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ON SEQUENTIAL PROPERTIES OF BANACH SPACES, SPACES OF MEASURES AND DENSITIES

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Abstract. We show that a conjunction of Mazur and Gelfand-Phillips properties of a Banach space E can be naturally expressed in terms of *weak** continuity of seminorms on the unit ball of E^*.

We attempt to carry out a construction of a Banach space of the form $C(K)$ which has the Mazur property but does not have the Gelfand-Phillips property. For this purpose we analyze the compact spaces on which all regular measures lie in the *weak** sequential closure of atomic measures, and the set-theoretic properties of generalized densities on the natural numbers.

Keywords: Gelfand-Phillips property, Mazur property, generalized density

MSC 2010: 46B26, 46E15, 46E27

1. Introduction

A Banach space E has the *Mazur property* if every $x^{**} \in E^{**}$ which is *weak** sequentially continuous on E^* is in fact *weak** continuous, and consequently is in E. Here a *weak** sequential continuity of a function $\varphi\colon E^* \to \mathbb{R}$ refers to the following familiar condition:

$$\lim_{n\to\infty} \varphi(x_n^*) = \varphi(x^*)$$

whenever x_n^* is a sequence converging to x^* in the *weak** topology of the space E^*.

Obviously every reflexive space E has the Mazur property; it is also not difficult to check that so does every separable E, since the ball in E^* is metrizable in the *weak** topology. There are several examples of Banach spaces E which have the

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Mazur property, though the *weak** topology of E^* is far from being metrizable; see Section 3 below. For such spaces it is the combination of linearity and sequential continuity that makes a given x^{**} *weak** continuous. In particular, it is easy to give an example of a separable E and a *weak** sequentially continuous but not continuous seminorm on E^*—see the remark after Lemma 2.2. Kazimierz Musiał posed the following problem (communicated privately).

Problem 1.1. Let E be a Banach space with the Mazur property, and suppose that $\varphi\colon E^* \to \mathbb{R}$ is a *weak** sequentially continuous function which is a seminorm on E^*. Is φ *weak** continuous on the unit ball B_{E^*}?

We show below (in Section 2) that the answer to the question above is affirmative if and only if the space E has the Gelfand-Phillips property. Let us now recall the latter notion.

A bounded subset A of a Banach space E is said to be *limited* if

$$\lim_{n\to\infty} \sup_{x\in A} |x_n^*(x)| = 0$$

for every *weak** null sequence $x_n^* \in E^*$. It is easy to check that every relatively norm compact set is limited. The space E is said to have the *Gelfand-Phillips property* if this may be reversed, i.e. if every limited subset of E is relatively norm compact.

We refer the reader to Section 3 for the discussion of the Mazur and Gelfand-Phillips properties of Banach spaces. Let us note here that in view of the solution to Problem 1 it is natural to ask about possible connections between those two properties. There are easy examples of Banach spaces with the Gelfand-Phillips property but without the Mazur property. However, the list of known examples might suggest that the Mazur property does imply the other one. In fact such a statement was announced in [10] but the argument mentioned there is incorrect (see the remark at the end of Section 3).

In the second part of the present paper we consider the following problem.

Problem 1.2. Is there a compact space K such that the underlying Banach space $C(K)$ has the Mazur property but does not have the Gelfand-Phillips property?

Our approach is based on some related results on the *weak** topology in the spaces of measures, presented in Section 4 and Section 5. In particular, Proposition 4.3 gives a technical criterion which guarantees that a Banach space of the form $C(K)$ has the Mazur property, while Theorem 5.1 singles out a certain class of compact spaces for which such a criterion is applicable.

Building on a result due to Schlumprecht [28], we give in the final section an affirmative solution to Problem 1.2. Our construction, however, relies on a set-theoretic assumption, whose consistency has not yet been established. This assumption is related to (generalized) densities on natural numbers, and leads to new cardinal invariants that are named in Section 6.*

In the sequel, by ω we mean the set of natural numbers, E always denotes a (real) Banach space, and K stands for a Hausdorff compact space. By $C(K)$ we denote the Banach space of continuous functions, and identify $C(K)^*$ with the space $M(K)$ of all signed Radon measures on K of finite variation. Moreover, we write $P(K)$ for the set of all probability measures from $M(K)$. For a given $t \in K$, $\delta_t \in P(K)$ is the Dirac measure at t.

2. On seminorms on E^*

Let us fix a Banach space E and a seminorm $\varphi\colon E^* \to \mathbb{R}_+$. Note that φ is *weak** (sequentially) continuous if and only if it is *weak** (sequentially) continuous at $0 \in E^*$. Indeed, if a net x_t^* converges to x^* then

$$-\varphi(x^* - x_t^*) \leqslant \varphi(x_t^*) - \varphi(x^*) \leqslant \varphi(x_t^* - x^*),$$

which, together with continuity at 0, implies that $\lim_t \varphi(x_t^*) = \varphi(x^*)$ (and we may replace a net by a sequence for the sequential version of the statement). Note also that a sequentially continuous seminorm is norm continuous, since $= \|x_n\| \to 0$ implies *weak** convergence.

Lemma 2.1. *If E has the Mazur property and a seminorm φ is weak* sequentially continuous then there is $A \subseteq E$ such that $\varphi(x^*) = \sup_{a\in A} x^*(a)$ for every $x^* \in E^*$.*

Proof. By the Hahn-Banach theorem, for every fixed x_0^* there is a linear functional z on E^* such that $|z| \leqslant \varphi$ and $z(x_0^*) = \varphi(x_0^*)$. If $\|x_n^*\| \to 0$ then $|z(x_n^*)| \leqslant \varphi(x_n^*) \to 0$; hence $z \in E^{**}$ and z is *weak** sequentially continuous. By the Mazur property z is in E, and this immediately gives the required formula. □

Lemma 2.2. *Let A be a bounded subset of a Banach space E and let us consider a seminorm $\varphi\colon E^* \to \mathbb{R}_+$ given by the formula*

$$\varphi(x^*) = \sup_{a\in A} |x^*(a)|.$$

Then

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* The authors are grateful to Tomek Bartoszyński, Adam Krawczyk and Michael Hrusak for discussions concerning that topic.

(i) *φ is weak* sequentially continuous if and only if A is limited;*
(ii) *φ is weak* continuous on B_{E^*} if and only if A is relatively norm compact.*

P r o o f. If A is limited then by definition φ is *weak** sequentially continuous at 0 and, by the remark above, it is *weak** sequentially continuous. We shall check (ii).

If A is relatively norm compact then for a given $\varepsilon > 0$ there is a finite ε-net $a_1, \dots, a_k \in A$. We have $|x_t^*(a)| \leqslant 2\varepsilon$ whenever $|x^*(a_i)| \leqslant \varepsilon$ for $i \leqslant k$ and $\|x^*\| \leqslant 1$. This means that φ is continuous on B_{E^*}.

Suppose that A is not relatively norm compact; then for some $\varepsilon > 0$ we can find a sequence a_n and a sequence x_n^* in B_{E*} such that $x_n^*(a_n) \geqslant \varepsilon$ and $x_n^*(a_i) = 0$ for every $n \in \omega$ and $i < n$. Consider $\psi(x^*) = \sup\limits_{n\in\omega} |x^*(a_n)|$; then $\psi(x_n^*) \geqslant \varepsilon$ for every n, but if x^* is a cluster point of $\{x_n^*\colon\ n \in \omega\}$ then we have $x^*(a_n) = 0$ for all n, so $\psi(x^*) = 0$. Since ψ is not *weak** continuous on B_{E^*}, it is not continuous at 0, and therefore $\varphi \geqslant \psi$ is not *weak** continuous either. □

We remark that in (ii) of Lemma 2.2 we do not have continuity on the whole of E^*; in fact we can easily give an example of a separable Banach space E and of a seminorm φ on E^* which is sequentially *weak** continuous but not *weak** continuous.

Let e_n denote the unit vector $(0, \dots, 1, \dots)$; we consider $E = l_1$ and $E^* = l_\infty$. Then

$$A = \{(1/k)e_k\colon\ k \in \omega\} \subseteq l_1$$

is relatively norm compact (hence limited), so if we consider $\varphi\colon\ l_\infty \to \mathbb{R}$ defined as in Lemma 2.2 then φ is *weak** sequentially continuous. To see that φ is not *weak** continuous note that 0 lies in the *weak** closure of the set $\{ne_n\colon\ n \in \omega\} \subseteq l_\infty$ while $\varphi(ne_n) = 1$ for every $n \in \omega$.

Theorem 2.3. *For a Banach space E the following assertions are equivalent:*
(a) *every weak* sequentially seminorm φ on E^* is weak* continuous on B_{E^*};*
(b) *E has the Mazur property and the Gelfand-Phillips property.*

P r o o f. (a) → (b) If $z \in E^{**}$ is *weak** sequentially continuous then $\varphi(x^*) = |z(x^*)|$ is a *weak** sequentially continuous seminorm; hence (a) implies the Mazur property.

For any bounded set $A \subseteq E$, we have a seminorm φ on E^* as in Lemma 2.2. If A is a limited subset of E then φ is *weak** sequentially continuous, so *weak** continuous on B_{E^*} by (a), and it follows from Lemma2.2 that A is relatively norm compact.

(b) → (a) By the Mazur property and Lemma 2.1, if φ is a *weak** sequentially continuous seminorm then $\varphi(x^*) = \sup\limits_{a\in A} x^*(a)$ for some $A \subseteq E$. Now Lemma 2.2 (i) tells us that A is limited so relatively norm compact by the Gelfand-Phillips property, and Lemma 2.2 (ii) completes the proof. □

3. Mazur versus Gelfand-Phillips

The Gelfand-Phillips property has attracted considerable attention over the last twenty years, which resulted in several interesting papers, see for instance Bourgain & Diestel [5], Drewnowski [6], Schlumprecht [28], Sinha & Arora [26], Freedman [9]. The class (GP) of spaces having this property is quite wide, and includes

(i) $l_1(\kappa)$ for every κ;

(ii) every E such that the ball in E^* is *weak** sequentially compact, or more generally

(iii) every E such that the ball in E^* contains *weak** sequentially precompact norming subset (see [6]);

(iv) $C(K)$ for every K which is Valdivia compact (this class includes all Corson compact and dyadic spaces, [26]).

Let us recall that a compact space K is *Corson compact* (*Valdivia compact*) if for some κ there is an embedding $g\colon K \to \mathbb{R}^\kappa$ such that $g[K] \subseteq \Sigma(\mathbb{R}^\kappa)$ ($g[K] \cap \Sigma(\mathbb{R}^\kappa)$ is dense in $g[K]$, respectively). Here $\Sigma(\mathbb{R}^\kappa)$ is the subspace of $\mathbb{R}^\kappa$ of elements having countable support. Corson and Valdivia compacta have numerous applications in functional analysis; we refer the reader to a survey paper Kalenda [12] for the background and further references on this topic.

The Mazur property is more isolated and rather difficult to handle. However, it appeared quite naturally in the theory of Pettis integration of Banach space valued functions, see Edgar [7] and Talagrand [27]; cf. Leung [18], Wilansky [29]. A recent paper by Kalenda [13] allows one to analyze the property from another perspective.

It is clear that E has the Mazur property if E^* has a *weak** angelic ball; therefore all weakly compactly generated Banach spaces are in (MP), see [7] for details. The space $l_1(\kappa)$ is in (MP) unless there are weakly inaccessible cardinals $\leqslant \kappa$, see [7]. (A cardinal number κ is weakly inaccessible if κ is a regular limit cardinal; for our purpose it is worth recalling that, consistently, such numbers do not exist.)

A Banach space $C(K)$ has the Mazur property under one of the following assumptions on a compact space K (see Plebanek [22]–[25]):

(i) K is first-countable;

(ii) K is Corson compact;

(iii) $K = \{0,1\}^\kappa$, and there are no weakly inaccessible cardinals $\leqslant \kappa$ (so for sure in case $\kappa = \omega_1$ and, consistently, for all κ).

It is well known that the class (GP) is not included in (MP): Let $K = [0, \omega_1]$, i.e. K is the space of ordinals $\alpha \leqslant \omega_1$ equipped with the interval topology. Then K is scattered and $C(K)$ has the Gelfand-Phillips property by a result due to Drewnowski [6] mentioned above. The space $C(K)$ does not have the Mazur property, since the formula $\varphi(\mu) = \mu(\{\omega_1\})$ defines $\varphi \in C(K)^{**} \setminus C(K)$ which is *weak** sequentially continuous.

Recall that if A is a limited set in any Banach space E then A is conditionally weakly compact (every sequence in A has a subsequence which is weakly Cauchy), and is even relatively weakly compact provided E contains no copy of l_1, see Bourgain & Diestel [5]. We remark below that the Mazur property always implies such a weak version of the Gelfand-Phillips property, considered by Leung [17].

Proposition 3.1. *If E has the Mazur property and the set $A \subseteq E$ is limited then A is relatively weakly compact.*

Proof. Otherwise, we can pick $z^{**} \in E^{**} \setminus E$ which lies in the *weak** closure of A. If $x_n^* \to 0$ in the *weak** topology then eventually $|x_n^*(a)| \leqslant \varepsilon$ for all $a \in A$, hence $|z^{**}(x_n^*)| \leqslant \varepsilon$. This means that z^{**} is *weak** sequentially continuous, a contradiction. □

All the facts on the classes (MP) and (GP) we have mentioned so far might suggest that simply the class (MP) is included in (GP). Such a result is claimed in [10] but Theorem 2 announced there is not correct. That result in particular says that if E has the Mazur property then the unit ball in E^* is *weak**-M-compact, i.e. according to the author's definition *for every bounded sequence x_n^*, its weak* closure contains a weak* converging subsequence*. This is not true: Consider $E = C(\{0,1\}^{\mathfrak{c}})$; then E has the Mazur property in most cases, for instance if $\mathfrak{c} = \omega_1, \omega_2, \ldots$. On the other hand, there is an embedding $g\colon \beta\omega \to \{0,1\}^{\mathfrak{c}}$ and if $\mu_n = \delta_{g(n)}$ then there are no *weak** converging sequences in their closure simply because $\beta\omega$ contains no nontrivial converging sequence.

4. Towards a counterexample

We will now investigate if there is a Banach space of the form $C(K)$ which is in (MP) but not in (GP). We plan to obtain a desired compact space K as a compactification of the natural numbers ω with the discrete topology. Such a compactification $K \supseteq \omega$ will be seen as the Stone space $\mathrm{ULT}(\mathfrak{A})$ of all ultrafilters on some algebra $\mathfrak{A}$ of subsets of ω.

Let $\mathfrak{A}$ be any Boolean algebra; for any $A \in \mathfrak{A}$ we write

$$\hat{A} = \{\mathcal{F} \in \mathrm{ULT}(\mathfrak{A})\colon\ A \in \mathcal{F}\};$$

recall that $\hat{A}$ is then a clopen subset of $\mathrm{ULT}(\mathfrak{A})$ and the family $\{\hat{A}\colon\ A \in \mathfrak{A}\}$ is by definition a base of the topology on $\mathrm{ULT}(\mathfrak{A})$.

If we want to violate the Gelfand-Phillips property in a space of the form $C(K)$, then we can use the following result due to Schlumprecht [28, Theorem 6]; here subsequential completeness of a sequence (f_n) in $C(K)$ means that every subsequence contains further subsequence which has a supremum in $C(K)$.

Theorem 4.1. *Let (f_n) be a normalized sequence in $C^+(K)$ of functions having pairwise disjoint supports. If (f_n) is subsequentially complete then $A = \{f_n\colon n \in \omega\} \subseteq C(K)$ is limited (and so $C(K)$ does not have the Gelfand-Phillips property since A is obviously not relatively norm compact).*

In what follows we shall say that a family $\mathcal{P}$ of infinite subsets of ω is a π-base if every infinite $B \subseteq \omega$ contains some $P \in \mathcal{P}$.

Corollary 4.2. *Suppose that $\mathfrak{A}$ is an algebra of subsets of ω containing all finite sets and some π-base. Then the Banach space $C(K)$, where $K = \mathrm{ULT}(\mathfrak{A})$, does not have the Gelfand-Phillips property.*

Proof. Given $n \in \omega$, then $\{n\} \in \mathfrak{A}$ and so $V_n = \widehat{\{n\}}$ is a clopen subset of K. Then the characteristic functions $f_n = \chi_{V_n}$ form a sequence as in Theorem 4.1—the subsequential completeness follows from the fact that $\mathfrak{A}$ contains a π-base. $\square$

We now turn to analyzing how to guarantee the Mazur property of the space $C(K)$ (we follow here Plebanek [23]). Every functional $z^{**} \in C(K)^{**}$ gives rise to a function

$$\varphi\colon K \to \mathbb{R}, \quad \varphi(t) = z^{**}(\delta_t) \text{ for } t \in K.$$

If z^{**} is *weak** sequentially continuous then φ is a sequentially continuous function on K, since the convergence $t_n \to t$ in K implies *weak** convergence $\delta_{t_n} \to \delta_t$. If we want to check that $C(K)$ enjoys the Mazur property we need to know that φ is in fact continuous. Moreover, one needs to check the formula

$$z^{**}(\mu) = \int \varphi \, \mathrm{d}\mu$$

for every probability Radon measure on K (then the formula extends easily to every signed Radon measure ν via the decomposition $\nu = \nu^+ - \nu^-$ and we finally have $z^{**} = \varphi \in C(K)$).

In the proof presented in [23] or [25] that $C(\{0,1\}^\kappa)$ has the Mazur property we could use a result due to Mazur himself [19] that every sequentially continuous function on $\{0,1\}^\kappa$ is continuous (provided there are no weakly inaccessible cardinals up to κ; see also [21]). For the construction below we shall need a new idea at this stage.

First let us recall that in any topological space X, if $B \subseteq X$ then the smallest sequentially closed subset of X containing B can be written as

$$\bigcup_{\xi<\omega_1} \mathrm{scl}_\xi(B),$$

where $\mathrm{scl}_0(B) = B$, and for $0 < \xi < \omega_1$, $\mathrm{scl}_\xi(B)$ is the set of limits of all converging sequences from $\bigcup_{\eta<\xi} \mathrm{scl}_\eta(B)$.

Given a compact space K, we consider the operation of sequential closure in the space $P(K)$ with its *weak** topology. For any $A \subseteq K$ we write

$$\mathrm{conv}\, A = \mathrm{conv}\{\delta_a\colon\ a \in A\},$$

for simplicity, i.e.conv A is the set of all probability measures supported by a finite subset of A. Moreover, we put

$$S(A) = \bigcup_{\xi<\omega_1} \mathrm{scl}_\xi(\mathrm{conv}\, A),$$

i.e. $S(A)$ is the smallest *weak** sequentially closed set in $P(K)$ containing all probability measures supported by finite subsets of A.

Proposition 4.3. *Let K be a compactification of ω, and suppose that*
(a) *for every $t \in K \setminus \omega$ and every $Y \subseteq \omega$, if $t \in \overline{Y}$ then $\delta_t \in S(Y)$;*
(b) *every $\mu \in P(K)$ belongs to $S(K)$.*
Then $C(K)$ has the Mazur property.

Proof. Let $z^{**} \in C(K)^{**}$ be *weak** sequentially continuous and let $\varphi\colon\ K \to \mathbb{R}$ be defined as above. We will check that φ is continuous on $\omega \cup \{x\}$ for every $x \in K$. This implies that φ is continuous on K, since ω is dense (by a purely topological lemma, see [27, Lemma 2.5.2]). The function φ is continuous at n for every $n \in \omega$, since n is isolated in K. Assume towards a contradiction that φ is not continuous on $\omega \cup \{x\}$ for some $x \in K \setminus \omega$. Then there is $Y \subseteq \omega$ such that $x \in \overline{Y}$ and, say, $\varphi(y) > \varphi(x) + \varepsilon$ for every $y \in Y$. But then, using linearity and sequential continuity of z^{**} we get $z^{**}(\mu) \geqslant z^{**}(\delta_x) + \varepsilon$ for every $\mu \in S(Y)$, a contradiction with $\delta_x \in S(Y)$.

We have $z^{**}(\mu) = \int \varphi\, \mathrm{d}\mu$ for every $\mu \in \mathrm{conv}\, K$, so by sequential continuity the same formula holds for every $\mu \in S(K)$, and therefore (b) guaranties $z^{**} = \varphi$. □

5. When $S(K) = P(K)$

Given a measure $\mu \in P(K)$ and a sequence (t_n) in K, (t_n) is said to be *μ-uniformly distributed* if

$$\frac{1}{n}\sum_{i\leqslant n}\delta_{t_i} \to \mu$$

in the *weak** topology of $M(K)$. Mercourakis [20] mentions several classes of compact spaces K for which every $\mu \in P(K)$ admits a uniformly distributed sequence. Note that for such spaces K we have in particular $S(K) = P(K)$. We shall name below another large class of spaces satisfying $S(K) = P(K)$, obtained from Boolean algebras via the Stone isomorphism.

Let us recall the notion of a minimally generated Boolean algebra introduced by Koppelberg [14], [15]. We say that a Boolean algebra $\mathfrak{B}$ is a minimal extension of $\mathfrak{A}$ if $\mathfrak{A} \subseteq \mathfrak{B}$ and there is no algebra $\mathfrak{C}$ such that $\mathfrak{A} \subsetneq \mathfrak{C} \subsetneq \mathfrak{B}$.

A Boolean algebra $\mathfrak{B}$ is minimally generated if there is a continuous sequence of algebras $(\mathfrak{A}_\alpha)_{\alpha\leqslant\kappa}$, such that $\mathfrak{A}_0 = \{0,1\}$, $\mathfrak{A}_{\alpha+1}$ is a minimal extension of $\mathfrak{A}_\alpha$ for every $\alpha < \kappa$ and $\mathfrak{A}_\kappa = \mathfrak{B}$.

The notion of a minimally generated algebra is a useful tool for various set-theoretic constructions, see e.g. Koszmider [16] and the references therein. It is also interesting from the measure-theoretic angle; it was shown in [4] that if K is a Stone space of a minimally generated algebra then the measures on K are small in various senses; for instance if the said algebra is generated in ω_1 steps then every $\mu \in P(K)$ is uniformly regular, which is a property which guarantees the existence of uniformly distributed sequences. We now present the following general result.

Theorem 5.1. *If K is the Stone space of a minimally generated algebra $\mathfrak{A}$ then $S(K) = P(K)$.*

It will be convenient to recall several definitions and facts before we prove 5.1. Let $\mathfrak{A}$ be a Boolean algebra and let K be its Stone space. Every (finitely additive) measure μ on $\mathfrak{A}$ can be transferred to the measure $\hat{\mu}$ on the algebra of clopen subsets of K via the formula $\hat{\mu}(\hat{A}) = \mu(A)$, and then extended to the unique Radon measure on K. Therefore we may treat finitely additive measures on $\mathfrak{A}$ rather than Radon measures on K. In this way our space $P(K)$ becomes simply the space $P(\mathfrak{A})$ of all probability (finitely additive) measures on $\mathfrak{A}$, where $P(\mathfrak{A})$ is equipped with the topology of convergence on all $A \in \mathfrak{A}$.

With every ultrafilter $\mathcal{F}$ on an algebra $\mathfrak{A}$ we can associate a 0–1 measure $\delta_{\mathcal{F}} \in P(\mathfrak{A})$, where $\delta_{\mathcal{F}}(A) = 1$ if $A \in \mathcal{F}$ and is 0 otherwise. We shall write $S(\mathfrak{A}) \subseteq P(\mathfrak{A})$ for the least sequentially closed set of measures containing convex combinations of 0–1 measures on $\mathfrak{A}$. To prove Theorem 5.1 we need to show that $P(\mathfrak{A}) = S(\mathfrak{A})$ whenever $\mathfrak{A}$ is minimally generated.

A measure μ on $\mathfrak{A}$ is *non-atomic* if for every $\varepsilon > 0$ there is a finite partition of 1 into elements of measure at most ε. Below we shall use the classical decomposition theorem (see e.g. Theorem 5.2.7 in [2]).

Theorem 5.2 (Hammer & Sobczyk)**.** *Every $\mu \in P(\mathfrak{A})$ can be uniquely decomposed into $\nu + \varphi$, where ν is non-atomic and $\varphi = \sum_i a_i \delta_{\mathcal{F}_i}$, $\mathcal{F}_i \in \mathrm{ULT}(\mathfrak{A})$.*

The following fact is proved in [4, Lemma 4.7]. Here we write μ^* and μ_* for the corresponding outer and inner measures; note that the condition $\mu_*(B) = \mu^*(B)$ means that we can find $A_0, A_1 \in \mathfrak{A}$ such that $A_0 \subseteq B \subseteq A_1$, $\mu(A_1) - \mu(A_0)$ being arbitrarily small.

Lemma 5.3. *If $\mathfrak{B}$ is an algebra that is minimally generated over algebra $\mathfrak{A}$ and $\mu \in P(\mathfrak{A})$ is non-atomic then $\mu_*(B) = \mu^*(B)$ for every $B \in \mathfrak{B}$. Consequently, every non-atomic $\mu \in P(\mathfrak{A})$ has the unique extension to $\tilde{\mu} \in P(\mathfrak{B})$.*

The next lemma can be checked by induction on α.

Lemma 5.4. *Let $\mu, \nu, \varphi \in P(\mathfrak{A})$. Suppose that $\mu = a\nu + b\varphi$ for some $a, b \geqslant 0$ with $a + b = 1$. Then for every $\alpha < \omega_1$ we have $\mu \in \mathrm{scl}_\alpha(\mathfrak{A})$ whenever $\nu, \varphi \in \mathrm{scl}_\alpha(\mathfrak{A})$.*

Lemma 5.5. *Let $\mathfrak{B}$ be minimally generated over $\mathfrak{A}$. Suppose that $\mu \in P(\mathfrak{A})$ is non-atomic and $(\mu_n)_n$ is a sequence of measures from $P(\mathfrak{A})$ converging to μ. Then μ has the unique extension to $\tilde{\mu}$ on $\mathfrak{B}$ and if $\tilde{\mu}_n$ is any extension of μ_n to $\mathfrak{B}$ for every n, then $\tilde{\mu}_n$ converge to $\tilde{\mu}$.*

P r o o f. Consider a non-atomic measure $\mu \in P(\mathfrak{A})$ and its extension $\tilde{\mu} \in P(\mathfrak{B})$ (which is unique by Lemma 5.3).

Take a sequence of measures $(\mu_n)_n$ from $P(\mathfrak{A})$ converging to μ, and let $\tilde{\mu}_n \in P(\mathfrak{B})$ be *any* extension of μ_n for every n (we do not assume that μ_n is non-atomic and thus $\tilde{\mu}_n$ need not be uniquely determined). We are to show that the sequence $\tilde{\mu}_n(B)$ converges to $\tilde{\mu}(B)$ for every $B \in \mathfrak{B}$. Indeed, fix $\varepsilon > 0$; by Lemma 5.3 there are $A_0, A_1 \in \mathfrak{A}$ such that $A_0 \subseteq B \subseteq A_1$ and $\mu(A_1) - \mu(A_0) < \frac{1}{2}\varepsilon$. Let n_0 be such that $\mu_n(A_0) > \mu(A_0) - \frac{1}{4}\varepsilon$ and $\mu_n(A_1) < \mu(A_1) + \frac{1}{4}\varepsilon$ for every $n > n_0$. Then $\mu_n(A_1) - \mu_n(A_0) < \varepsilon$ and $\mu_n(A_0) < \tilde{\mu}_n(B) < \mu_n(A_1)$ and $\mu_n(A_0) < \tilde{\mu}(B) < \mu_n(A_1)$ for every $n > n_0$. It follows that $|\tilde{\mu}_n(B) - \tilde{\mu}(B)| < \varepsilon$ for every $n > n_0$. □

Lemma 5.6. *If $\mathfrak{B}$ is minimally generated over $\mathfrak{A}$ and $\mu \in S(\mathfrak{A})$, then μ has an extension to $\tilde{\mu} \in S(\mathfrak{B})$.*

Proof. Let $\mathfrak{B}$ be minimally generated over $\mathfrak{A}$. We show that for every α, if $\mu \in \mathrm{scl}_\alpha(\mathfrak{A})$, then it has an extension to $\tilde{\mu} \in \mathrm{scl}_\alpha(\mathfrak{B})$.

Assume that $\mu \in \mathrm{scl}_0(\mathfrak{A})$, i.e. $\mu = a_0\delta_{\mathcal{F}_0} + \ldots + a_k\delta_{\mathcal{F}_k}$ for some $a_i \in \mathbb{R}$, $\mathcal{F}_i \in \mathrm{ULT}(\mathfrak{A})$ for $i \leqslant k$. Clearly, $\tilde{\mu} = a_0\delta_{\mathcal{F}_0'} + \ldots + a_k\delta_{\mathcal{F}_k'}$, where $\mathcal{F}_i'$ is any extension of $\mathcal{F}_i$ to an ultrafilter on $\mathfrak{B}$ for every $i \leqslant k$, extends μ and $\tilde{\mu} \in \mathrm{scl}_0(\mathfrak{B})$.

Suppose now that every $\mu \in \mathrm{scl}_\beta(\mathfrak{A})$ has an extension to $\tilde{\mu} \in \mathrm{scl}_\beta(\mathfrak{B})$ for every $\beta < \alpha$ and consider $\mu \in \mathrm{scl}_\alpha(\mathfrak{B})$. Use Theorem 5.2 to decompose μ into non-atomic and purely atomic parts; suppose for instance that $\mu = \frac{1}{2}(\nu + \varphi)$, where ν is non-atomic and φ is purely atomic (the general case will follow by an obvious modification of coefficients).

Let $\tilde{\nu} \in P(\mathfrak{B})$ be the unique extension of ν and let $\hat{\varphi} \in P(\mathfrak{B})$ be any extension of φ to the strictly atomic measure. Let $\tilde{\mu} = \frac{1}{2}(\tilde{\nu} + \hat{\varphi})$. By Lemma 5.4 it is enough to show that $\tilde{\nu} \in \mathrm{scl}_\alpha(\mathfrak{B})$ as $\hat{\varphi} \in \mathrm{scl}_1(\mathfrak{B})$.

Since $\nu \in \mathrm{scl}_\alpha(\mathfrak{A})$, there is a sequence $(\nu_n)_n$ from $\bigcup\limits_{\beta<\alpha} \mathrm{scl}_\beta(\mathfrak{A})$ converging to ν. By the inductive assumption, for every n there is an extension $\tilde{\nu}_n \in \bigcup\limits_{\beta<\alpha} \mathrm{scl}_\beta(\mathfrak{B})$ of ν_n. By Lemma 5.5, $\tilde{\nu}_n$ converges to $\tilde{\nu}$. Thus, $\tilde{\nu} \in \mathrm{scl}_\alpha(\mathfrak{B})$ and we are done. □

Proof of Theorem 5.1. Fix a sequence of minimal extensions $\mathfrak{A}_\alpha$, $\alpha \leqslant \kappa$ generating $\mathfrak{A}$. Assume towards a contradiction that $P(\mathfrak{A}) \setminus S(\mathfrak{A}) \neq \emptyset$ while $P(\mathfrak{A}_\alpha) = S(\mathfrak{A}_\alpha)$ for every $\alpha < \kappa$.

It follows from Theorem 5.2 and Lemma 5.4 that we can pick a non-atomic measure $\mu \in P(\mathfrak{A}) \setminus S(\mathfrak{A})$. Then for each $\alpha < \kappa$ the restriction μ_α of μ to $\mathfrak{A}_\alpha$ cannot be non-atomic (if μ_α were non-atomic then we would have $\mu \in S(\mathfrak{A})$ by Lemma 5.6 and Lemma 5.3).

We have shown that κ is the first $\alpha \leqslant \kappa$ at which μ is non-atomic on $\mathfrak{A}_\alpha$. But this plainly implies that κ has countable cofinality. Therefore we can write $\mathfrak{A}$ as $\bigcup\limits_{n\in\omega} \mathfrak{B}_n$, where $\mathfrak{B}_{n+1}$ is minimally generated over $\mathfrak{B}_n$ for every n and, putting $\nu_n = \mu|\mathfrak{B}_n$, we have $\nu_n \in S(\mathfrak{B}_n)$ for every n. Every ν_n extends to some $\nu_n' \in S(\mathfrak{A})$ by Lemma 5.3. Finally, we get $\mu = \lim\limits_{n\to\infty} \nu_n' \in S(\mathfrak{A})$, a contradiction. □

6. Condensing filters on ω

In this section we investigate for which algebras $\mathfrak{A}$ of subsets of ω the Stone space $K = \mathrm{ULT}(\mathfrak{A})$ satisfies $S(K) = P(K)$ (i.e. condition (i) of Proposition 4.3). As we shall see this problem is naturally connected with the properties of densities on ω. Some of the concepts and remarks presented here, in particular the one concerning densities of the form d_φ have been suggested by Tomek Bartoszyński, Adam Krawczyk and Michael Hrusak.

We shall denote by $[\omega]^\omega$ the family of all infinite subsets of ω; $[\omega]$ will stand for the whole power set of ω (note that the symbol P is already in use). For $A, B \subseteq \omega$ we write $A \subseteq^* B$ if A is almost included in B, i.e. if the set $A \setminus B$ is finite. Recall that the *asymptotic density* of a set $A \subseteq \omega$, denoted usually by $d(A)$, is defined as

$$d(A) = \lim_{n\to\infty} \frac{|A \cap n|}{n},$$

provided the limit exists.

We start by the following simple example which illustrates the main idea.

Example 6.1. There is an algebra $\mathfrak{A} \subseteq [\omega]$ containing all finite sets and such that in the space $K = \mathrm{ULT}(\mathfrak{A})$ (which contains ω as a dense discrete subset) there is $\mathcal{F} \in K$ such that $\delta_\mathcal{F} \in S(\omega)$ while $\mathcal{F}$ is not in the sequential closure of ω.

Proof. Let $\mathcal{F}$ be the filter of all sets $A \subseteq \omega$ of density 1; let $\mathfrak{A}$ be the algebra generated by $\mathcal{F}$, that is $\mathfrak{A} = \{A \subseteq \omega \colon d(A) = 1 \text{ or } d(A) = 0\}$. Consider now $\mathcal{F} \in K = \mathrm{ULT}(\mathfrak{A})$.

Every infinite $B \subseteq \omega$ contains an infinite subset A of density zero, and this easily implies that ω contains no converging sequences; in particular, ω is a sequentially closed subset of K. On the other hand, $\delta_\mathcal{F} \in S(\omega)$, simply because

$$\delta_\mathcal{F} = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \delta_i.$$

□

Recall that if $\mathcal{F} \subseteq [\omega]^\omega$ is any family closed under finite intersections then a set $A \in [\omega]^\omega$ is called a *pseudo-intersection* of $\mathcal{F}$ if $A \subseteq^* F$ for every $F \in \mathcal{F}$. From the topological point of view, if $\mathcal{F} \in \mathrm{ULT}(\mathfrak{A})$ has a pseudo-intersection A then elements of A form a sequence converging to $\mathcal{F}$ in the Stone space of $\mathfrak{A}$. The cardinal number $\mathfrak{p}$ is defined so that whenever we have a family $\mathcal{F} \subseteq [\omega]^\omega$ of *fewer* than $\mathfrak{p}$, and $\mathcal{F}$ is closed under finite unions, then $\mathcal{F}$ has a pseudo-intersection. We can imitate those classical concepts as follows.

For an infinite set $B \subseteq \omega$ fix a strictly increasing enumeration $b_1 < b_2 < b_3 < \dots$ of its elements. Then for any A define the *relative density of A in B* by

$$d_B(A) = d(\{n\colon\ b_n \in A\}),$$

provided the limit exists.

Definition 6.2. We say that $B \in [\omega]^\omega$ is a *condenser* of a family $\mathcal{A} \subseteq [\omega]^\omega$ if $d_B(A) = 1$ for every $A \in \mathcal{A}$.

It is clear that if B is a pseudo-intersection of $\mathcal{F}$ then B is a condenser of $\mathcal{F}$; letting $\mathcal{F}$ be the filter $\{A \subseteq \omega\colon\ d(A) = 1\}$ we have an example of a filter having ω as a condenser, but having no pseudo-intersection. We note that a condenser of a filter need not be its element.

Example 6.3. There is a filter $\mathcal{F}$ whose all condensers lie outside $\mathcal{F}$.

Proof. Let $G_n = (2^{n-1}, 2^n] \cap \omega$ for every $n \geqslant 0$. Let $\mathcal{F}$ be a filter generated by the family

$$\Big\{ \bigcup_{n \in D} G_n\colon\ d(D) = 1 \Big\}.$$

If B is any selector of the family $\{G_n\colon\ n \geqslant 0\}$ then $d_B(F) = 1$ for every $F \in \mathcal{F}$. However, it is easy to check that for every $F \in \mathcal{F}$ there is $F_1 \in \mathcal{F}$ such that $d_F(F_1) = \frac{1}{2}$. □

The relevance of condensers comes from the observation that if $\mathcal{F} \in \mathrm{ULT}(\mathfrak{A})$ has a condenser B then $\delta_{\mathcal{F}} \in S(B)$. In fact, we may consider here a slightly more general notion of density. For a function $\varphi\colon\ \omega \to \mathbb{R}^+$ define

$$d_\varphi(A) = \lim_{n \to \infty} \frac{\sum_{i \in A \cap n} \varphi(i)}{\sum_{i \in n} \varphi(i)},$$

provided the limit exists. We say that a density d_φ *condenses* a filer $\mathcal{F}$ if $d_\varphi(A) = 1$ for every $A \in \mathcal{F}$. We have the following obvious lemma.

Lemma 6.4. *Let $\mathfrak{A}$ be an algebra of subsets of ω containing all finite sets. Suppose that $\mathcal{F} \in \mathrm{ULT}(\mathfrak{A})$ is such a filter that for some φ the density d_φ condenses $\mathcal{F}$. Then $\delta_{\mathcal{F}} \in S(\omega)$.*

Let us write $\mathfrak{k}$ ($\mathfrak{k}^*$) for the minimal cardinal number κ for which there is a family $\mathcal{A} = \{A_\xi\colon\ \xi < \kappa\} \subseteq [\omega]^\omega$ such that $A_\xi \subseteq^* A_\eta$ whenever $\eta < \xi < \kappa$, and $\mathcal{A}$ has no condenser, that is there is no $\varphi\colon\ \omega \to \mathbb{R}^+$ with $d_\varphi(A_\xi) = 1$ for every $\xi < \kappa$). We have the following obvious inequalities

$$\omega_1 \leqslant \mathfrak{p} \leqslant \mathfrak{k} \leqslant \mathfrak{k}^* \leqslant \mathfrak{c},$$

but it is not known if any of the relations $\mathfrak{p} < \mathfrak{k}$, $\mathfrak{k} < \mathfrak{k}^*$ is consistent with the usual axioms of the set theory. One can check that $\mathfrak{k}^* \leqslant \mathfrak{b}$ and that $\mathfrak{k}^* < \mathfrak{b}$ is relatively consistent, basing on some results on the cardinal number $\mathfrak{b}$, see Blass [3]. Ideals of the form $\{A \subseteq \omega\colon\ d_\varphi(A) = 0\}$, where $\varphi\colon\ \omega \to \mathbb{R}^+$, are sometimes called the Erdős-Ulam ideals; cardinal invariants of such ideals on ω are considered by Hrusak [11] and Farkas & Soukup [8].

The following problem may be stated independently of the Banach space properties we are discussing.

Problem 6.5. Is it consistent that there is a Boolean algebra $\mathfrak{A}$ of subsets of natural numbers such that

(i) no ultrafilter on $\mathfrak{A}$ has a pseudo-intersection;

(ii) every ultrafilter on $\mathfrak{A}$ has a condenser (or at least is condensed by some density)?

Equivalently, we ask here if there is a compactification K of ω such that for every $t \in K \setminus \omega$, t is not a limit of a sequence from ω while δ_t is the the limit of a sequence of purely atomic measures supported by ω.

7. A possible example

Recall that $\mathcal{A} \subseteq [\omega]^\omega$ is a *m.a.d. family* if it is maximal pairwise almost disjoint.

The cardinal number $\mathfrak{h}$ mentioned below is *the distributivity number*, i.e. the smallest cardinality of a collection $\mathcal{T}$ of m.a.d. families whose union is splitting, i.e. for every infinite $A \subseteq \omega$ there is $T \in \bigcup \mathcal{T}$ such that both $A \cap T$ and $A \setminus T$ are infinite. It is known that $\mathfrak{p} \leqslant \mathfrak{h} \leqslant \mathfrak{b}$, see [3].

Theorem 7.1 (Balcar, Pelant, Simon [1]). *There is a family of infinite sets $\mathcal{S} \subseteq [\omega]^\omega$ such that*

- *$\mathcal{S}$ is a $\subseteq^*$-tree of height $\mathfrak{h}$,*
- *each level of $\mathcal{S}$, except of the root (which is ω), is a m.a.d. family,*
- *every infinite $A \subseteq \omega$ contains an element from $\mathcal{S}$.*

A $\subseteq^*$-tree satisfying the above properties is often called a *base matrix tree.* We can assume that each of its nodes has $\mathfrak{c}$ immediate successors.

Theorem 7.2. *Let $\mathcal{S}$ be a base matrix tree of Theorem 7.1, and let $\mathfrak{A}$ be an algebra of subsets of ω generated by $\mathcal{S}$ together with all finite sets. Further, let K be the Stone space of all ultrafilters on $\mathfrak{A}$.*

(i) *The Banach space $C(K)$ does not have the Gelfand-Phillips property.*
(ii) *The space K satisfies $S(K) = P(K)$.*
(iii) *If $\mathfrak{h} < \mathfrak{k}^*$ then for every $t \in K \setminus \omega$ and every $Y \subseteq \omega$, if $t \in \overline{Y}$ then $\delta_t \in S(Y)$.*
(iv) *Consequently, assuming $\mathfrak{h} < \mathfrak{k}^*$ the space $C(K)$ has the Mazur property.*

Proof. Part (i) follows from Corollary 4.2 since $\mathcal{S}$ contains a π-base. Part (ii) follows from Theorem 5.1 since the algebra $\mathfrak{A}$, as a tree algebra, is minimally generated, see e.g. [4] (actually, $S(K) = P(K)$ can be also derived from a result due to Sapounakis that every measure on K has a uniformly distributed sequence, see [20]).

Let us write $\mathcal{S}$ as the union of the tree levels $\mathcal{L}_\xi$, $\xi < \mathfrak{h}$, so that every $\mathcal{L}_\xi$ is an almost disjoint family, and every $A \in \mathcal{L}_\xi$ has $\mathfrak{c}$ immediate almost disjoint successors.

We now check (iii). Let $t \in K \setminus \omega$ be such that $t \in \overline{Y}$ for some $Y \subseteq \omega$. We write $t = \mathcal{F}$ when thinking of t as of an ultrafilter on $\mathfrak{A}$.

Suppose that $\mathcal{F} \cap \mathcal{L}_\xi \neq \emptyset$ for every $\xi < \mathfrak{h}$; then $\mathcal{F}$ is generated by a family A_ξ, $\xi < \mathfrak{h}$, where $A_\xi \in \mathcal{F} \cap \mathcal{L}_\xi$, forming a branch. Then the sets $A_\xi \cap Y$ are infinite and form a $\subseteq^*$-decreasing family, so by our assumption $\mathfrak{h} < \mathfrak{k}^*$ there is a function $\varphi\colon Y \to \mathbb{R}_+$ such that the corresponding density d_φ satisfies $d_\varphi(A_\xi) = 1$ for every $\xi < \mathfrak{h}$. This implies that δ_t is the limit of measures from $\operatorname{conv} Y$, see Lemma 6.4.

Suppose now that $A \in \mathcal{F} \cap \mathcal{L}_\xi$ while no $B \in \mathcal{L}_{\xi+1}$ is in $\mathcal{F}$. Since $t = \mathcal{F}$ lies in the closure of Y we can choose a sequence of almost disjoint $B_n \in \mathcal{L}_{\xi+1}$ such that $B_n \subseteq^* A$ and $B_n \cap Y$ is infinite for every n. For every n we can pick an ultrafilter $\mathcal{F}_n$ on $\mathfrak{A}$ containing $B_n \cap Y$ and such that $\mathcal{F}_n$ is generated by some branch of the tree $\mathcal{S}$. Writing $t_n = \mathcal{F}_n$ we have $\delta_{t_n} \in S(Y)$ by the above argument. But we have $t_n \to t$ in the space K, so δ_t is also in $S(Y)$ as the limit of δ_{t_n}.

The remaining case is that the first γ for which $\mathcal{F} \cap \mathcal{L}_\gamma = \emptyset$ is the limit ordinal but then we can argue in a similar manner: for $\xi < \gamma$ pick $A_\xi \in \mathcal{F} \cap \mathcal{L}_\xi$; there must be a sequence of distinct $B_n \in \mathcal{L}_\gamma$ such that each $B_n \subseteq^* A_\xi$ for $\xi < \gamma$ and $B_n \cap Y$ is infinite. Again we get $t = \mathcal{F}$ as the limit of branches.

Finally, (iv) follows from (iii) and Lemma 4.3. □

Unfortunately, it is not known if the assumption $\mathfrak{h} < \mathfrak{k}^*$ appearing in part (iii) of Theorem 7.2 is consistent with ZFC.

At least, we can show that it is consistent with ZFC that there exists a Boolean algebra with properties similar to those of the above theorem and of Problem 6.5. We have to relax the property that all ultrafilters have to possess condensers. Instead of this, we will demand that all ultrafilters have to be feeble.

Definition 7.3. A filter $\mathcal{F} \subseteq P(\omega)$ is *feeble* provided there is a finite-to-one function $f\colon \omega \to \omega$ such that $f[F]$ is co-finite for every $F \in \mathcal{F}$.

Note that the assumption $\mathfrak{h} < \mathfrak{b}$ is consistent with ZFC. Namely, in standard Hechler's model $\mathfrak{h} = \aleph_1$ whereas $\mathfrak{b} = \mathfrak{c}$ (see [3]).

Theorem 7.4. *Assume $\mathfrak{h} < \mathfrak{b}$. Then there is a Boolean algebra $\mathfrak{A} \subseteq P(\omega)$ such that*

(i) *no ultrafilter on $\mathfrak{A}$ has a pseudo-intersection;*
(ii) *every ultrafilter on $\mathfrak{A}$ is feeble.*

The following fact reveals the connection between feebleness and condensers and shows that the above object is somehow similar to that of Problem 6.5.

Fact 7.5. *If a filter $\mathcal{F}$ has a condenser, then it is feeble.*

Proof. Assume P is a condenser of $\mathcal{F}$ and fix a co-infinite $N \subseteq \omega$ of density 1. Fix increasing enumerations $p_1 < p_2 < \ldots$ of elements of P and $n_1 < n_2 < \ldots$ of elements of N. Let $f\colon \omega \to \omega$ be such that $f|_{\omega \setminus P}$ is any bijection onto $\omega \setminus N$ and $f(p_k) = n_k$. Clearly, f is a bijection and $f[F]$ has density 1 for every $F \in \mathcal{F}$.

Notice that the function $g(n) = [\log_2(n)]$ proves the feebleness of the density filter. Indeed, it is finite-to-one and if A is co-infinite, then $d_*(g^{-1}[A]) < \frac{1}{2}$.

Therefore, $g \circ f$ witnesses that $\mathcal{F}$ is feeble. □

The proof of Theorem 7.4 resembles that of Theorem 7.2, but we need several definitions and lemmas. We find it convenient to say that for a filter $\mathcal{F} \subseteq P(\omega)$ with a pseudo-intersection, a family $\mathcal{A} \subseteq P(\omega)$ is *a m.a.d. family below* $\mathcal{F}$ if it is a maximal infinite family such that $\mathcal{A}$ is pairwise almost disjoint and consists of pseudo-intersections of $\mathcal{F}$.

Lemma 7.6. *Let $\mathcal{A}$ be a m.a.d. family below a filter $\mathcal{F}$ and let $f\colon \omega \to \omega$ be a bijection. Then there is a refinement $\mathcal{B}$ of $\mathcal{A}$ (i.e. for every $B \in \mathcal{B}$ there is $A \in \mathcal{A}$ such that $B \subseteq^* A$) such that $\mathcal{B}$ is a m.a.d. family below $\mathcal{F}$ and $f[B]$ has density 0 for every $B \in \mathcal{B}$.*

Proof. Let $\mathcal{B}$ be a maximal family such that

- $\mathcal{B}$ is pairwise almost disjoint,
- $\mathcal{B}$ is a refinement of $\mathcal{A}$,
- if $B \in \mathcal{B}$, then $f[B]$ has density 0.

The family $\mathcal{B}$ is a m.a.d. family below $\mathcal{F}$. Indeed, assume that there is an infinite $N \notin \mathcal{B}$ such that $N \cap B$ is finite for every $B \in \mathcal{B}$. Clearly, $N \cap A$ is infinite for some $A \in \mathcal{A}$. Hence, every infinite $M \subseteq A \cap N$ such that $f[M]$ is of density 0 contradicts the maximality assumption. □

Lemma 7.7. *There is a base matrix tree $\mathcal{S}$ such that if $\mathcal{T}$ is a linearly ordered by $\subseteq^*$ subfamily of $\mathcal{S}$ (a tower), then there is a bijection $f_{\mathcal{T}}\colon \omega \to \omega$ such that*
(a) *$f_{\mathcal{T}}[S]$ has density 1 for every $S \in \mathcal{T}$;*
(b) *$f_{\mathcal{T}}[S]$ has density 0 if $S \subsetneq^* T$ for every $T \in \mathcal{T}$.*

Proof. Let $\mathcal{S}'$ be a base matrix tree with all branches cofinal. Denote by $(\mathcal{L}'_\xi)_{\xi<\mathfrak{h}}$ the levels of $\mathcal{S}'$. We define the levels $(\mathcal{L}_\xi)_{\xi<\mathfrak{h}}$ of $\mathcal{S}$ inductively modifying the levels of $\mathcal{S}'$. Let $\mathcal{L}_0 = \mathcal{L}'_0$ and $\xi < \mathfrak{h}$.

Assume that we have defined $\mathcal{L}_\alpha$ for every $\alpha < \xi$. Consider a tower $\mathcal{T} = (T_\alpha)_{\alpha<\xi}$, where $T_\alpha \in \mathcal{L}_\alpha$. Since $\mathcal{T}$ has a pseudo-intersection, there is a bijection $f_{\mathcal{T}}\colon \omega \to \omega$ such that $f_{\mathcal{T}}[T_\alpha]$ has density 1 for every $\alpha < \xi$.

Consider the maximal almost disjoint family $\mathcal{A}$ which refines $\mathcal{L}'_\xi$ and is below $\mathcal{T}$. Use Lemma 7.6 to find a refinement $\mathcal{B}_{\mathcal{T}}$ of $\mathcal{A}$ such that $f_{\mathcal{T}}[B]$ has density 0 for every $B_{\mathcal{T}} \in \mathcal{B}$.

Repeat this procedure for every tower $\mathcal{T}$ of height ξ and enumerate $L_\xi = \bigcup_{\mathcal{T}} \mathcal{B}_{\mathcal{T}}$.

□

The following theorem due to Solomon is proved in [3, Theorem 9.10].

Theorem 7.8. *Every filter generated by less than $\mathfrak{b}$ sets is feeble.*

Before proving the main theorem notice that if a Boolean algebra is generated by a base matrix tree, then it does not have an ultrafilter with a pseudo-intersection. Otherwise, we could easily find an infinite subset of the pseudo-intersection which does not contain any element of the tree.

Proof of Theorem 7.4. Let $\mathfrak{A}$ be the Boolean algebra generated by $\mathcal{S}$ from Lemma 7.7. We can repeat the proof of Theorem 7.2 to show that $\mathfrak{A}$ satisfies the demanded conditions.

Let $\mathcal{F}$ be an ultrafilter on $\mathfrak{A}$. Following the terminology of the proof of Theorem 7.2 we have to deal with two cases. If there is no $\alpha < \mathfrak{h}$ such that $\mathcal{F} \cap \mathcal{L}_\alpha = \emptyset$, then $\mathcal{F}$ is generated by $\mathfrak{h}$ many sets. So, by Theorem 7.8 it is feeble, since $\mathfrak{h} < \mathfrak{b}$.

Otherwise, there is $\alpha < \mathfrak{h}$ such that $\mathcal{F} \cap \mathcal{L}_\alpha = \emptyset$. Consider the family

$$\mathcal{T} = \{T \in \mathcal{S} \cap \mathcal{F}\colon\ T \in \mathcal{L}_\beta,\ \beta < \alpha\}.$$

Since it is a tower, we can find a bijection $f_{\mathcal{T}}$ as in Lemma 7.7. Then $f_{\mathcal{T}}[F]$ is of density 1 for every $F \in \mathcal{F}$. If $g\colon \omega \to \omega$ is a function witnessing the feebleness of the density filter (e.g. that from the proof of Fact 7.5), then $g \circ f_{\mathcal{T}}$ proves that $\mathcal{F}$ is feeble. □

Unfortunately, the Boolean algebra from Theorem 7.4 cannot be used directly to produce a Banach space with the Mazur property and without the Gelfand-Phillips property.

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