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A NEW TYPE OF ORTHOGONALITY FOR NORMED PLANES

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Abstract. In this paper we introduce a new type of orthogonality for real normed planes which coincides with usual orthogonality in the Euclidean situation. With the help of this type of orthogonality we derive several characterizations of the Euclidean plane among all normed planes, all of them yielding also characteristic properties of inner product spaces among real normed linear spaces of dimensions $d \geqslant 3$.

Keywords: chordal orthogonality, Feuerbach circle, inner product space, James orthogonality, Minkowski plane, normed linear space, normed plane, orthocentricity, Wallace line

MSC 2010: 46B20, 46C15, 52A21

1. Introduction

Let $(\mathbb{M}, \|\cdot\|)$ be a two-dimensional normed vector space, usually also called a (*normed* or) *Minkowski plane*. Let $\mathcal{C} = \{x \in \mathbb{M} \colon \|x\| = 1\}$ be the *unit circle* of $(\mathbb{M}, \|\cdot\|)$, i.e., a closed convex curve centered at the origin 0. If $\mathcal{C}$ does not contain a non-degenerate line segment, then the normed plane $(\mathbb{M}, \|\cdot\|)$ is called *strictly convex*, and if $\mathcal{C}$ is a smooth curve, then the plane is said to be *smooth*. Any homothetical copy of $\mathcal{C}$ is called a *circle* of $(\mathbb{M}, \|\cdot\|)$ and denoted by C. Denote the *line segment* between two different points $x, y \in (\mathbb{M}, \|\cdot\|)$ by $[x, y]$, the *line* through them by $\langle x, y\rangle$, and the *triangle* with (non-collinear) vertices $x, y, z \in (\mathbb{M}, \|\cdot\|)$ by $\triangle xyz$. Any triangle in a strictly convex Minkowski plane possesses *at most* one circumcircle; see [12, Proposition 14]. In a smooth Minkowski plane every triangle has *at least* one circumcircle; see again [12, Proposition 41].

As one can see in the paper [1], the orthogonality of segments that are, in addition, circle chords has been an interesting subject of recent research in Minkowski geometry, in view of extending problems from classical convexity to normed linear

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spaces. Having this in mind, we will introduce a new type of orthogonality defined with help of circle chords in normed planes. This is also inspired by the fact that, for any non-Euclidean norm, chords in the usual orthogonal position are in general not orthogonal with respect to all known orthogonality types defined for normed planes. (For various orthogonality types in Minkowski spaces we refer to [2, § 3–4 and § 7–8], [4], and [13, § 3.5].) This new type of orthogonality is called *chordal orthogonality*, and it coincides with usual orthogonality in the Euclidean subcase. Based on several properties of chordal orthogonality, which are derived in this paper, we prove some characterizations of the Euclidean plane among all Minkowski planes. E.g., if this type of orthogonality is always symmetric, then the corresponding normed plane is Euclidean. These characteristic properties of the Euclidean plane yield also new characterizations of inner product spaces among all d-dimensional Minkowski spaces, $d \geqslant 3$.

2. Definition and basic properties of chordal orthogonality

Let a circle C in a Minkowski plane $(\mathbb{M}, \|\cdot\|)$ be given, and let $[p_1, q_1]$ and $[p_2, q_2]$ be two chords of C. We say that $[p_1, q_1]$ is *chordal-orthogonal* to $[p_2, q_2]$ if the line through q_2 and through the point p_2^*, which is opposite to p_2 in C, is parallel to $\langle p_1, q_1\rangle$. In case that $p_2^* \equiv q_2$, we say that $[p_1, q_1]$ is *chordal-orthogonal* to $[p_2, q_2]$ if there exists a supporting line of C at q_2 which is parallel to $\langle p_1, q_1\rangle$. If q_2^* is the opposite point of q_2 in C, then $p_2^* q_2 p_2 q_2^*$ is a parallelogram; see Figure 1.

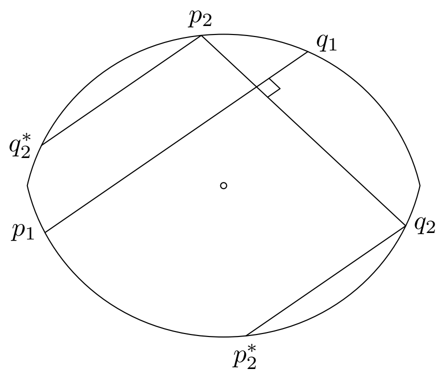

Figure 1

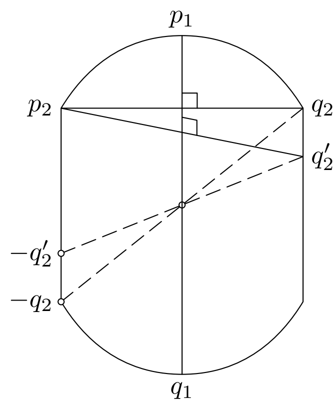

Figure 2

This implies that

$$\langle p_2^*, q_2\rangle \parallel \langle p_1, q_1\rangle \iff \langle p_2, q_2^*\rangle \parallel \langle p_1, q_1\rangle,$$

showing the correctness of the definition of chordal orthogonality.

We will use the notion of chordal orthogonality to describe relations between geometric objects related to triangles (including their circumcircles), and from now on we consider chordal orthogonality in a natural way only with respect to the unit circle $\mathcal{C}$, using the symbol $\perp_{\mathcal{C}}$ for it. In other words, when writing $[p_1, q_1] \perp_{\mathcal{C}} [p_2, q_2]$, we automatically mean that $[p_1, q_1]$ and $[p_2, q_2]$ are chords of the unit circle. It is easy to check that in the Euclidean case the relation $[p_1, q_1] \perp_{\mathcal{C}} [p_2, q_2]$ yields orthogonality in the usual Euclidean sense.

Directly from the definitions of chordal orthogonality one gets the following properties of the notion introduced above.

Proposition 2.1. *Let two parallel chords $[p_1, q_1]$ and $[p_1', q_1']$ of the unit circle $\mathcal{C}$ be given. Then*

$$[p_1, q_1] \perp_{\mathcal{C}} [p_2, q_2] \Longleftrightarrow [p_1', q_1'] \perp_{\mathcal{C}} [p_2, q_2].$$

Proposition 2.2. *Let three chords $[p_1, q_1]$, $[p_1', q_1']$, and $[p_2, q_2]$ of the unit circle $\mathcal{C}$ be given such that p_2 and q_2 are not opposite in $\mathcal{C}$. If $[p_1, q_1] \perp_{\mathcal{C}} [p_2, q_2]$ and $[p_1', q_1'] \perp_{\mathcal{C}} [p_2, q_2]$, then $[p_1, q_1] \parallel [p_1', q_1']$.*

Remark 2.1. The above proposition means that for any chord $[p_2, q_2]$ of the unit circle $\mathcal{C}$ and for any point p_1 of $\mathcal{C}$ different from p_2 and q_2 there exists at most one chord through p_1 which is chordal-orthogonal to $[p_2, q_2]$.

Note that by a *diameter chord* of the unit circle we mean a chord passing through the center of $\mathcal{C}$.

Proposition 2.3. *Let a chord $[p, q]$ of the unit circle $\mathcal{C}$ be given. Every diameter chord of $\mathcal{C}$ which is chordal-orthogonal to $[p, q]$ bisects $[p, q]$, i.e., intersects $[p, q]$ at its midpoint.*

We note that any interior point of the unit circle is the midpoint of some chord; see [12, Lemma 13].

Proposition 2.4. *For any interior point $p \neq 0$ of the unit circle $\mathcal{C}$ the diameter chord through this point is chordal-orthogonal to the chord with midpoint p.*

Proposition 2.5. *For any chord $[p_1, q_1]$ of the unit circle $\mathcal{C}$, where $\mathcal{C}$ is strictly convex, and for any point p_2 on $\mathcal{C}$, there exists a unique point q_2 on $\mathcal{C}$ such that $[p_1, q_1] \perp_{\mathcal{C}} [p_2, q_2]$.*

P r o o f. Let p_2^* be the point opposite to p_2 in $\mathcal{C}$, and let G be the line through p_2^* parallel to $[p_1, q_1]$. A line and a circle in a strictly convex Minkowski plane have at most two common points (cf. [12, Proposition 11]). If G and $\mathcal{C}$ intersect only at p_2^*, then G is a supporting line of $\mathcal{C}$, and therefore $[p_1, q_1] \perp_{\mathcal{C}} [p_2, p_2^*]$. In case that $G \cap \mathcal{C} = \{p_2^*, q_2\}$, then $[p_1, q_1] \perp_{\mathcal{C}} [p_2, q_2]$. □

Remark 2.2. In Figure 2 one can see a counterexample to Proposition 2.5 for the case that the normed plane $(\mathbb{M}, \|\cdot\|)$ is not strictly convex.

Note that, in general, the notion of chordal orthogonality is not symmetric. Nevertheless, the next proposition shows that symmetry can be achieved for some special position of the chords. This proposition also shows that any angle $\angle xzy$ inscribed in $\mathcal{C}$ with x, y as opposite points of $\mathcal{C}$ is "right" in the sense of chordal orthogonality, reminding the reader of the famous Theorem of Thales. (It should be noticed that such an angle is "right" also in the sense of James orthogonality, i.e., the vector $x - z$ is James orthogonal to the vector $z - y$. Remember that x is said to be *James orthogonal* to y, denoted by $x \# y$, if $\|x - y\| = \|x + y\|$; see [7].)

Proposition 2.6. *For pairwise different points $x, y, z \in \mathcal{C}$ with $z \neq -x$ and $z \neq -y$ the relations*

$$[x, z] \perp_{\mathcal{C}} [z, y] \text{ and } [z, y] \perp_{\mathcal{C}} [x, z] \tag{1}$$

hold if and only if x and y are opposite points of $\mathcal{C}$.

Proof. $\Longleftarrow$: This implication follows immediately from the definition of chordal orthogonality.

$\Longrightarrow$: The first relation of (1) implies

$$[-y, z] \parallel [x, z] \Longleftrightarrow z \in \langle x, -y \rangle \text{ or } x \equiv -y. \tag{2}$$

By the second relation in (1) we get

$$[-x, z] \parallel [z, y] \Longleftrightarrow z \in \langle -x, y \rangle \text{ or } -x \equiv y.$$

Assume that $z \in \langle x, -y \rangle$ and $z \in \langle -x, y \rangle$. Therefore the lines $\langle x, -y \rangle$ and $\langle -x, y \rangle$ coincide. Since a line through the center of the unit circle $\mathcal{C}$ intersects $\mathcal{C}$ at exactly two points, our assumption is impossible. □

It turns out that if in a strictly convex Minkowski plane at least one of the relations of (1) holds, then the "$\Longrightarrow$" part of Proposition 2.6 is also true; see Proposition 2.7. Figure 3 shows a counterexample in a Minkowski plane that is not strictly convex.

Proposition 2.7. *In a strictly convex Minkowski plane, let three pairwise distinct points* $x, y, z \in \mathcal{C}$ *with* $z \neq -x$ *and* $z \neq -y$ *be given. If* $[x,z] \perp_{\mathcal{C}} [z,y]$ *or* $[z,y] \perp_{\mathcal{C}} [x,z]$*, then* x *and* y *are opposite with respect to the unit circle.*

Proof. If $[x,z] \perp_{\mathcal{C}} [z,y]$, then (2) holds. Since a line and a circle in a strictly convex Minkowski plane have at most two common points (see again [12, Proposition 11]), it follows that $x \equiv -y$. □

Let $\triangle p_1p_2p_3$ be an arbitrary triangle in a strictly convex Minkowski plane with circumcenter p and circumradius λ. In [3], Asplund and Grünbaum introduced the so-called *$\mathcal{C}$-orthocenter* of $\triangle p_1p_2p_3$ if, in addition, the plane is smooth. Namely, they proved that if C_i are circles with radius λ passing through p_j and p_k, with $\{i,j,k\} = \{1,2,3\}$ and different from the circumcircle of $\triangle p_1p_2p_3$, then $\bigcap_{i=1}^{3} C_i$ is not empty and consists of precisely one point h. They called it the *$\mathcal{C}$-orthocenter* of $\triangle p_1p_2p_3$, for which the equality

$$h = p_1 + p_2 + p_3 - 2p \tag{3}$$

holds. We note that this is also true when the plane is not necessarily smooth; see [10]. Also it should be noticed that in the Euclidean case this $\mathcal{C}$-orthocenter coincides with the classical orthocenter, i.e., with the intersection point of the triangle altitudes. A survey on extensions of the related *three-circles theorem* is [9], and we refer also to [14], where orthocentricity concepts in normed planes are studied.

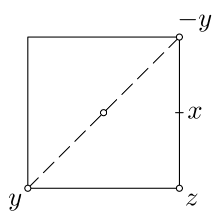

Figure 3

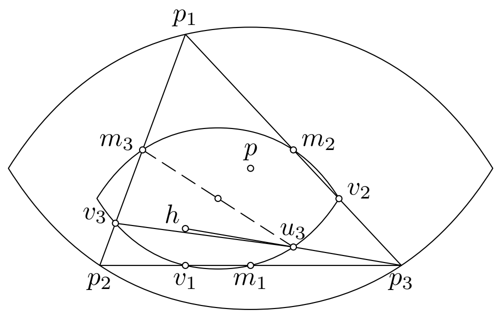

Figure 4

Remark 2.3. In [10] relations between the $\mathcal{C}$-orthocenter and James orthogonality for triangles with circumcircle in strictly convex normed planes are derived. For example, one has $p_1 - h \# p_2 - p_3$, $p_2 - h \# p_3 - p_1$, and $p_3 - h \# p_1 - p_2$.

The notion of the *nine-point circle*[1] of a triangle in the Euclidean plane (i.e., the circle through the foot points of the three altitudes of any triangle, the midpoints

[1] This circle is also known as the Feuerbach circle.

of their three sides, and the midpoints of the segments from the vertices to the triangle orthocenter) was also generalized by Asplund and Grünbaum in [3]. They proved that for any triangle $\triangle p_1p_2p_3$ in a strictly convex, smooth Minkowski plane with circumcenter p and $\mathcal{C}$-orthocenter h, the midpoints of the triangle sides and the midpoints of the segments $[p_i, h]$ $(i = 1, 2, 3)$ lie on the same circle whose center is the midpoint of $[h, p]$, and they called this Minkowskian generalization of the classical Feuerbach circle also the *six-point circle* of $\triangle p_1p_2p_3$. We note again that this is also true without demanding the smoothness of the plane $(\mathbb{M}, \|\cdot\|)$, if $\triangle p_1p_2p_3$ can be inscribed in a circle; see [10]. Figure 4 shows the six-point circle of a triangle in a strictly convex Minkowski plane which is not smooth. In addition, the six-point circle contains the three intersection points of $\langle h, p_i\rangle$ and $\langle p_j, p_k\rangle$, $\{i, j, k\} = \{1, 2, 3\}$, only in the Euclidean case. For more properties of six-point circles in strictly convex Minkowski planes we refer to [10] and [11]. Now, using the notion of chordal orthogonality, we will prove that the second intersection point of any triangle side and the six-point circle (the first is the midpoint of the side) can, somehow, also be interpreted as the foot point of a "triangle altitude".

Theorem 2.1. *In a strictly convex Minkowski plane $(\mathbb{M}, \|\cdot\|)$, let a triangle $\triangle p_1p_2p_3$ with circumcenter p and $\mathcal{C}$-orthocenter h be given. Let m_1, m_2, m_3 be the midpoints of $[p_2, p_3]$, $[p_3, p_1]$, and $[p_1, p_2]$, respectively, and let the unit circle $\mathcal{C}$ be the six-point circle of $\triangle p_1p_2p_3$. If $\mathcal{C} \cap \langle p_i, p_j\rangle = \{m_k, v_k\}$, where $\{i, j, k\} = \{1, 2, 3\}$, and $\mathcal{C} \cap [p_k, h] = \{u_k\}$, $k = 1, 2, 3$, then*

$$[m_k, v_k] \perp_{\mathcal{C}} [v_k, u_k].$$

P r o o f. Since the six-point circle of $\triangle p_1p_2p_3$ coincides with the unit circle $\mathcal{C}$, we have $p = -h$. Thus, by (3) we get

$$p_1 + p_2 + p_3 + h = 0.$$

Therefore the midpoint u_3 of $[p_3, h]$ is $-\frac{1}{2}(p_1 + p_2)$. This means that u_3 and m_3 are opposite points of $\mathcal{C}$. Hence

$$[m_3, v_3] \perp_{\mathcal{C}} [v_3, u_3]$$

by Proposition 2.6. □

3. Characterizations of the Euclidean plane via chordal orthogonality

In this section we will use chordal orthogonality to present four new characterizations of the Euclidean plane among arbitrary normed planes and, in addition, two characterizations of the Euclidean plane among all strictly convex normed planes.

Theorem 3.1. *Let* $(\mathbb{M}, \|\cdot\|)$ *be an arbitrary Minkowski plane. If for any chords* $[p_1, q_1]$, $[p_2, q_2]$, $[p'_1, q'_1]$, *and* $[p'_2, q'_2]$ *of the unit circle of* $(\mathbb{M}, \|\cdot\|)$, *satisfying*

$$[p_1, q_1] \perp_{\mathcal{C}} [p_2, q_2] \text{ and } [p'_1, q'_1] \perp_{\mathcal{C}} [p'_2, q'_2],$$

the implication $[p_1, q_1] \parallel [p'_1, q'_1] \implies [p_2, q_2] \parallel [p'_2, q'_2]$ *holds, then the plane is Euclidean.*

P r o o f. Let $[x_1, y_1]$ and $[x_2, y_2]$ be two parallel chords of the unit circle with midpoints z_1 and z_2, respectively. Proposition 2.6 implies that $[x_1, y_1] \perp_{\mathcal{C}} [y_1, -x_1]$ and $[x_2, y_2] \perp_{\mathcal{C}} [y_2, -x_2]$. Therefore $[y_1, -x_1] \parallel [y_2, -x_2]$. Thus we get $[0, z_1] \parallel [0, z_2]$, which means that the plane is Euclidean; see [2, p. 28, (3.5′)]. □

Theorem 3.2. *If in an arbitrary normed plane the relation of chordal orthogonality is always symmetric, then this plane is Euclidean.*

P r o o f. Let $[p_1, q_1]$, $[p_2, q_2]$, $[p'_1, q'_1]$, and $[p'_2, q'_2]$ be chords of the unit circle such that $[p_1, q_1] \perp_{\mathcal{C}} [p_2, q_2]$, $[p'_1, q'_1] \perp_{\mathcal{C}} [p'_2, q'_2]$, and $[p_1, q_1] \parallel [p'_1, q'_1]$. By the assumed symmetry of chordal orthogonality we get

$$[p_2, q_2] \perp_{\mathcal{C}} [p_1, q_1]. \tag{4}$$

Besides, Proposition 2.1 yields $[p_1, q_1] \perp_{\mathcal{C}} [p'_2, q'_2]$. Therefore

$$[p'_2, q'_2] \perp_{\mathcal{C}} [p_1, q_1]. \tag{5}$$

If p_1 and q_1 are not opposite in the unit circle $\mathcal{C}$, then by (4), (5), and Proposition 2.2 we obtain

$$[p_2, q_2] \parallel [p'_2, q'_2]. \tag{6}$$

In case that p_1 and q_1 are opposite, the points p'_1 and q'_1 cannot be opposite. By similar arguments we get

$$[p_2, q_2] \perp_{\mathcal{C}} [p'_1, q'_1],$$
$$[p'_2, q'_2] \perp_{\mathcal{C}} [p'_1, q'_1],$$

implying also (6). According to Theorem 3.1 this means that the plane is Euclidean. □

Lemma 3.1. *In an arbitrary normed plane we have*

$$[p_1, q_1] \perp_{\mathcal{C}} [p_2, q_2] \Longleftrightarrow x = \frac{1}{2}(p_1 + q_1 + p_2 + q_2) \in \langle p_1, q_1 \rangle.$$

Proof. If $[p_1, q_1] \perp_{\mathcal{C}} [p_2, q_2]$, then $[p_2, -q_2] \parallel [p_1, q_1]$. Therefore $p_2 + q_2 = \lambda(p_1 - q_1)$, where $\lambda \in \mathbb{R}$. Thus the point x satisfies

$$x = \Big(\frac{1}{2} + \frac{\lambda}{2}\Big)p_1 + \Big(\frac{1}{2} - \frac{\lambda}{2}\Big)q_1 \Longleftrightarrow x \in \langle p_1, q_1 \rangle.$$

Conversely, if $x \in \langle p_1, q_1 \rangle$, then for some $\mu \in \mathbb{R}$ we have

$$x = \mu p_1 + (1 - \mu)q_1 \Longleftrightarrow \frac{1}{2}(p_1 + q_1 + p_2 + q_2) = \mu p_1 + (1 - \mu)q_1 \Longleftrightarrow$$
$$\frac{1}{2}(p_2 + q_2) = \Big(\mu - \frac{1}{2}\Big)p_1 + \Big(\frac{1}{2} - \mu\Big)q_1 = \Big(\mu - \frac{1}{2}\Big)(p_1 - q_1) \Longleftrightarrow [p_1, q_1] \perp_{\mathcal{C}} [p_2, q_2].$$

□

The next theorem is an immediate consequence of Lemma 3.1 and Theorem 3.2.

Theorem 3.3. *If for any two chords $[p_1, q_1]$ and $[p_2, q_2]$ of the unit circle of an arbitrary normed plane, satisfying $[p_1, q_1] \perp_{\mathcal{C}} [p_2, q_2]$, the intersection point of the corresponding lines $\langle p_1, q_1 \rangle$ and $\langle p_2, q_2 \rangle$ is*

$$x = \frac{1}{2}(p_1 + q_1 + p_2 + q_2), \tag{7}$$

then the plane is Euclidean.

Remark 3.1. We note that if two chords $[p_1, q_1]$ and $[p_2, q_2]$ of any circle in the Euclidean plane are orthogonal (in the usual Euclidean sense), then for $\{x\} = \langle p_1, q_1 \rangle \cap \langle p_1, q_1 \rangle$ relation (7) also holds; see [6]. This means that Theorem 3.3 is a characterization of the Euclidean plane among all Minkowski planes. (Note that also the two theorems before Theorem 3.3 are characterizations of the Euclidean plane among all Minkowski planes.)

Our next theorem gives another characterization of the Euclidean plane by using the so-called Wallace line of a given triangle.

Theorem 3.4. *In an arbitrary Minkowski plane* $(\mathbb{M}, \|\cdot\|)$*, let a triangle* $\triangle x_1x_2x_3$ *inscribed in the unit circle* $\mathcal{C}$ *of* $(\mathbb{M}, \|\cdot\|)$ *be given. Let points* y, y_1, y_2, y_3 *lie on* $\mathcal{C}$ *and satisfy* $[y, y_1] \perp_{\mathcal{C}} [x_2, x_3]$*,* $[y, y_2] \perp_{\mathcal{C}} [x_3, x_1]$*, and* $[y, y_3] \perp_{\mathcal{C}} [x_1, x_2]$*. Then the points*

$$u = \frac{1}{2}(x_1 + x_2 + y + y_3),\ v = \frac{1}{2}(x_2 + x_3 + y + y_1),\ w = \frac{1}{2}(x_3 + x_1 + y + y_2)$$

are collinear if and only if the plane $(\mathbb{M}, \|\cdot\|)$ *is Euclidean.*

Proof. $\Longrightarrow$: Let $[p_1, q_1]$ and $[p_2, q_2]$ be two chords of the unit circle $\mathcal{C}$ of $(\mathbb{M}, \|\cdot\|)$ such that $[p_1, q_1] \perp_{\mathcal{C}} [p_2, q_2]$. Lemma 3.1 implies that the point

$$x = \frac{1}{2}(p_1 + q_1 + p_2 + q_2)$$

lies on the line $\langle p_1, q_1\rangle$. If we prove that $x \in \langle p_2, q_2\rangle$, then Theorem 3.3 will imply that the plane $(\mathbb{M}, \|\cdot\|)$ is Euclidean. Consider the triangle $\triangle(-p_1)p_2q_2$; see Figure 5. It is inscribed in $\mathcal{C}$, and $[p_1, q_1] \perp_{\mathcal{C}} [p_2, q_2]$. Moreover, by Proposition 2.6 we get $[p_1, p_2] \perp_{\mathcal{C}} [p_2, -p_1]$ and $[p_1, q_2] \perp_{\mathcal{C}} [-p_1, q_2]$.

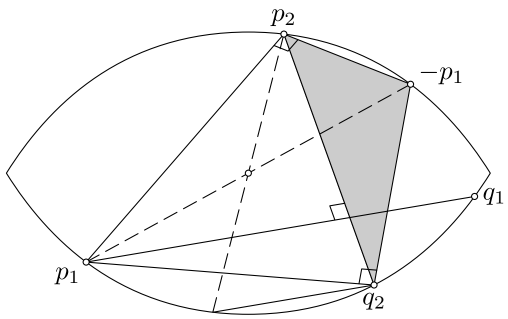

Figure 5

Therefore the points

$$u = \frac{1}{2}(p_2 + q_2 + p_1 + q_1) = x,$$
$$v = \frac{1}{2}(p_2 - p_1 + p_1 + p_2) = p_2,$$
$$w = \frac{1}{2}(-p_1 + q_2 + p_1 + q_2) = q_2$$

are collinear.

$\Longleftarrow$: If the plane $(\mathbb{M}, \|\cdot\|)$ is Euclidean, then the chords $[y, y_i]$ and $[x_j, x_k]$ with $\{i, j, k\} = \{1, 2, 3\}$ are orthogonal in the usual Euclidean sense. But then the collinearity of the points u, v, and w follows from statement $(3, 1)$ in [6]. □

Remark 3.2. In the Euclidean case, the line determined by u, v, and w is known as the Wallace (or Simson) line of the triangle $\triangle x_1x_2x_3$; see again [6, §3].

Theorem 3.5. *In a strictly convex Minkowski plane, let a triangle inscribed in its unit circle be given. If any side of this triangle is chordal-orthogonal to the chord through its opposite vertex and its $\mathcal{C}$-orthocenter (if such a chord exists), then the plane is Euclidean.*

Proof. Let $[p_1, p_2]$ and $[p_3, p_4]$ be two parallel chords of the unit circle. If h is the $\mathcal{C}$-orthocenter of $\triangle p_1p_2p_3$ and $\langle p_3, h\rangle \cap \mathcal{C} = \{p_3^*\}$, then

$$[p_1, p_2] \perp_{\mathcal{C}} [p_3, p_3^*] \Longleftrightarrow [p_3, p_4] \perp_{\mathcal{C}} [p_3, p_3^*].$$

This means that $p_3^* = -p_4$; see Proposition 2.7. Therefore p_3, h, and $-p_4$ are collinear. Hence by (3) we have

$$h = (1-\lambda)p_3 + \lambda(-p_4) \Longleftrightarrow p_1 + p_2 + p_3 = (1-\lambda)p_3 - \lambda p_4 \Longleftrightarrow p_1 + p_2 = -\lambda(p_3 + p_4),$$

where $\lambda \in \mathbb{R}$. Thus it is proved that the midpoints of $[p_1, p_2]$, $[p_3, p_4]$, and the origin 0 are collinear, which means that the plane is Euclidean; see [2, p. 28, (3.5′)]. □

The above theorem shows the following: If we say that a side of any triangle and the line through the opposite vertex and its $\mathcal{C}$-orthocenter are *Asplund-Grünbaum orthogonal* (see once more [3]), then *this type of orthogonality and chordal orthogonality coincide only in the Euclidean plane.* The next theorem shows that, in some sense, also the James orthogonality and the chordal orthogonality coincide only in the Euclidean plane.

Theorem 3.6. *Let four points p_1, q_1, p_2, q_2 in a strictly convex Minkowski plane be given such that $p_1 - q_1 \# p_2 - q_2$. Let $\{u_1, v_1\} = \langle p_1q_1\rangle \cap \mathcal{C}$, $\{u_2, v_2\} = \langle p_2q_2\rangle \cap \mathcal{C}$, and $[u_1, v_1] \perp_{\mathcal{C}} [u_2, v_2]$ (if all the points u_1, v_1, u_2, v_2 exist). Then the plane is Euclidean.*

Proof. This theorem follows immediately from Theorem 3.5 and Remark 2.3. □

Remark 3.3. Let $K \subset \mathbb{R}^d$, $d \geqslant 3$, be a convex body (i.e., a compact convex set with interior points in $\mathbb{R}^d$). Since every convex body K with center 0, all whose intersections with 2-subspaces are ellipses, is itself a d-ellipsoid (see, e.g., [5] and [8]), it is clear that the theorems given in this section yield characterizations of inner product spaces among all d-dimensional Minkowski spaces or, in the last two cases, among all d-dimensional strictly convex Minkowski spaces.

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