

$$t_0 = 0, \quad t_1 = \eta, \quad t_{n+2} = t_{n+1} + \frac{l(t_{n+1} - t_n)^2}{2(1 - l_0 t_{n+1})} \tag{3.3}$$

is non-decreasing, bounded above by

$$t^{\star\star} = \frac{2\eta}{2 - \delta}, \tag{3.4}$$

where

$$\delta = \frac{4l}{l + \sqrt{l^2 + 8l_0 l}}, \tag{3.5}$$

and converges to some $t^\star \in [0, t^{\star\star}]$.

Moreover, the following estimates hold for all $n \geqslant 0$*:*

$$0 < t_{n+2} - t_{n+1} \leqslant \frac{\delta}{2}(t_{n+1} - t_n) \leqslant \ldots \leqslant \Big(\frac{\delta}{2}\Big)^{n+1}\eta. \tag{3.6}$$

P r o o f. We follow the proof of Lemma 2.1.

The estimate corresponding to (2.10) is given by

$$\Big(l\Big(1-\frac{\delta}{2}\Big)\Big(\frac{\delta}{2}\Big)^k + \delta l_0\Big(1 - \Big(\frac{\delta}{2}\Big)^{k+1}\Big)\Big)\eta \leqslant \delta\Big(1-\frac{\delta}{2}\Big), \tag{3.7}$$

which leads to the definition of functions

$$f(s) = (2l_0 s^2 + l_0 s - l), \tag{3.8}$$

$$f_k(s) = (l s^{k-1} + 2l_0(1 + s + s^2 + \ldots + s^k))\eta - 2 \quad (k \geqslant 1), \tag{3.9}$$

which implies that

$$f_{k+1}(s) = f(s)s^{k-1}\eta + f_k(s). \tag{3.10}$$

Then we can set

$$\delta_0 = \frac{l\eta}{1 - l_0\eta}, \quad s_\infty = 1 - l_0\eta.$$

It is then simple algebra to show that conditions (2.1), and (2.3) reduce to (3.2).

That completes the proof of Lemma 3.1. □

In the next result we provide more estimates on the distances $t_{n+1} - t_n$ and $t^\star - t_n$ ($n \geqslant 0$):

Proposition 3.2. *Under the hypotheses of Lemma* 3.1, *the following estimates hold for all* $n \geqslant 0$*:*

$$t_{n+1} - t_n \leqslant \Big(\frac{\delta}{2}\Big)^n (2h_0)^{2^n - 1}\eta \tag{3.11}$$

and

$$t^\star - t_n \leqslant \Big(\frac{\delta}{2}\Big)^n \frac{(2h_0)^{2^n-1}\eta}{1 - (2h_0)^{2^n}} \quad (2h_0 < 1). \tag{3.12}$$

P r o o f. In order to show (3.11) we need the estimate

$$\frac{1-\left(\frac{\delta}{2}\right)^{k+1}}{1-\frac{\delta}{2}}\eta \leqslant \frac{1}{l}\left(1-\left(\frac{\delta}{2}\right)^{k-1}\frac{l}{2b}\right) \qquad (k \geqslant 1). \tag{3.13}$$

For $k=1$, (3.13) becomes

$$\left(1+\frac{\delta}{2}\right)\eta \leqslant \frac{2b-l}{2bl_0}$$

or

$$\left(1+\frac{2l}{l+\sqrt{l^2+8l_0l}}\right)\eta \leqslant \frac{4l_0-l+\sqrt{l^2+8l_0l}}{l_0(4l_0+l+\sqrt{l^2+8l_0l})}.$$

In view of (3.2), it suffices to show that

$$\frac{l_0(4l_0+l+\sqrt{l^2+8l_0l})(3l+\sqrt{l^2+8l_0l})}{(l+\sqrt{l^2+8l_0l})(4l_0-l+\sqrt{l^2+8l_0l})} \leqslant b,$$

which is true as equality.

Let us now assume estimate (3.13) is true for all integers smaller or equal to k. We must show (3.13) holds for k being replaced by $k+1$:

$$\frac{1-\left(\frac{\delta}{2}\right)^{k+2}}{1-\frac{\delta}{2}}\eta \leqslant \frac{1}{l_0}\left(1-\left(\frac{\delta}{2}\right)^{k}\frac{l}{2b}\right) \qquad (k \geqslant 1)$$

or

$$\left(1+\frac{\delta}{2}+\left(\frac{\delta}{2}\right)^2+\ldots+\left(\frac{\delta}{2}\right)^{k+1}\right)\eta \leqslant \frac{1}{l_0}\left(1-\left(\frac{\delta}{2}\right)^{k}\frac{l}{2b}\right). \tag{3.14}$$

By the induction hypothesis, to show (3.14) it suffices to prove

$$\frac{1}{l_0}\left(1-\left(\frac{\delta}{2}\right)^{k-1}\frac{l}{2b}\right)+\left(\frac{\delta}{2}\right)^{k+1}\eta \leqslant \frac{1}{l_0}\left(1-\left(\frac{\delta}{2}\right)^{k}\frac{l}{2b}\right)$$

or

$$\left(\frac{\delta}{2}\right)^{k+1}\eta \leqslant \frac{1}{l}\left(\left(\frac{\delta}{2}\right)^{k-1}-\left(\frac{\delta}{2}\right)^{k}\right)\frac{l}{2b}$$

or

$$\delta^2\eta \leqslant \frac{l(2-\delta)}{bl_0}.$$

In view of (3.2) it suffices to show that

$$\frac{bl_0\delta^2}{l(2-\delta)} \leqslant b,$$

which holds as equality by the choice of δ given by (3.5). This completes the induction for estimates (3.13).

We shall show (3.11) using induction on $k \geqslant 0$. The estimate (3.11) is true for $k = 0$ by (3.2), (3.3), (3.5). In order to show the estimate (3.11) for $k = 1$, since $t_2 - t_1 = \frac{1}{2}l(t_1 - t_0)^2/(1 - l_0 t_1)$, it suffices to prove

$$\frac{l\eta^2}{2(1-l_0\eta)} \leqslant 2\delta b\eta^2$$

or

$$\frac{l}{1-l_0\eta} \leqslant \frac{8bl}{l+\sqrt{l^2+8l_0 l}} \quad (\eta \neq 0)$$

or

$$\eta \leqslant \frac{1}{l_0}\Big(1 - \frac{l+\sqrt{l^2+8l_0 l}}{8b}\Big) \quad (l_0 \neq 0, l \neq 0).$$

But by (3.2)

$$\eta \leqslant \frac{4}{l+4l_0+\sqrt{l^2+8l_0 l}}.$$

It then suffices to show

$$\frac{4}{l+4l_0+\sqrt{l^2+8l_0 l}} \leqslant \frac{1}{l_0}\Big(1 - \frac{l+\sqrt{l^2+8l_0 l}}{8b}\Big)$$

or

$$\frac{l+\sqrt{l^2+8l_0 l}}{8b} \leqslant 1 - \frac{4l_0}{l+4l_0+\sqrt{l^2+8l_0 l}}$$

or

$$\frac{l+\sqrt{l^2+8l_0 l}}{8b} \leqslant \frac{l+\sqrt{l^2+8l_0 l}}{l+4l_0+\sqrt{l^2+8l_0 l}},$$

which is true by (3.2), as the equality.

Let us assume (3.14) holds for all integers smaller or equal to k. We shall show (3.14) holds for k replaced by $k+1$.

Using (3.2) and the induction hypothesis, we have in turn

$$
\begin{aligned}
t_{k+2}-t_{k+1} &= \frac{l}{2(1-l_0t_{k+1})}(t_{k+1}-t_k)^2 \\
&\leqslant \frac{l}{2(1-l_0t_{k+1})}\Big(\Big(\frac{\delta}{2}\Big)^k(2h_0)^{2^k-1}\eta\Big)^2 \\
&\leqslant \frac{l}{2(1-l_0t_{k+1})}\Big(\Big(\frac{\delta}{2}\Big)^{k-1}(2h_0)^{-1}\eta\Big)\Big(\Big(\frac{\delta}{2}\Big)^{k+1}(2h_0)^{2^{k+1}-1}\eta\Big) \\
&\leqslant \Big(\frac{\delta}{2}\Big)^{k+1}(2h_0)^{2^{k+1}-1}\eta,
\end{aligned}
$$

since

$$
\frac{l}{2(1-l_0t_{k+1})}\Big(\Big(\frac{\delta}{2}\Big)^{k-1}(2h_0)^{-1}\eta\Big)\leqslant 1 \qquad (k\geqslant 1). \tag{3.15}
$$

Indeed, we can show instead of (3.15) that

$$
t_{k+1}\leqslant \frac{1}{l_0}\Big(1-\Big(\frac{\delta}{2}\Big)^{k-1}\frac{l}{2b}\Big),
$$

which is true, since (3.6) and the induction hypothesis imply

$$
\begin{aligned}
t_{k+1} &\leqslant t_k+\frac{\delta}{2}(t_k-t_{k-1}) \\
&\leqslant t_1+\frac{\delta}{2}(t_1-t_0)+\ldots+\frac{\delta}{2}(t_k-t_{k-1}) \\
&\leqslant \eta+\Big(\frac{\delta}{2}\Big)\eta+\ldots+\Big(\frac{\delta}{2}\Big)^k\eta=\frac{1-\Big(\frac{\delta}{2}\Big)^{k+1}}{1-\frac{\delta}{2}}\eta\leqslant\frac{1}{l_0}\Big(1-\Big(\frac{\delta}{2}\Big)^{k-1}\frac{l}{2b}\Big).
\end{aligned}
$$

This completes the induction for the estimate (3.12).

Using the estimate (3.14) for $j\geqslant k$, we obtain in turn

$$
\begin{aligned}
t_{j+1}-t_k &= (t_{j+1}-t_j)+(t_j-t_{j-1})+\ldots+(t_{k+1}-t_k) \\
&\leqslant \Big(\Big(\frac{\delta}{2}\Big)^j(2h_0)^{2^j-1}+\Big(\frac{\delta}{2}\Big)^{j-1}(2h_0)^{2^{j-1}-1}+\ldots+\Big(\frac{\delta}{2}\Big)^k(2h_0)^{2^k-1}\Big)\eta \\
&\leqslant \Big(1+(2h_0)^{2^k}+\Big((2h_0)^{2^k}\Big)^2+\ldots\Big)\Big(\frac{\delta}{2}\Big)^k(2h_0)^{2^k-1}\eta \\
&= \Big(\frac{\delta}{2}\Big)^k\frac{(2h_0)^{2^k-1}\eta}{1-(2h_0)^{2^k}}.
\end{aligned} \tag{3.16}
$$

The estimate (3.12) follows from (3.16) by letting $j\longrightarrow\infty$.

This completes the proof of Proposition 3.2. □

We shall study Newton's method (3.1) for couples (F, x_0) belonging to a class $\mathcal{C}(l, l_0, \eta, \delta)$ defined as follows (see also the corresponding Definition 2.2).

Definition 3.3. Let l, l_0, η, δ be non-negative parameters satisfying the hypotheses of Lemma 3.1.

We say that a triplet (F, x_{-1}, x_0) belongs to the class $\mathcal{C}(l, l_0, \eta, \delta)$ if

(h$_1$) F is a nonlinear operator defined on a convex subset $\mathcal{D}$ of a Banach space $\mathcal{X}$ with values in a Banach space $\mathcal{Y}$;

(h$_2$) F is Fréchet-differentiable on the interior $\mathcal{D}^0$ of $\mathcal{D}$, and there exists $x_0 \in \mathcal{D}$ such that

the linear operator $A = F'(x_0)$ is invertible, its inverse A^{-1} is bounded,

$$\|A^{-1}F(x_0)\| \leqslant \eta;$$
$$\|A^{-1}[F'(x) - F'(x_0)]\| \leqslant l_0\|x - x_0\|;$$

and

$$\|A^{-1}[F'(x) - F'(y)]\| \leqslant l\|x - y\|$$

for all $x, y \in \mathcal{D}$;

(h$_3$) the set $\mathcal{D}_c = \{x \in \mathcal{D};\ F \text{ is continuous at } x\}$ contains the closed ball $\overline{U}(x_0, t^\star)$, where $t^\star$ is given in Lemma 3.1.

We present the semilocal convergence theorem for Newton's method (3.1):

Theorem 3.4. *If $(F, x_0) \in \mathcal{C}(l, l_0, \eta, \delta)$, then the sequence $\{x_n\}$ $(n \geqslant 0)$ generated by Newton's method* (3.1) *is well defined, remains in $\overline{U}(x_0, t^\star)$ for all $n \geqslant 0$ and converges to a unique solution $x^\star \in \overline{U}(x_0, t^\star)$ of the equation $F(x) = 0$.*

Moreover, the following estimates hold for all $n \geqslant 0$:

$$\|x_{n+1} - x_n\| \leqslant t_{n+1} - t_n,$$

and

$$\|x_n - x^\star\| \leqslant t^\star - t_n$$

where the sequence $\{t_n\}$ $(n \geqslant 0)$ is given by (3.3).

Furthermore, if there exists $R > t^\star$ such that

$$U(x_0, R) \subseteq \mathcal{D}$$

and

$$l_0(t^\star + R) \leqslant 2,$$

then the solution $x^\star$ is unique in $U(x_0, R)$.

P r o o f. The proof as being identical to that of Theorem 1 in [6] is omitted. Note that in [6], we simply used sufficient convergence conditions different from the ones in Lemma 3.1. This is the only difference between the proofs. □

Remark 3.5. The famous for its simplicity and clarity Newton-Kantorovich hypothesis corresponding to (3.2) is given in [3], [5], [13]:

$$2h = 2l\eta \leqslant 1. \tag{3.17}$$

It then follows from (3.2) and (3.17) that

$$h \leqslant \frac{1}{2} \Longrightarrow h_0 \leqslant \frac{1}{2}$$

but not necessarily vice versa, unless $l = l_0$. Comments similar to the ones in Remark 2.5 can be made for Newton's method.

We complete this study with three numerical examples.

Example 3.6. Define the scalar function F by $F(x) = c_0 x + c_1 + c_2 \sin e^{c_3 x}$, $x_0 = 0$, where c_i, $i = 1, 2, 3$ are given parameters. Then it can easily be seen that for c_3 large and c_2 sufficiently small, l/l_0 can be arbitrarily large. That is, (3.2) may be satisfied but not (3.17).

Example 3.7. Let $\mathcal{X} = \mathcal{Y} = \mathbb{R}$, $x_0 = 1$, $U_0 = \{x\colon\ |x - x_0| \leqslant 1 - \gamma\}$, $\gamma \in [0, \frac{1}{2})$, and define the function F on U_0 by

$$F(x) = x^3 - \gamma. \tag{3.18}$$

Using condition (h_2) of Definition 3.3, we get

$$\eta = \frac{1}{3}(1 - \gamma), \quad l_0 = 3 - \gamma, \quad \text{and} \quad l = 2\ (2 - \gamma).$$

The Kantorovich condition (3.17) is violated, since

$$\frac{4}{3}(1 - \gamma)(2 - \gamma) > 1 \quad \text{for all} \quad \gamma \in \Big[0, \frac{1}{2}\Big).$$

Hence, there is no guarantee that Newton's method (1.2) converges to $x^* = \sqrt[3]{\gamma}$, starting at $x_0 = 1$.

However, our condition (3.2) is true for all $\gamma \in I = [.450339002, \frac{1}{2})$. Hence, the conclusions of our Theorem 3.4 can be applied to solve equation (3.18) for all $\gamma \in I$.

Example 3.8. Let $\mathcal{X} = \mathcal{Y} = \mathcal{C}[0,1]$ be the space of real-valued continuous functions defined on the interval $[0,1]$ with the norm

$$\|x\| = \max_{0\leqslant s\leqslant 1} |x(s)|.$$

Let $\theta \in [0,1]$ be a given parameter. Consider the "cubic" integral equation

$$u(s) = u^3(s) + \lambda u(s) \int_0^1 q(s,t)u(t)\,\mathrm{d}t + y(s) - \theta. \tag{3.19}$$

Here the kernel $q(s,t)$ is a continuous function of two variables defined on $[0,1] \times [0,1]$; the parameter λ is a real number called the "albedo" for scattering; $y(s)$ is a given continuous function defined on $[0,1]$ and $x(s)$ is the unknown function sought in $\mathcal{C}[0,1]$. Equations of the form (3.19) arise for gasses [5], [8]. For simplicity, we choose $u_0(s) = y(s) = 1$, and $q(s,t) = s/(s+t)$ for all $s \in [0,1]$ and $t \in [0,1]$, with $s+t \neq 0$. If we let $\mathcal{D} = U(u_0, 1-\theta)$ and define the operator F on $\mathcal{D}$ by

$$F(x)(s) = x^3(s) - x(s) + \lambda x(s) \int_0^1 q(s,t)x(t)\,\mathrm{d}t + y(s) - \theta \tag{3.20}$$

for all $s \in [0,1]$, then every zero of F satisfies equation (3.19). We have the estimate

$$\max_{0\leqslant s\leqslant 1} \left| \int \frac{s}{s+t}\,\mathrm{d}t \right| = \ln 2.$$

Therefore, if we set $\xi = \|F'(u_0)^{-1}\|$, then it follows from condition (h_2) of Definition 3.3 that

$$\eta = \xi(|\lambda| \ln 2 + 1 - \theta),$$

$$l = 2\xi(|\lambda| \ln 2 + 3(2-\theta)) \quad \text{and} \quad l_0 = \xi(2|\lambda| \ln 2 + 3(3-\theta)).$$

It follows from Theorem 3.4 that if condition (3.2) holds, then problem (3.19) has a unique solution near u_0. This assumption is weaker than the one given before using the Newton-Kantorovich hypothesis (3.17).

Note also that $l_0 < l$ for all $\theta \in [0,1]$.

Example 3.9. Consider the nonlinear boundary value problem [5]

$$\begin{cases} u'' = -u^3 - \gamma u^2, \\ u(0) = 0, \quad u(1) = 1. \end{cases}$$

It is well known that this problem can be formulated as the integral equation

$$u(s) = s + \int_0^1 Q(s,t)(u^3(t) + \gamma u^2(t))\,\mathrm{d}t, \tag{3.21}$$

where Q is the Green function

$$Q(s,t) = \begin{cases} t(1-s), & t \leqslant s, \\ s(1-t), & s < t. \end{cases}$$

We observe that

$$\max_{0\leqslant s\leqslant 1} \int_0^1 |Q(s,t)| = \frac{1}{8}.$$

Let $\mathcal{X} = \mathcal{Y} = \mathcal{C}[0,1]$ with the norm

$$\|x\| = \max_{0\leqslant s\leqslant 1} |x(s)|.$$

Then problem (3.21) is in the form (1.1), where, $F\colon\ \mathcal{D} \longrightarrow \mathcal{Y}$ is defined as

$$[F(x)](s) = x(s) - s - \int_0^1 Q(s,t)(x^3(t) + \gamma x^2(t))\,\mathrm{d}t.$$

It is easy to verify that the Fréchet derivative of F is defined in the form

$$[F'(x)v](s) = v(s) - \int_0^1 Q(s,t)(3x^2(t) + 2\gamma x(t))v(t)\,\mathrm{d}t.$$

If we set $u_0(s) = s$ and $\mathcal{D} = U(u_0, R)$, then since $\|u_0\| = 1$, it is easy to verify that $U(u_0, R) \subset U(0, R+1)$. It follows that $2\gamma < 5$, hence

$$\begin{aligned} \|I - F'(u_0)\| &\leqslant \frac{3\|u_0\|^2 + 2\gamma\|u_0\|}{8} = \frac{3+2\gamma}{8}, \\ \|F'(u_0)^{-1}\| &\leqslant \frac{1}{1 - \dfrac{3+2\gamma}{8}} = \frac{8}{5-2\gamma}, \\ \|F(u_0)\| &\leqslant \frac{\|u_0\|^3 + \gamma\|u_0\|^2}{8} = \frac{1+\gamma}{8}, \\ \|F(u_0)^{-1}F(u_0)\| &\leqslant \frac{1+\gamma}{5-2\gamma}. \end{aligned}$$

On the other hand, for $x, y \in \mathcal{D}$ we have

$$[(F'(x) - F'(y))v](s) = -\int_0^1 Q(s,t)(3x^2(t) - 3y^2(t) + 2\gamma(x(t) - y(t)))v(t)\,\mathrm{d}t.$$

Consequently,

$$
\begin{aligned}
\|F'(x) - F'(y)\| &\leqslant \frac{\|x-y\|(2\gamma + 3(\|x\| + \|y\|))}{8} \\
&\leqslant \frac{\|x-y\|(2\gamma + 6R + 6\|u_0\|)}{8} \\
&= \frac{\gamma + 6R + 3}{4}\|x-y\|, \\
\|F'(x) - F'(u_0)\| &\leqslant \frac{\|x-u_0\|(2\gamma + 3(\|x\| + \|u_0\|))}{8} \\
&\leqslant \frac{\|x-u_0\|(2\gamma + 3R + 6\|u_0\|)}{8} \\
&= \frac{2\gamma + 3R + 6}{8}\|x-u_0\|.
\end{aligned}
$$

Therefore, the conditions of Theorem 3.4 hold with

$$
\eta = \frac{1+\gamma}{5-2\gamma}, \quad l = \frac{\gamma + 6R + 3}{4}, \quad l_0 = \frac{2\gamma + 3R + 6}{8}.
$$

Note also that $l_0 < l$.

Conclusion

We provided a semilocal convergence analysis for the Secant and Newton's methods in order to approximate a locally unique solution of an equation in a Banach space.

Using a combination of Lipschitz and center-Lipschitz conditions, instead of only Lipschitz conditions [13], we provided an analysis with the following advantages over the work in [13]: larger convergence domain, and weaker sufficient convergence conditions. Note that these advantages are obtained under the same computational cost as in [13], since in practice the computation of the Lipschitz constant l requires the computation of l_0. Hence, the applicability of these methods has been extended. Numerical examples further validating the results are also provided.

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