Valentin Gutev

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WEAK SELECTIONS AND WEAK ORDERABILITY OF FUNCTION SPACES

Valentin Gutev, Durban

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Abstract. It is proved that for a zero-dimensional space X, the function space $C_p(X,2)$ has a Vietoris continuous selection for its hyperspace of at most 2-point sets if and only if X is separable. This provides the complete affirmative solution to a question posed by Tamariz-Mascarúa. It is also obtained that for a strongly zero-dimensional metrizable space E, the function space $C_p(X,E)$ is weakly orderable if and only if its hyperspace of at most 2-point sets has a Vietoris continuous selection. This provides a partial positive answer to a question posed by van Mill and Wattel.

Keywords: Vietoris hyperspace, continuous selection, function space, weakly orderable space

MSC 2010: 54B20, 54C35, 54C65, 54F05

1. Introduction

Let Y be a topological space, and let $\mathscr{F}(Y)$ be the set of all nonempty closed subsets of Y. Also, let $\mathscr{D} \subset \mathscr{F}(Y)$. A map $\varphi\colon \mathscr{D} \to Y$ is a *selection* for $\mathscr{D}$ if $\varphi(S) \in S$ for every $S \in \mathscr{D}$. A selection $\varphi\colon \mathscr{D} \to Y$ is *continuous* if it is continuous with respect to the relative Vietoris topology τ_V on $\mathscr{D}$. Let us recall that τ_V is generated by all collections of the form

$$\langle \mathscr{V} \rangle = \Big\{ S \in \mathscr{F}(Y)\colon\ S \subset \bigcup \mathscr{V} \ \text{ and } \ S \cap V \neq \emptyset \text{ whenever } V \in \mathscr{V} \Big\},$$

where $\mathscr{V}$ runs over the finite families of open subsets of Y.

In the sequel, all spaces are assumed to be at least Hausdorff. In this paper we are interested in continuous selections for $\mathscr{D}$, when $\mathscr{D}$ is the family

$$\mathscr{F}_2(Y) = \{S \in \mathscr{F}(Y)\colon\ |S| \leqslant 2\}$$

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of at most 2-point subsets of Y. Suppose that $\varphi\colon \mathscr{F}_2(Y) \to Y$ is a selection. Then it introduces a natural order-like relation $\preceq_\varphi$ on Y [7] by defining that $y \preceq_\varphi z$ if and only if $\varphi(\{y,z\}) = y$. For convenience, we write $y \prec_\varphi z$ if $y \preceq_\varphi z$ and $y \neq z$. This relation is very similar to a linear order on Y in that it is both total and antisymmetric, but, unfortunately, it may fail to be transitive. In this regard, one of the fundamental questions in the theory of continuous selections for at most 2-point subsets is the following one.

Question 1 (van Mill and Wattel, [8]). Let Y be a space which has a continuous selection for $\mathscr{F}_2(Y)$. Does there exist a linear order $\preceq$ on Y such that, for each $y \in Y$, the sets $\{z \in Y\colon\ z \preceq y\}$ and $\{z \in Y\colon\ y \preceq z\}$ are both closed?

Recall that a space Y is *orderable* (or, *linearly orderable*) if the topology of Y coincides with the open interval topology on Y generated by a linear ordering on Y. Following [8], we say that a space Y is *weakly orderable* if there exists a coarser orderable topology on Y. In this terminology, Question 1 states the conjecture whether a space Y is weakly orderable provided that it has a continuous selection for $\mathscr{F}_2(Y)$. In view of that, a selection $\varphi\colon \mathscr{F}_2(Y) \to Y$ is often called a *weak selection* for Y.

Recently, Michael Hrušák and Iván Martínez-Ruiz solved Question 1 in the negative by constructing a separable, first countable locally compact space which admits a continuous weak selection but is not weakly orderable [6]. For a detailed discussion on several classes of spaces where Question 1 was resolved in the affirmative, we refer the interested reader to [5].

In the present paper we are interested in continuous weak selections for spaces of continuous functions. To this end, for sets X and E, we will use E^X to denote the set of all maps from X to E. If E is a topological space, then E^X will be always endowed with the Tychonoff product topology. For spaces X and E we will use $C_p(X,E)$ to denote the space of all continuous maps $f\colon\ X \to E$ equipped with the pointwise convergence topology, i.e. with the topology inherited from the product space E^X. In case E is the real line $\mathbb{R}$, as usual, we write $C_p(X)$ instead of $C_p(X,\mathbb{R})$. It should be mentioned that $C_p(X)$ is a dense linear subspace of $\mathbb{R}^X$; consequently, it inherits many properties of $\mathbb{R}^X$. Several selection properties of $C_p(X,E)$-spaces were investigated in [9]. In particular, it was obtained that $C_p(X)$ has a continuous weak selection if and only if X is a singleton ([9, Proposition 2.1]), also that if X is zero-dimensional and E is a strongly zero-dimensional metrizable space, then $C_p(X,E)$ is weakly orderable if and only if X is separable ([9, Theorem 4.5]). Here E is *strongly zero-dimensional* if its covering dimension is zero, while X is *zero-dimensional* if it has a base of clopen sets.

On this basis, the following question was posed in [9].

Question 2 ([9]). Let X be a zero-dimensional space. Is it true that X is separable provided $C_p(X,2)$ has a continuous weak selection?

We are now ready to state also the main purpose of this paper. Namely, in this paper we provide the complete affirmative solution to Question 2, see Theorem 3.2 and Corollary 3.3. Also, we demonstrate that for a strongly zero-dimensional metrizable space E, the function space $C_p(X,E)$ is weakly orderable if and only if it has a continuous weak selection, Theorem 4.1. This generalizes [9, Theorem 4.5] and provides a further partial positive answer to Question 1.

For the proper understanding of our results, let us emphasise that all they involve in one or another way the requirement that the function space $C_p(X,E)$ is dense in the product space E^X. This requirement is quite natural in view of the $C_p(X)$-spaces, also when it comes to deriving properties of X by means of properties of $C_p(X,E)$. In the realm of such function spaces, we demonstrate that if $C_p(X,E)$ has a continuous weak selection, then both X and E must be totally disconnected, Theorem 2.1. That is, the natural environment for continuous weak selections for $C_p(X,E)$-spaces is when X and E are both totally disconnected.

2. Function spaces and continuous weak selections

Throughout this section and in the sequel, all spaces are assumed to have at least 2 distinct points. The following theorem will be proved.

Theorem 2.1. *Let X and E be spaces such that $C_p(X,E)$ is dense in E^X and has a continuous weak selection. Then both X and E must be totally disconnected.*

To prepare for the proof of Theorem 2.1, let us recall some terminology. For a space Y, a weak selection φ for Y, and points $y,z\in Y$, define the following $\preceq_\varphi$-*intervals*:

$$(y,z)_{\preceq_\varphi}=\{t\in Y:\ y\prec_\varphi t\prec_\varphi z\},\quad \text{and}$$
$$[y,z]_{\preceq_\varphi}=\{t\in Y:\ y\preceq_\varphi t\preceq_\varphi z\},$$

where $\preceq_\varphi$ is the order-like relation generated by φ. Since $\preceq_\varphi$ is not necessarily transitive, both the intervals $(y,z)_{\preceq_\varphi}$ and $(z,y)_{\preceq_\varphi}$ can be nonempty, similarly for $[y,z]_{\preceq_\varphi}$ and $[z,y]_{\preceq_\varphi}$. Let us explicitly mention that if φ is continuous, then both the $\preceq_\varphi$-open intervals $(y,z)_{\preceq_\varphi}$ and $(z,y)_{\preceq_\varphi}$ are open in Y, while the corresponding $\preceq_\varphi$-closed intervals $[y,z]_{\preceq_\varphi}$ and $[z,y]_{\preceq_\varphi}$ are closed in Y, see [7]. We will freely rely on this fact.

For a space E and $y \in E$, we will use $\mathscr{C}[y]$ to denote the *component* of the point y in E, and $\mathscr{C}^*[y]$—the corresponding *quasi-component*, i.e.

$$\mathscr{C}[y] = \bigcup\{C \subset E\colon\ y \in C \text{ and } C \text{ is connected}\}, \quad \text{and}$$
$$\mathscr{C}^*[y] = \bigcap\{C \subset E\colon\ y \in C \text{ and } C \text{ is clopen}\}.$$

Note that E is totally disconnected if and only if $\mathscr{C}^*[y] = \{y\}$ for every $y \in E$.

Finally, let us recall that a canonical open set of $C_p(X, E)$ is of the form

$$\{f \in C_p(X, E)\colon\ f(x_k) \in U_k \text{ for every } k \leqslant n\},$$

where $n < \omega$ and, for each $k \leqslant n$, $x_k \in X$ and U_k is an open subset of E.

Lemma 2.2. *Let X and E be spaces such that $C_p(X, E)$ is dense in E^X and has a continuous weak selection. Then E is totally disconnected.*

P r o o f. Identifying each point $y \in E$ with the constant map $y(x) = y$, $x \in X$, we have that E is naturally embedded in $C_p(X, E)$. Consequently, E has a continuous weak selection because so does $C_p(X, E)$. To show that E is totally disconnected, we have to showing that all quasi-components of E are singletons. Since E has a continuous weak selection, by [4, Theorem 4.1], $\mathscr{C}[y] = \mathscr{C}^*[y]$ for every $y \in E$. Hence, our proof is reduced to show that each connected component of E is a singleton. Suppose that this fails, i.e. that there exists a point $y \in E$ such that its connected component $\mathscr{C}[y]$ in E is not a singleton. Also, let φ be a continuous weak selection for $C_p(X, E)$, and let $\preceq_\varphi$ be the order-like relation generated by φ. Take two distinct points $y_1, y_2 \in \mathscr{C}[y]$, say $y_1 \prec_\varphi y_2$. Then, by [2, Lemma 2.5], the $\preceq_\varphi$-interval $(y_1, y_2)_{\preceq_\varphi}$ is a nonempty open subset of $C_p(X, E)$ and $(y_1, y_2)_{\preceq_\varphi} \subset \mathscr{C}[y] \subset E$. That is,

$$f \text{ is constant for every } f \in (y_1, y_2)_{\preceq_\varphi}. \tag{2.1}$$

Take a point $z \in \mathscr{C}[y]$ such that the constant map $z(x) = z$, $x \in X$, corresponding to this point is in $(y_1, y_2)_{\preceq_\varphi}$. Since $(y_1, y_2)_{\preceq_\varphi}$ is open in $C_p(X, E)$ and z is constant, there now exists an open set $U \subset (y_1, y_2)_{\preceq_\varphi}$ with $z \in U$, and a finite set $F \subset X$ with $|F| \geqslant 2$, such that

$$\mathscr{U} = \{f \in C_p(X, E)\colon\ f(F) \subset U\} \subset (y_1, y_2)_{\preceq_\varphi}.$$

According to [2, Lemma 2.5] once again, the $\preceq_\varphi$-interval $[y_1, y_2]_{\preceq_\varphi}$ is a connected subset of $C_p(X, E)$, while U is open in $[y_1, y_2]_{\preceq_\varphi}$ being open in $(y_1, y_2)_{\preceq_\varphi}$. Consequently, U has no isolated points. Thus U is infinite, while F is a finite set. Hence, for every $x \in F$ there exists a nonempty open subset $U_x \subset U$ such that the family $\{U_x\colon\ x \in F\}$ is pairwise disjoint. Consider the set

$$\Omega = \{h \in E^X\colon\ h(x) \in U_x,\ \text{whenever } x \in F\}.$$

Then Ω is a nonempty open subset of E^X, hence $\Omega \cap C_p(X,E) \neq \emptyset$ because $C_p(X,E)$ is dense in E^X. Take an $f \in \Omega \cap C_p(X,E)$. Then $f(x) \in U_x \subset U$ for every $x \in F$ and, therefore, $f \in \mathscr{U} \subset (y_1, y_2)_{\preceq_\varphi}$. Since $|F| \geqslant 2$ and $\{U_x\colon\ x \in F\}$ is pairwise disjoint, f must have at least 2 distinct values. However, according to (2.1), f must be a constant map. The contradiction so obtained demonstrates that the components of E must be singletons. □

We conclude the preparation for the proof of Theorem 2.1 with the following simple observation; its verification is left to the reader.

Proposition 2.3. *Let X be a space, and let E be a totally disconnected space. Then X is totally disconnected provided $C_p(X,E)$ is dense in E^X.*

Proof of Theorem 2.1. Let X and E be as in that theorem. Then, by Lemma 2.2, E must be totally disconnected which, by Proposition 2.3, implies that X must be totally disconnected as well. □

3. Separability and continuous weak selections

In the present section we provide the complete affirmative solution to Question 2. To prepare for this, suppose that φ is a weak selection for Y. Following [1], for subsets $B, C \subset Y$, we write that $B \preceq_\varphi C$ (respectively, $B \prec_\varphi C$) if $y \preceq_\varphi z$ (respectively, $y \prec_\varphi z$) for every $y \in B$ and $z \in C$. Obviously, $B \prec_\varphi C$ implies $B \cap C = \emptyset$. For some other properties of this relation, see [1].

Here we are interested in weak selections for $C_p(X,E)$ when $E = 2 = \{0,1\}$ is the 2-point space. According to [3, Theorem 3.1], if φ is a continuous weak selection for $C_p(X,2)$ and $f, g \in C_p(X,2)$ with $f \prec_\varphi g$, then there are open sets $\mathscr{U}, \mathscr{V} \subset C_p(X,2)$ such that $f \in \mathscr{U}$, $g \in \mathscr{V}$, and $\mathscr{U} \prec_\varphi \mathscr{V}$. However, in this case, a canonical open neighbourhood of an $h \in C_p(X,2)$ is given by

$$\{l \in C_p(X,2)\colon\ l \restriction M = h \restriction M\},$$

where M runs over the finite subsets of X. Hence, we have the following simple criterion for continuity of weak selections for $C_p(X,2)$.

Proposition 3.1. *Let X be a space, φ a continuous weak selection for $C_p(X,2)$, $\preceq_\varphi$ the order-like relation generated by φ, and let $f, g \in C_p(X,2)$ be such that $f \prec_\varphi g$. Then there exists a finite set $M \subset X$ such that if $h, l \in C_p(X,2)$, $h \restriction M = f \restriction M$ and $l \restriction M = g \restriction M$, then $h \prec_\varphi l$.*

Now, we have the following theorem.

Theorem 3.2. *Let X be a zero-dimensional space such that $C_p(X,2)$ has a continuous weak selection. Then X must be separable.*

Proof. Suppose that X is not separable. Then X must be infinite, and being zero-dimensional, it has a sequence $\{U_n\colon n < \omega\}$ of pairwise disjoint nonempty clopen sets. Take an $f \in C_p(X,2)$ and, for every $n < \omega$, define $g_n \in C_p(X,2)$ by letting $g_n \restriction X \setminus U_n = f \restriction X \setminus U_n$ and $g_n(x) = 1 - f(x)$, $x \in U_n$. Then

$$f = \lim_{n\to\infty} g_n. \tag{3.1}$$

Indeed, take a finite set $M \subset X$. Then there is an $m < \omega$ such that $M \cap U_n = \emptyset$ for every $n > m$. Hence, $g_n \restriction M = f \restriction M$ for every $n > m$, which completes the verification.

Now, by hypothesis, $C_p(X,2)$ has a continuous weak selection φ. Let $\preceq_\varphi$ be the order-like relation on $C_p(X,2)$ generated by φ. Since $f \neq g_0$, by Proposition 3.1 there exists a finite set $M_0 \subset X$ such that if $h, l \in C_p(X,2)$, $h \restriction M_0 = f \restriction M_0$ and $l \restriction M_0 = g_0 \restriction M_0$, then $h \prec_\varphi l$ if $f \prec_\varphi g_0$ and $l \prec_\varphi h$ if $g_0 \prec_\varphi f$. Note that $M_0 \cap U_0 \neq \emptyset$ because, by construction, $g_0 \restriction X \setminus U_0 = f \restriction X \setminus U_0$. Set $\alpha(0) = 0$, and let

$$\begin{aligned}\alpha(1) &= \min\{k < \omega\colon\ M_0 \cap U_k = \emptyset\}\\ &= \min\{\alpha(0) < k < \omega\colon\ M_0 \cap U_k = \emptyset\}.\end{aligned}$$

Since $f \neq g_{\alpha(1)}$, just like before, Proposition 3.1 implies the existence of a finite set M_1 such that if $h, l \in C_p(X,2)$, $h \restriction M_1 = f \restriction M_1$ and $l \restriction M_1 = g_{\alpha(1)} \restriction M_1$, then $h \prec_\varphi l$ if $f \prec_\varphi g_{\alpha(1)}$ and $l \prec_\varphi h$ if $g_{\alpha(1)} \prec_\varphi f$. Again, we have that $M_1 \cap U_{\alpha(1)} \neq \emptyset$. Thus, by induction, we get a sequence $\{M_n\colon n < \omega\}$ of nonempty finite subsets of X and an increasing sequence $\{\alpha(n)\colon n < \omega\}$ of natural numbers such that for every $n < \omega$,

(a) $\alpha(n+1) = \min\{\alpha(n) < k < \omega\colon\ M_n \cap U_k = \emptyset\}$,

(b) if $h, l \in C_p(X,2)$, $h \restriction M_n = f \restriction M_n$ and $l \restriction M_n = g_{\alpha(n)} \restriction M_n$, then $h \prec_\varphi l$ if $f \prec_\varphi g_{\alpha(n)}$ and $l \prec_\varphi h$ if $g_{\alpha(n)} \prec_\varphi f$.

Let $M = \bigcup\{M_n\colon n < \omega\}$, and let $Y = \overline{M}$. Since X is not separable, we have that $X \setminus Y \neq \emptyset$, hence there exists a nonempty clopen subset O with $O \cap Y = \emptyset$. Define $h \in C_p(X,2)$ by $h \restriction X \setminus O = f \restriction X \setminus O$, and $h(x) = 1 - f(x)$, $x \in O$. Also, for every

$n < \omega$, define $h_n \restriction X \setminus O = g_{\alpha(n)} \restriction X \setminus O$ and $h_n(x) = 1 - f(x)$, $x \in O$. Thus, we get a sequence $\{h_n \in C_p(X, 2)\colon\ n < \omega\}$ which, by (3.1), is convergent to h, and clearly $h \neq f$. Suppose that $f \prec_\varphi h$, the other case being symmetric. Then there exists an $m < \omega$ such that $f \prec_\varphi h_n$ for every $n > m$, because $\lim\limits_{n\to\infty} \varphi(\{f, h_n\}) = \varphi(\{f, h\})$. Since $O \cap M_n = \emptyset$ for every $n < \omega$, by (b) and the definition of the maps h_n's we now have that

$$f \prec_\varphi g_{\alpha(n)} \quad \text{whenever } n > m.$$

On the other hand, $h_{n+1} \restriction M_n = f \restriction M_n$ because $h_{n+1}(x) = g_{\alpha(n+1)}(x) = f(x)$ for $x \in Y \setminus U_{\alpha(n+1)}$ and, by (a), $M_n \cap U_{\alpha(n+1)} = \emptyset$. Hence, by (b), we also have that $h_{n+1} \prec_\varphi g_{\alpha(n)}$ for every $n > m$. Since φ is continuous, this finally implies that

$$h = \lim_{n\to\infty} h_{n+1} \preceq_\varphi \lim_{n\to\infty} g_{\alpha(n)} = f.$$

However, $h \neq f$, so $h \prec_\varphi f$. Thus, we get that $h \prec_\varphi f$ and $f \prec_\varphi h$, which is clearly impossible. The contradiction so obtained completes the proof. □

Corollary 3.3. *Let X be a zero-dimensional space, and let E be a space such that $C_p(X, E)$ has a continuous weak selection. Then X must be separable.*

P r o o f. Since E has at least 2 distinct points (recall that we consider only such spaces), $C_p(X, 2)$ is naturally embedded as a (closed) subset of $C_p(X, E)$. Consequently, $C_p(X, 2)$ has a continuous weak selection and, by Theorem 3.2, X must be separable. □

4. Weak orderability and continuous weak selections

In this last section of the paper, we provide a complete characterisation of the weak orderability of function spaces by means of continuous weak selections. In order to state our result, let us recall that the *density* of a space X, denoted by $d(X)$, is the least cardinal δ such that X has a dense subset of cardinality δ. In particular, X is separable if $d(X) \leqslant \omega$. On the other hand, if A is a dense subset of X, then $A \cap U \neq \emptyset$ for every nonempty open $U \subset X$. Here we consider the following related cardinal invariant. By the *clopen density* of a space X, denoted by $d_*(X)$, we will mean the least cardinal δ such that X has a subset A of cardinality δ and with the property that $A \cap G \neq \emptyset$ for every nonempty clopen subset $G \subset X$. Observe that X is connected if and only if $d_*(X) = 1$, while $d_*(X) = d(X)$ for every zero-dimensional space X.

The following theorem will be proved.

Theorem 4.1. *Let E be a strongly zero-dimensional metrizable space. Then for a space X, the following assertions are equivalent:*
(a) *$C_p(X, E)$ has a continuous weak selection.*
(b) *X has a countable clopen density.*
(c) *$C_p(X, E)$ is weakly orderable.*

To prepare for the proof of Theorem 4.1, for a space X consider the equivalence relation $\sim$ on X defined for $x, y \in X$ by $x \sim y$ if and only if x and y cannot be separated by a clopen subset of X, i.e. if $\mathscr{C}^*[x] = \mathscr{C}^*[y]$. Let $X_* = X/\sim$ be the quotient set, and let $q\colon X \to X_*$ be the corresponding quotient map. Then X_* is a totally disconnected space if endowed with the natural quotient topology $\mathscr{T}_\sim$ generated by this equivalence relation. We are interested in considering the coarser topology $\mathscr{T}_*$ on X_* which is generated by the clopen subsets of $\mathscr{T}_\sim$. Thus, $\mathscr{T}_* \subset \mathscr{T}_\sim$, and X_* endowed with this topology is zero-dimensional. In the sequel, we will refer to this resulting space merely as X_*, and will call it the *quasi-clopen-modification* of X.

Suppose that E is a space. Since $q\colon X \to X_*$ is continuous (we have that $\mathscr{T}_* \subset \mathscr{T}_\sim$), it generates a natural map $q^\#\colon C_p(X_*, E) \to C_p(X, E)$, $q^\#(g) = g \circ q$ for $g \in C_p(X_*, E)$, which is clearly continuous and injective. In fact, $q^\#$ is an embedding, which follows easily from the fact that if

$$\mathscr{U}_* = \{g \in C_p(X_*, E)\colon\ g(z_k) \in U_k \text{ for every } k \leqslant n < \omega\}$$

for some open sets U_k, $k \leqslant n$, in E, then $\mathscr{U}_* = (q^\#)^{-1}(\mathscr{U})$, where

$$\mathscr{U} = \{f \in C_p(X, E)\colon\ f(x_k) \in U_k \text{ for every } k \leqslant n < \omega\}$$

for some points $x_k \in X$ with $q(x_k) = z_k$, $k \leqslant n$.

Proposition 4.2. *Let X be a space, E a zero-dimensional space, and let X_* be the quasi-clopen modification of X. Then for every $f \in C_p(X, E)$ there exists an $f_* \in C_p(X_*, E)$ such that $q^\#(f_*) = f_* \circ q = f$. In particular, $q^\#$ is bijective and, hence, a homeomorphism.*

P r o o f. Take an $f \in C_p(X, E)$ and $x, y \in X$, and then observe that $f(x) = f(y)$ provided $x \sim y$. Indeed, if $f(x) \neq f(y)$, then there is a clopen set $U \subset E$ with $f(x) \in U$ and $f(y) \notin U$, because E is zero-dimensional. That is, $x \not\sim y$. Thus, we may define $f_*\colon X_* \to E$ by letting $f_*(q(x)) = f(x)$, $x \in X$. To see that f_* is continuous, take a clopen set $U \subset E$. Then, $G = f^{-1}(U)$ is clopen in X, hence $G = q^{-1}(q(G))$. Consequently, $G_* = q(G)$ is open in X_* and, clearly, $G_* = f_*^{-1}(U)$. Thus, f_* is continuous, and $q^\#$ is surjective. This implies that $q^\#$ is bijective because it is injective. Hence, it is a homeomorphism, which completes the proof. □

We finalize the preparation for the proof of Theorem 4.1 with the following simple observation; its verification is left to the reader.

Proposition 4.3. *Let X be a space, and let X_* be the quasi-clopen modification of X. Then $d(X_*) = d_*(X)$.*

P r o o f o f T h e o r e m 4.1. Let X_* be the quasi-clopen modification of X. Then X_* is zero-dimensional while, by Proposition 4.2, $C_p(X_*, E)$ is homeomorphic to $C_p(X, E)$. To show that (a) $\Rightarrow$ (b), suppose that $C_p(X, E)$ has a continuous weak selection. Then $C_p(X_*, E)$ has a continuous weak selection, so, by Corollary 3.3, X_* must be separable. Hence, by Proposition 4.3, X has a countable clopen density. To show that (b) $\Rightarrow$ (c), take in mind that, by Proposition 4.3, X_* is a separable space. Then, by [9, Corollary 4.2] (see, also, [9, Theorem 4.1]), $C_p(X_*, E)$ is weakly orderable, hence so is $C_p(X, E)$. To see finally that (c) $\Rightarrow$ (a), use [7, Lemma 7.5.1] that every weakly orderable space has a continuous weak selection. □

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Author's address: V a l e n t i n G u t e v, School of Mathematical Sciences, University of KwaZulu-Natal, Westville Campus, Private Bag X54001, Durban 4000, South Africa, e-mail: gutev@ukzn.ac.za.