Ali Abkar; B. Jafarzadeh

Weighted sub-Bergman Hilbert spaces in the unit disk

Czechoslovak Mathematical Journal, Vol. 60 (2010), No. 2, 435–443

Persistent URL: http://dml.cz/dmlcz/140580

Czechoslovak Mathematical Journal, 60 (135) (2010), 435–443

WEIGHTED SUB-BERGMAN HILBERT SPACES IN THE UNIT DISK

A. Abkar, Qazvin, B. Jafarzadeh, Mahshahr

(Received December 31, 2008)

Abstract. We study sub-Bergman Hilbert spaces in the weighted Bergman space A^2_α. We generalize the results already obtained by Kehe Zhu for the standard Bergman space A^2.

Keywords: weighted Bergman space, sub-Bergman Hilbert space, weighted Toeplitz operator, reproducing kernel

MSC 2010: 47B35, 46E22 (30H05)

1. Introduction

Let $\mathbb{D}$ denote the unit disk in the complex plane. For $\alpha > -1$ we define the weighted Bergman space A^2_α as the space of all analytic functions f in $\mathbb{D}$ such that

$$\int_{\mathbb{D}} |f(z)|^2 \,\mathrm{d}A_\alpha(z) < +\infty$$

where $\mathrm{d}A_\alpha(z) = (\alpha+1)\pi^{-1}(1-|z|^2)^\alpha \,\mathrm{d}x\,\mathrm{d}y$ denotes the normalized area measure. It is well-known that A^2_α is a Hilbert space of analytic functions. The weighted Bergman projection $P_\alpha\colon L^2(\mathbb{D}, \mathrm{d}A_\alpha) \to A^2_\alpha$ is defined by

$$P_\alpha f(z) = \int_{\mathbb{D}} f(w) K_\alpha(z,w) \,\mathrm{d}A_\alpha(w),$$

where

$$K_\alpha(z,w) = \frac{1}{(1-z\overline{w})^{2+\alpha}}, \quad (z,w) \in \mathbb{D}\times\mathbb{D}$$

is the reproducing kernel for the space A^2_α. For $\varphi \in L^\infty(\mathbb{D})$, the weighted Toeplitz operator on A^2_α is defined by

$$T^\alpha_\varphi f = P_\alpha(\varphi f).$$

When $\alpha = 0$, we omit the superscript and simply write T_φ instead of T^0_φ; using this convention, P, $\mathrm{d}A$, and $K(z,w)$ stand respectively for P_α, $\mathrm{d}A_\alpha$, and $K_\alpha(z,w)$ in the standard (unweighted) Bergman space case $\alpha = 0$.

Let H_1 and H_2 be two Hilbert spaces, and let $T\colon H_1 \to H_2$ be a bounded operator. The range of T with the inner product

$$\langle Tx, Ty\rangle_{H_2} = \langle x, y\rangle_{H_1}, \qquad x, y \in H_1 \ominus \ker T,$$

is denoted by $\mathcal{M}(T)$. The Hilbert space

$$\mathcal{H}(T) = \mathcal{M}((I - TT^*)^{1/2})$$

is called the complemented space to $\mathcal{M}(T)$.

Recall that $H^\infty = H^\infty(\mathbb{D})$ denotes the Banach space of all bounded analytic functions on the unit disk; we denote its unit ball by $(H^\infty)_1$. We consider a function $\varphi \in (H^\infty)_1$ and study the spaces $\mathcal{H}(T^\alpha_\varphi)$ and $\mathcal{H}(T^\alpha_{\overline{\varphi}})$. These are Hilbert spaces in the weighted Bergman space A^2_α, and are called *sub-Bergman Hilbert spaces*. For simplicity, we denote them by $\mathcal{H}_\alpha(\varphi)$ and $\mathcal{H}_\alpha(\overline{\varphi})$ respectively. For $\alpha = 0$, these spaces were studied by Kehe Zhu in his two subsequent papers [5] and [6]. Indeed, Zhu's work was inspired by the pioneering work of Donald Sarason in introducing the phrase "*sub-Hardy Hilbert spaces*" in [2]. For the history and importance of the sub-Hardy and sub-Bergman Hilbert spaces we refer the reader to the just mentioned papers.

In [5], Zhu proved that $\mathcal{H}(\varphi)$ equals $\mathcal{H}(\overline{\varphi})$ and that both the spaces contain H^∞. He then was able to show that if $\varphi = B$ is a finite Blaschke product, then $\mathcal{H}(B) = H^2$, the Hardy space on the unit disk (see [6]). Here we will see that

$$H^\infty \subset \mathcal{H}_\alpha(\varphi) = \mathcal{H}_\alpha(\overline{\varphi}),$$

for α positive, moreover, if φ equals a finite Blaschke product B, then

$$\mathcal{H}_\alpha(B) = \mathcal{H}_\alpha(\overline{B}) = A^2_{\alpha-1}.$$

We should mention that S. Sultanic in a recent paper obtained the same results by using a very computational method (see [4]).

2. The spaces $\mathcal{H}_\alpha(\varphi)$ and $\mathcal{H}_\alpha(\overline{\varphi})$

This section is devoted to the proof of the fact that the sub-Bergman Hilbert spaces $\mathcal{H}_\alpha(\varphi)$ and $\mathcal{H}_\alpha(\overline{\varphi})$ coincide as sets, and that their norms are equivalent. Moreover, both the spaces contain H^∞.

Proposition 2.1. *Let $\varphi \in (H^\infty)_1$ and $\alpha > -1$. The reproducing kernels of $\mathcal{H}_\alpha(\varphi)$ and $\mathcal{H}_\alpha(\overline{\varphi})$ are given, respectively, by*

$$K_\varphi^\alpha(z,w) = \frac{1-\varphi(z)\overline{\varphi(w)}}{(1-z\overline{w})^{\alpha+2}}, \qquad z,w \in \mathbb{D}$$

and

$$K_{\overline{\varphi}}^\alpha(z,w) = \int_{\mathbb{D}} \frac{1-|\varphi(u)|^2}{(1-z\overline{u})^{\alpha+2}(1-u\overline{w})^{\alpha+2}}\,\mathrm{d}A_\alpha(u).$$

Proof. Suppose that for $w \in \mathbb{D}$, K_w^α are the reproducing kernels of A_α^2. According to I–3 of [2], the reproducing kernels of $\mathcal{H}_\alpha(\varphi)$ are given by

$$(I - T_\varphi^\alpha T_{\overline{\varphi}}^\alpha)K_w^\alpha, \qquad w \in \mathbb{D}.$$

Note that for every $z \in \mathbb{D}$ we have

$$\begin{aligned} T_{\overline{\varphi}}^\alpha K_w^\alpha(z) &= \int_{\mathbb{D}} K_\alpha(z,u)\overline{\varphi(u)}K_w^\alpha(u)\,\mathrm{d}A_\alpha(u) \\ &= \overline{\int_{\mathbb{D}} K_z^\alpha(u)\varphi(u)K_\alpha(w,u)\,\mathrm{d}A_\alpha(u)} \\ &= \overline{T_\varphi^\alpha K_z^\alpha(w)} = \overline{\varphi(w)}K_w^\alpha(z) \end{aligned}$$

so that $T_{\overline{\varphi}}^\alpha K_w^\alpha = \overline{\varphi(w)}K_w^\alpha$, and hence

$$\begin{aligned} K_\varphi^\alpha(z,w) &= (I - T_\varphi^\alpha T_{\overline{\varphi}}^\alpha)K_w^\alpha(z) \\ &= (1 - \overline{\varphi(w)}\varphi)K_w^\alpha(z) \\ &= \frac{1-\varphi(z)\overline{\varphi(w)}}{(1-z\overline{w})^{\alpha+2}}, \qquad z,w \in \mathbb{D}. \end{aligned}$$

As for the second part, we note that according to I–3 of [2], the reproducing kernel of $\mathcal{H}_\alpha(\overline{\varphi})$ has the form

$$K_{\overline{\varphi},w}^\alpha = (I - T_{\overline{\varphi}}^\alpha T_\varphi^\alpha)K_w^\alpha = T_{1-|\varphi|^2}K_w^\alpha.$$

Since for every $z \in \mathbb{D}$ we have

$$\begin{aligned} K_{\overline{\varphi}}^\alpha(z,w) = K_{\overline{\varphi},w}^\alpha(z) &= T_{1-|\varphi|^2}K_w^\alpha(z) \\ &= \int_{\mathbb{D}} \frac{1-|\varphi(u)|^2}{(1-z\overline{u})^{\alpha+2}(1-u\overline{w})^{\alpha+2}}\,\mathrm{d}A_\alpha(u), \end{aligned}$$

the result follows. □

Proposition 2.1. *Let $\varphi \in (H^\infty)_1$ and $\alpha > -1$. Then every element of $\mathcal{H}_\alpha(\overline{\varphi})$ has the representation*

$$f(z) = \int_{\mathbb{D}} \frac{1-|\varphi(w)|^2}{(1-z\overline{w})^{\alpha+2}}\, g(w)\,\mathrm{d}A_\alpha(w),$$

where g is an analytic function satisfying

$$\int_{\mathbb{D}} |g(z)|^2 \left(1-|\varphi(z)|^2\right)\,\mathrm{d}A_\alpha(z) < +\infty.$$

Proof. Put $\mathrm{d}A_{\alpha,\varphi}(z) = (1-|\varphi(z)|^2)\,\mathrm{d}A_\alpha(z)$, and let $A^2_{\alpha,\varphi}$ be the subspace of $L^2(\mathbb{D}, \mathrm{d}A_{\alpha,\varphi})$ consisting of all analytic functions. Define an operator

$$S^\alpha_\varphi\colon\ A^2_{\alpha,\varphi} \to A^2_\alpha$$

by $S^\alpha_\varphi g = P_\alpha((1-|\varphi|^2)g)$. It follows that $\|S^\alpha_\varphi\|_{A^2_\alpha} \leqslant \|g\|_{A^2_{\alpha,\varphi}}$, moreover, for every $f \in A^2_\alpha$ and every $g \in A^2_{\alpha,\varphi}$ we have

$$\begin{aligned}\langle (S^\alpha_\varphi)^* f, g\rangle_{A^2_{\alpha,\varphi}} &= \langle f, P_\alpha\left((1-|\varphi|^2)g\right)\rangle_{A^2_\alpha}\\ &= \langle f, (1-|\varphi|^2)g\rangle_{L^2(\mathbb{D},\mathrm{d}A_\alpha)} = \langle f, g\rangle_{A^2_{\alpha,\varphi}}.\end{aligned}$$

This means that $(S^\alpha_\varphi)^*$ is the inclusion operator. Note that for every $w \in \mathbb{D}$ we have $S^\alpha_\varphi (S^\alpha_\varphi)^* K^\alpha_w \in \mathcal{M}(S^\alpha_\varphi)$. On the other hand, given $f \in \mathcal{M}(S^\alpha_\varphi)$, there exists $g \in A^2_{\alpha,\varphi} \ominus \ker S^\alpha_\varphi$ such that $S^\alpha_\varphi g = f$. Therefore

$$\begin{aligned}\langle f, S^\alpha_\varphi (S^\alpha_\varphi)^* K^\alpha_w\rangle_{\mathcal{M}(S^\alpha_\varphi)} &= \langle g, (S^\alpha_\varphi)^* K^\alpha_w\rangle_{A^2_{\alpha,\varphi}}\\ &= \langle f, K^\alpha_w\rangle_{A^2_\alpha}\\ &= f(w),\end{aligned}$$

which means that $S^\alpha_\varphi (S^\alpha_\varphi)^* K^\alpha_w$ are the reproducing kernels of $\mathcal{M}(S^\alpha_\varphi)$. It now follows that for every $z, w \in \mathbb{D}$ we have

$$\begin{aligned}S^\alpha_\varphi (S^\alpha_\varphi)^* K^\alpha_w(z) &= P_\alpha\left((1-|\varphi|^2)K^\alpha_w\right)(z)\\ &= \int_{\mathbb{D}} \frac{1-|\varphi(u)|^2}{(1-z\overline{u})^{\alpha+2}(1-u\overline{w})^{\alpha+2}}\,\mathrm{d}A_\alpha(u).\end{aligned}$$

This together with Proposition 2.1 implies that $S^\alpha_\varphi (S^\alpha_\varphi)^* K^\alpha_w$ are the reproducing kernels of $\mathcal{H}_\alpha(\overline{\varphi})$, too. Now, from the uniqueness property we conclude that $\mathcal{M}(S^\alpha_\varphi) = \mathcal{H}_\alpha(\overline{\varphi})$. In particular, for every $f \in \mathcal{H}_\alpha(\varphi)$ there is a $g \in A^2_{\alpha,\varphi}$ such that $f = S^\alpha_\varphi g$. □

The next proposition now follows from I–8 and I–9 of [2].

Proposition 2.3. *Let $\varphi \in (H^\infty)_1$, $\alpha > -1$ and $f \in A^2_\alpha$. Then*

(a) *$f \in \mathcal{H}_\alpha(\varphi)$ if and only if $T^\alpha_{\overline{\varphi}} f \in \mathcal{H}_\alpha(\overline{\varphi})$ and in this case*

$$\|f\|^2_{\mathcal{H}_\alpha(\varphi)} = \|f\|^2_{A^2_\alpha} + \|T^\alpha_{\overline{\varphi}} f\|^2_{\mathcal{H}_\alpha(\overline{\varphi})},$$

(b) *$f \in \mathcal{H}_\alpha(\overline{\varphi})$ if and only if $T^\alpha_{\varphi} f \in \mathcal{H}_\alpha(\varphi)$ and in this case*

$$\|f\|^2_{\mathcal{H}_\alpha(\overline{\varphi})} = \|f\|^2_{A^2_\alpha} + \|T^\alpha_{\varphi} f\|^2_{\mathcal{H}_\alpha(\varphi)},$$

(c) $\mathcal{M}(T^\alpha_\varphi) \cap \mathcal{H}_\alpha(\varphi) = \varphi \mathcal{H}_\alpha(\overline{\varphi})$.

Proposition 2.4. *Let $\varphi \in (H^\infty)_1$ and $\alpha > 0$. Then every $\psi \in H^\infty$ is a multiplier on both $\mathcal{H}_\alpha(\varphi)$ and $\mathcal{H}_\alpha(\overline{\varphi})$, moreover, $\|T^\alpha_\psi\| \leqslant \|\psi\|_\infty$.*

P r o o f. Assume that $\|\psi\|_\infty = 1$. By Proposition 2.1, the functions

$$\frac{1 - \psi(z)\overline{\psi(w)}}{(1 - z\overline{w})^{1+\alpha/2}}, \qquad \frac{1 - \varphi(z)\overline{\varphi(w)}}{(1 - z\overline{w})^{1+\alpha/2}}$$

are reproducing kernels of $\mathcal{H}_{\alpha/2-1}(\psi)$ and $\mathcal{H}_{\alpha/2-1}(\varphi)$, respectively. According to Lemma 3.11 of [5] the product

$$\begin{aligned} K(z,w) &= \frac{(1 - \psi(z)\overline{\psi(w)})(1 - \varphi(z)\overline{\varphi(w)})}{(1 + z\overline{w})^{\alpha+2}} \\ &= (1 - \psi(z)\overline{\psi(w)}) K^\alpha_\varphi(z,w) \end{aligned}$$

is again a reproducing kernel on $\mathbb{D}$. It now follows from a theorem of Beatrous and Burbea (see [3], or Theorem 2.2 of [5]) that ψ is a contractive multiplier on $\mathcal{H}_\alpha(\varphi)$. To see that ψ is a multiplier on $\mathcal{H}_\alpha(\overline{\varphi})$ we assume $f \in \mathcal{H}_\alpha(\overline{\varphi})$. According to Proposition 2.4, $\varphi f \in \mathcal{H}_\alpha(\varphi)$ and hence $\psi(\varphi f) \in \mathcal{H}_\alpha(\varphi)$. Thus $\psi f \in \mathcal{H}_\alpha(\overline{\varphi})$, by Proposition 2.4. Finally, we note that

$$\begin{aligned} \|\psi f\|^2_{\mathcal{H}_\alpha(\overline{\varphi})} &= \|\psi f\|^2_{A^2_\alpha} + \|\psi \varphi f\|^2_{\mathcal{H}_\alpha(\varphi)} \\ &= \|\psi\|^2_\infty (\|f\|^2_{A^2_\alpha} + \|\varphi f\|^2_{\mathcal{H}_\alpha(\varphi)}) \\ &= \|f\|^2_{\mathcal{H}_\alpha(\overline{\varphi})}. \end{aligned}$$

□

Theorem 2.5. *Let $\varphi \in (H^\infty)_1$ and $\alpha > 0$. Then $\mathcal{H}_\alpha(\varphi) = \mathcal{H}_\alpha(\overline{\varphi})$ with equivalence of norms.*

Proof. Assume that $\varphi \neq 0$, otherwise $\mathcal{H}_\alpha(\varphi) = \mathcal{H}_\alpha(\overline{\varphi}) = A^2_\alpha$. By the preceding proposition, $\varphi\mathcal{H}_\alpha(\varphi) \subset \mathcal{H}_\alpha(\varphi)$. On the other hand, $\varphi\mathcal{H}_\alpha(\varphi) \subset \varphi A^2_\alpha = \mathcal{M}(T^\alpha_\varphi)$. It now follows from Proposition 2.3 that

$$\varphi\mathcal{H}_\alpha(\varphi) \subset \mathcal{M}(T^\alpha_\varphi) \cap \mathcal{H}_\alpha(\varphi) = \varphi\mathcal{H}_\alpha(\overline{\varphi}).$$

This implies that $\mathcal{H}_\alpha(\varphi) \subset \mathcal{H}_\alpha(\overline{\varphi})$. As for the reverse inclusion, let T denote the operator of multiplication by φ on $L^2(\mathbb{D}, \mathrm{d}A_\alpha)$. It is well-known that T is bounded and $T^*f = \overline{\varphi}f$. Now for every f and g in $L^2(\mathbb{D}, \mathrm{d}A_\alpha)$ we have

$$\begin{aligned}\langle T^*Tf, g\rangle &= \int_{\mathbb{D}} \varphi(z)f(z)\overline{\varphi(z)}\,\overline{g(z)}\,\mathrm{d}A_\alpha(z)\\ &= \langle \overline{\varphi}f, \overline{\varphi}g\rangle\\ &= \langle TT^*f, g\rangle.\end{aligned}$$

This shows that T is a normal operator, from which it follows that its restriction to A^2_α is subnormal:

$$T^\alpha_\varphi T^\alpha_{\overline{\varphi}} = T^\alpha_\varphi (T^\alpha_\varphi)^* \leqslant (T^\alpha_\varphi)^* T^\alpha_\varphi = T^\alpha_{\overline{\varphi}} T^\alpha_\varphi.$$

This implies the inclusion $\mathcal{H}_\alpha(\overline{\varphi}) \subset \mathcal{H}_\alpha(\varphi)$ from which the equality $\mathcal{H}_\alpha(\varphi) = \mathcal{H}_\alpha(\overline{\varphi})$ follows. Finally, let $I_1\colon \mathcal{H}_\alpha(\varphi) \to \mathcal{H}_\alpha(\overline{\varphi})$ and $I_2\colon \mathcal{H}_\alpha(\overline{\varphi}) \to \mathcal{H}_\alpha(\varphi)$ denote the identity operators. By Proposition 2.3, both I_1 and I_2 are bounded, so that the norms on $\mathcal{H}_\alpha(\varphi)$ and $\mathcal{H}_\alpha(\overline{\varphi})$ are equivalent. □

Theorem 2.6. *Let $\varphi \in (H^\infty)_1$ and $\alpha > 0$. Then $H^\infty \subset \mathcal{H}_\alpha(\varphi) = \mathcal{H}_\alpha(\overline{\varphi})$.*

Proof. According to the preceding theorem, it remains to verify that $H^\infty \subset \mathcal{H}_\alpha(\overline{\varphi})$. To this end, it suffices to show that $\mathcal{H}_\alpha(\overline{\varphi})$ contains a nonzero constant function (see Proposition 2.4). Let E denote the proper subspace of $A^2_{\alpha,\varphi}$ generated by $\{z^n\}_{n\geqslant 1}$. Consider $g \in A^2_{\alpha,\varphi} \ominus E$ with $\|g\|_{A^2_{\alpha,\varphi}} = 1$. Put

$$f(z) = \langle g, 1\rangle_{A^2_{\alpha,\varphi}} = \int_{\mathbb{D}} g(u)(1 - |\varphi(u)|^2)\,\mathrm{d}A_\alpha(u).$$

According to Proposition 2.2, the constant function f belongs to $\mathcal{H}_\alpha(\overline{\varphi})$. However, f does not vanish identically, otherwise we get

$$\langle g, 1\rangle_{A^2_{\alpha,\varphi}} = 0, \qquad g \in E^\perp$$

from which we obtain $1 \in E$, a contradiction. □

3. Finite Blaschke products

In this section we intend to describe $\mathcal{H}_\alpha(B)$ and $\mathcal{H}_\alpha(\overline{B})$ where B is a finite Blaschke product. For the standard Bergman space A^2_α, this was done by Zhu in [6]. He proved that $\mathcal{H}_\alpha(B) = \mathcal{H}_\alpha(\overline{B}) = H^2$, the Hardy space. The following theorem says that for $\alpha > 0$, the spaces $\mathcal{H}_\alpha(B)$ and $\mathcal{H}_\alpha(\overline{B})$ equal $A^2_{\alpha-1}$, the Hilbert space associated with the reproducing kernel

$$K_w^{\alpha-1}(z) = \frac{1}{(1 - z\overline{w})^{\alpha+1}}.$$

Note that for $\alpha = 0$, the function $(1 - z\overline{w})^{-1}$ is the reproducing kernel for the Hardy space.

Theorem 3.1. *Let B be a finite Blaschke product and $\alpha > 0$. Then*

$$\mathcal{H}_\alpha(B) = \mathcal{H}_\alpha(\overline{B}) = A^2_{\alpha-1}.$$

Proof. We first verify that $\mathcal{H}_\alpha(\overline{B}) \subset A^2_{\alpha-1}$. Let $f \in \mathcal{H}_\alpha(\overline{B})$. By Proposition 2.2 we have

$$f(z) = Tg(z) = \int_{\mathbb{D}} \frac{1 - |B(w)|^2}{(1 - z\overline{w})^{\alpha+2}} \, g(w) \, \mathrm{d}A_\alpha(w),$$

where g is an analytic function satisfying

$$\int_{\mathbb{D}} |g(z)|^2 \left(1 - |B(z)|^2\right) \, \mathrm{d}A_\alpha(z) < +\infty.$$

According to Lemma 1 of [5], there exists a $C > 0$ such that

$$1 - |B(z)|^2 \leqslant C(1 - |z|^2), \qquad z \in \mathbb{D},$$

from which it follows that $g \in A^2_{\alpha+1}$. Moreover, for every $z \in \mathbb{D}$ we have

$$(1 - |z|^2)^{-1}|f(z)| \leqslant C(1 - |z|^2)^{-1} \int_{\mathbb{D}} \frac{(1 - |w|^2)^{\alpha+1}}{|1 - z\overline{w}|^{\alpha+2}} |g(w)| \, \mathrm{d}A(w).$$

Put $\mathrm{d}\mu(z) = (1 - |z|^2)^{\alpha+1} \, \mathrm{d}A(z)$. By Theorem 1.9 of [1] the operator

$$\Lambda g(z) = (1 - |z|^2)^{-1} \int_{\mathbb{D}} \frac{(1 - |w|^2)^{\alpha+1}}{|1 - z\overline{w}|^{\alpha+2}} g(w) \, \mathrm{d}A(w)$$

is bounded on $L^2(\mathbb{D}, \mathrm{d}\mu)$. Therefore we can find a constant C_1 such that

$$\int_{\mathbb{D}} |f(z)|^2 (1 - |z|^2)^{-2} \, \mathrm{d}\mu(z) \leqslant C_1 \|g\|_{L^2(\mathbb{D}, \mathrm{d}\mu)} = \frac{C_1}{\alpha + 2} \|g\|_{A^2_{\alpha+1}}.$$

This argument shows that $f \in A^2_{\alpha-1}$, or $\mathcal{H}_\alpha(\overline{B}) \subset A^2_{\alpha-1}$. So far we have proved that $\mathcal{H}_\alpha(\overline{B})$ equals the range of the operator $T\colon A^2_{\alpha,B} \to A^2_{\alpha-1}$. We now consider the operator $S\colon A^2_{\alpha-1} \to A^2_{\alpha,B}$ defined by

$$h(z) = Sf(z) = \int_{\mathbb{D}} \frac{f(w)}{(1-z\overline{w})^{\alpha+2}} \,\mathrm{d}A_{\alpha-1}(w).$$

Note that for $f \in A^2_{\alpha-1}$ we have

$$\begin{aligned} f(z) + \frac{zf'(z)}{\alpha+1} &= \int_{\mathbb{D}} \frac{f(w)}{(1-z\overline{w})^{\alpha+1}} \,\mathrm{d}A_{\alpha-1}(w) + \frac{z}{\alpha+1}\int_{\mathbb{D}} \frac{(\alpha+1)\overline{w}f(w)}{(1-z\overline{w})^{\alpha+2}} \,\mathrm{d}A_{\alpha-1}(w) \\ &= \int_{\mathbb{D}} \frac{f(w)}{(1-z\overline{w})^{\alpha+2}} \,\mathrm{d}A_{\alpha-1}(w) = Sf(z), \end{aligned}$$

from which it follows that for $f(z) = \sum\limits_{n=0}^{\infty} a_n z^n$, we have

$$Sf(z) = \sum_{n=0}^{\infty} \frac{n+\alpha+1}{\alpha+1} a_n z^n.$$

By Lemma 1 of [5] we know that $1-|B(z)|^2 \asymp 1-|z|^2$, so that

$$\begin{aligned} \|Sf\|^2_{A^2_{\alpha,B}} \asymp \|Sf\|^2_{A^2_{\alpha+1}} &= \sum_{n=0}^{\infty} \frac{n!\Gamma(\alpha+3)(n+\alpha+1)^2}{\Gamma(n+\alpha+3)(\alpha+1)^2}|a_n|^2 \\ &\geqslant \sum_{n=0}^{\infty} \frac{n!\Gamma(\alpha+1)}{\Gamma(n+\alpha+1)}|a_n|^2 = \|f\|^2_{A^2_{\alpha-1}}, \end{aligned}$$

which means that S is bounded from below. Since S is invertible, the image of the unit ball of $A^2_{\alpha-1}$ under S contains a ball of radius $r > 0$ centered at zero. Therefore for every unit vector $g \in A^2_{\alpha,B}$ we have

$$\begin{aligned} \|Tg\|_{A^2_{\alpha-1}} &= \sup\{|\langle Tg, f\rangle_{A^2_{\alpha-1}}|\colon \|f\|_{A^2_{\alpha-1}} \leqslant 1\} \\ &= \sup\left\{\left|\int_{\mathbb{D}} g(w)\overline{Sf(w)}(1-|B(w)|^2)\,\mathrm{d}A_\alpha(w)\right|\colon \|f\|_{A^2_{\alpha-1}} \leqslant 1\right\} \\ &\geqslant \sup\left\{\left|\int_{\mathbb{D}} g(w)\overline{h(w)}(1-|B(w)|^2)\,\mathrm{d}A_\alpha(w)\right|\colon \|h\|_{A^2_{\alpha,B}} \leqslant r\right\} \\ &\geqslant \sup\left\{\left|\int_{\mathbb{D}} g(w)\overline{h(w)}(1-|B(w)|^2)\,\mathrm{d}A_\alpha(w)\right|\colon \|h\|_{A^2_{\alpha,B}} = r\right\} \\ &= r\|g\|_{A^2_{\alpha,B}} \\ &= r. \end{aligned}$$

This means that T is bounded from below so that its range, $\mathcal{H}_\alpha(\overline{B})$, is closed in $A^2_{\alpha-1}$. Since $\mathcal{H}_\alpha(\overline{B})$ contains H^∞ by Theorem 2.6 and H^∞ is dense in the weighted Bergman space $A^2_{\alpha-1}$, we conclude that $\mathcal{H}_\alpha(\overline{B}) = A^2_{\alpha-1}$. □

Acknowledgements. This work constitutes part of the second author's Master thesis done under supervision of the first named author at the University of Tehran. Here we record our thanks to Mathematics Department of the University of Tehran.

References

[1] *H. Hedenmalm, B. Korenblum and K. Zhu*: Theory of Bergman Spaces. Graduate Texts in Mathematics, Vol. 199, Springer-Verlag, New York, 2000.
[2] *D. Sarason*: Sub-Hardy Hilbert Spaces in the Unit Disk. John Wiley and Sons Inc., New York, 1994.
[3] *S. Saitoh*: Thoery of Reproducing Kernels and its Applications. Pitman Research Notes in Mathematics, Vol. 189, New York, 1988.
[4] *S. Sultanic*: Sub-Bergman Hilbert spaces. J. Math. Anal. Appl. *324* (2006), 639–649.
[5] *K. Zhu*: Sub-Bergman Hilbert spaces in the unit disk. Indiana Univ. Math. J. *45* (1996), 165–176.
[6] *K. Zhu*: Sub-Bergman Hilbert spaces in the unit disk II. J. Func. Anal. *202* (2003), 327–341.
[7] *K. Zhu*: Operator Theory in Function Spaces. Dekker, New York, 1990.

Authors' addresses: A. Abkar, Department of Mathematics, Imam Khomeini International University, Qazvin 34194, Iran, e-mail: abkar@ikiu.ac.ir; B. Jafarzadeh, Department of Mathematics, Islamic Azad University of Mahshahr, Mahshahr, Iran, e-mail: Bagher60@hotmail.com.