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EXISTENCE OF PERFECT MATCHINGS IN A PLANE BIPARTITE GRAPH

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Abstract. We give a necessary and sufficient condition for the existence of perfect matchings in a plane bipartite graph in terms of elementary edge-cut, which extends the result for the existence of perfect matchings in a hexagonal system given in the paper of F. Zhang, R. Chen and X. Guo (1985).

Keywords: elementary edge-cut, hexagonal system, perfect matching, plane bipartite graph

MSC 2010: 05C70, 05C75

1. Introduction

A *matching* of a graph G is a set of edges of G such that no two of them have common ends. A *perfect matching* of a graph G is a matching of G which covers all its vertices. Let S be a set of vertices of a graph G. The set of vertices of G adjacent to at least one vertex of S is called the *neighbor set* of S in G and denoted by $N(S)$. Hall's theorem tells when a bipartite graph has a perfect matching.

Theorem 1.1 [2]**.** *Let G be a bipartite graph with bipartition (V_1, V_2). Then G has a matching from V_1 to V_2 if and only if $|N(A)| \geqslant |A|$ for every $A \subseteq V_1$. In particular, G has a perfect matching if and only if $|V_1| = |V_2|$ and $|N(A)| \geqslant |A|$ for every $A \subset V_1$.*

A *hexagonal system* is a plane bipartite graph which is often used to represent a benzenoid hydrocarbon. It is a 2-connected subgraph of a hexagonal lattice such that each finite face is a unit regular hexagon. It is well-known that a hexagonal system is the skeleton of a benzenoid hydrocarbon molecule if and only if it has a perfect matching. Sachs [3] provided a necessary condition for the existence of perfect matchings in a hexagonal system in terms of orthogonal edge-cut and conjectured that it is also a sufficient condition. Zhang, Chen and Guo [4] gave counterexamples to Sachs's conjecture and provided a necessary and sufficient condition for the existence of perfect matchings in a hexagonal system in terms of the elementary edge-cut. In this paper we extend the result for the existence of perfect matchings from a hexagonal system to that of a plane bipartite graph in terms of the elementary edge-cut.

2. Preliminaries

In this section we introduce the basic terminology and results. If S is a set of vertices of a graph G, then we use $\langle S\rangle$ to denote the induced subgraph of G generated by S. Let G be a bipartite graph. Then we can color the vertices of G with black and white such that adjacent vertices obtain different colors. We use $W(G)$ (or $B(G)$) to denote the set of vertices of G colored white (black). A *plane graph* is a graph in the plane where any two edges are either disjoint or meet only at a common end vertex. Each interior region of a plane graph G is called a *finite face* of G, and the exterior region of G is called the *infinite face* of G. The *dual graph* of a plane graph G is denoted by G^*. Each vertex f^* of G^* corresponds to a (finite or infinite) face f of G and is placed inside f; each edge e^* of G^* corresponds to an edge e of G which is adjacent to two faces f_1 and f_2 of G, and the edge e^* crosses only the edge e of G and joins the vertices f_1^* and f_2^* of G^*. We call e^* the *dual edge* of e. By definition, a dual graph of a connected plane graph is also a connected plane graph, and it may contain self-loops or multiple edges.

Let $\mathcal{C}$ be a set of edges of a connected graph G. Then $\mathcal{C}$ is called an *edge-cut* of G if $G\setminus\mathcal{C}$ is not connected. It is well-known [1] that edges in a plane graph G form a *minimal edge-cut* of G if and only if the corresponding dual edges form a cycle in G^*.

Let H be a hexagonal system drawn in a position with some edges in vertical direction. A straight line segment C with end points P_1 and P_2 is called a *cut segment* if it satisfies the following conditions:

(i) C is orthogonal to one of the three edge directions of H,
(ii) each of P_1 and P_2 is the center of an edge of H,
(iii) every point of C is either an interior or a boundary point of some cell of H,
(iv) the graph obtained from H by deleting all edges intersected by C has exactly two components.

Let $\mathcal{C}$ denote the set of edges of H intersected by C, then $\mathcal{C}$ is called an *orthogonal edge-cut* of H, see Fig. 1 (a). By defintion, each orthogonal edge-cut $\mathcal{C}$ of a hexagonal system H has the property that all vertices next to the cut segment on one side of the segment are black while those on the other side are white. Two components of $H \setminus \mathcal{C}$ are called the black bank $H_b(\mathcal{C})$ and the white bank $H_w(\mathcal{C})$ of $\mathcal{C}$ respectively.

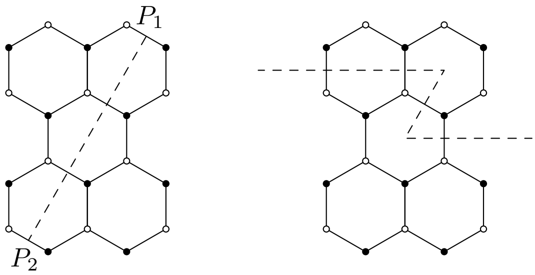

Figure 1. (a) Orthogonal (Elementary) edge-cut (b) Elementary edge-cut

Theorem 2.1 [3]. *Let H be a hexagonal system such that $|B(H)| = |W(H)|$. If H has a perfect matching, then $0 \leqslant |B(H_b(\mathcal{C}))| - |W(H_b(\mathcal{C})) = |W(H_w(\mathcal{C}))| - |B(H_w(\mathcal{C}))| \leqslant |C|$ for each orthogonal edge-cut $\mathcal{C}$ of H.*

Zhang, Chen and Guo [4] gave examples showing that the converse of the above theorem is not true. They provided a necessary and sufficient condition for the existence of perfect matchings in a hexagonal system in the following theorem.

Theorem 2.2 [4]. *Let H be a hexagonal system such that $|B(H)| = |W(H)|$. Then H has a perfect matching if and only if $|B(G')| \geqslant |W(G')|$ for each edge-cut $\{e_1, \ldots, e_t\}$ of H satisfying the following three conditions:*

(i) *$G \setminus \{e_1, \ldots, e_t\}$ has exactly two connected components G' and G'',*

(ii) *$V(e_i) \cap V(G') \subset B(H)$ and $V(e_i) \cap V(G'') \subset W(H)$ for each e_i $(1 \leqslant i \leqslant t)$,*

(iii) *edges e_1 and e_t lie on the boundary of H, and e_i, e_{i+1} are edges of some hexagonal unit cell for each $1 \leqslant i \leqslant t-1$.*

The concept of an elementary edge-cut of a plane bipartite graph was first introduced in [5]. An *elementary edge-cut* $\mathcal{C}$ of a connected plane bipartite graph G is a minimal edge-cut of G such that $G \setminus \mathcal{C}$ contains exactly two components and all edges of $\mathcal{C}$ are incident with white vertices of one component of G, which is called the white bank of $\mathcal{C}$ and denoted by $G_w(\mathcal{C})$; the other component of G is called the black bank of $\mathcal{C}$, and denoted by $G_b(\mathcal{C})$, see Fig. 1 and Fig. 2.

Lemma 2.3. *Let H be a hexagonal system. Then an edge-cut $\mathcal{C}$ of H is an elementary edge-cut if and only if it can be ordered so that it satisfies conditions* (i), (ii) *and* (iii) *of Theorem* 2.2.

Proof. If an edge-cut $\mathcal{C}$ of H satisfies the above three conditions, then $\mathcal{C}$ is a minimal edge-cut. Otherwise, there is an edge $e_i \in \mathcal{C}$ such that $(H \setminus \mathcal{C}) \cup \{e_i\}$ is not connected. Then $(H \setminus \mathcal{C}) \cup \{e_i\}$ has two components G_1 and G_2 since $H \setminus \mathcal{C}$ has two components G' and G''. Without loss of generality, we can assume $V(G_1) = V(G')$ and $V(G_2) = V(G'')$. It follows that both end vertices of e_i are contained in the same component, say G_1, of $(H\setminus\mathcal{C})\cup\{e_i\}$. Then $V(e_i)\cap V(G'') = V(e_i)\cap V(G_2) = \emptyset$. This contradicts condition (ii). Hence, $\mathcal{C}$ is a minimal edge-cut and so an elementary edge-cut of H. On the other hand, if $\mathcal{C}$ is an elementary edge-cut of H, then it is trivial that $\mathcal{C}$ satisfies (i) and (ii). By the proof of Theorem 2.2 [4], we can see that $\mathcal{C}$ satisfies (iii) as follows: Suppose that $\mathcal{C}$ has no edges on the boundary of H, then one component G' of $H \setminus \mathcal{C}$ is again a hexagonal system which is surrounded by hexagons in H. By the fact [3] that the boundary of any hexagonal system has at least 6 edges whose both end vertices have degree two, it follows that the component G' is neither a black bank nor a white bank of $\mathcal{C}$, which is a contradiction. Hence, $\mathcal{C}$ has at least one edge on the boundary of H. Since $\mathcal{C}$ is a minimal edge-cut of H, its corresponding dual edges form a cycle in H^*. Therefore, $\mathcal{C}$ has exactly two edges on the boundary of H and satisfies condition (iii). $\square$

It is clear that an orthogonal edge-cut of a hexagonal system is also an elementary edge-cut. However, an elementary edge-cut of a hexagonal system is not necessarily an orthogonal edge-cut.

3. Main results

Theorem 3.1. *Let G be a connected plane bipartite graph with $|B(G)| = |W(G)|$ and maximum degree $\Delta(G) \geqslant 3$. Then G has a perfect matching if and only if $|B(G_b(\mathcal{C}))| \geqslant |W(G_b(\mathcal{C}))|$ for every elementary edge-cut $\mathcal{C}$ of G.*

Proof. The main idea of the proof is similar to that of Theorem 2.2 [4]. We give it here for completeness. Necessity. Let $\mathcal{C}$ be an elementary edge-cut of G. Choose $S = W(G_b(\mathcal{C}))$. Then $N(S) = B(G_b(\mathcal{C}))$. Since G has a perfect matching, $|B(G_b(\mathcal{C}))| = |N(S)| \geqslant |S| = |W(G_b(\mathcal{C}))|$ by Hall's Theorem 1.1.

We will prove sufficiency by contradiction. Suppose that G does not have a perfect matching. By Hall's Theorem, there exists a nonempty subset $S \subseteq W(G)$ such that $|S| > |N(S)|$. It is clear that $S \neq W(G)$ since $|W(G)| = |B(G)|$. Without loss of generality, we can assume that $\langle S \cup N(S)\rangle$ is connected and S is maximal, that is, S cannot be a proper subset of $S^* \subseteq W(G)$ such that $|S^*| > |N(S^*)|$ and $\langle S^* \cup N(S^*)\rangle$ is connected. We claim that $|N(S)| < |S| \leqslant |N(S)| + \Delta(G) - 2$. Otherwise, $|S| > |N(S)| + \Delta(G) - 2$. Choose a vertex v not in S and adjacent to a vertex of $N(S)$ and let $S^* = S \cup \{v\}$. Then $\langle S^* \cup N(S^*)\rangle$ is connected and $|N(S^*)| \leqslant |N(S)| + \Delta(G) - 1 < |S| + 1 = |S^*|$. This contradicts the maximality of S. Therefore, the claim is valid.

Let $G' = \langle S \cup N(S)\rangle$ and $G'' = G - G'$. Let $\mathcal{C}$ be the edges of G between G' and G''. It is easy to see that $\mathcal{C}$ is an edge-cut of G. Note that $W(G') = S$ and $B(G') = N(S)$. Hence, G' is the black bank of $\mathcal{C}$ and G'' is the union of white banks of $\mathcal{C}$.

Next, we show that G'' has exactly one component. Recall that $|W(G)| = |B(G)|$ and $|W(G')| - |B(G')| = |S| - |N(S)| > 0$. Then $|B(G'')| - |W(G'')| = |S| - |N(S)| > 0$. Assume that $G''_1, G''_2, \ldots, G''_t$ are components of G''. Then $|B(G'')| - |W(G'')| = \sum_{i=1}^{t}(|B(G''_i)| - |W(G''_i)|) > 0$. We claim that $|B(G''_i)| - |W(G''_i)| > 0$ for each $1 \leqslant i \leqslant t$. Otherwise, if there is some $1 \leqslant i_0 \leqslant t$ such that $|B(G''_{i_0})| - |W(G''_{i_0})| \leqslant 0$, then

$$|S \cup W(G''_{i_0})| = |S| + |W(G''_{i_0})| > |N(S)| + |B(G''_{i_0})| = |N(S) \cup B(G''_{i_0})|.$$

Let $S^* = S \cup W(G''_{i_0})$. Then $N(S^*) = N(S) \cup B(G''_{i_0})$ and $|S^*| > |N(S^*)|$. It is easy to see that $\langle S^* \cup N(S^*)\rangle$ is connected. This contradicts the maximality of S. Hence, $|B(G_i{''})| - |W(G''_i)| \geqslant 1$ for each $1 \leqslant i \leqslant t$. If G'' has more than one component, that is, $t > 1$, then $|S| - |N(S)| > \sum_{i=1}^{t-1}(|B(G''_i)| - |W(G''_i)|)$. It follows that

$$\left|S \cup \left(\bigcup_{i=1}^{t-1} W(G''_i)\right)\right| = |S| + \sum_{i=1}^{t-1}|W(G''_i)| > |N(S)| + \sum_{i=1}^{t-1}|B(G''_i)|$$
$$= \left|N(S) \cup \left(\bigcup_{i=1}^{t-1} B(G''_i)\right)\right|.$$

Let $S^* = S \cup \left(\bigcup_{i=1}^{t-1} W(G''_i)\right)$. Then $N(S^*) = N(S) \cup \left(\bigcup_{i=1}^{t-1} B(G''_i)\right)$ and $|S^*| > |N(S^*)|$. It is easy to see that $\langle S^* \cup N(S^*)\rangle$ is connected. This contradicts the maximality of S.

Therefore, $G \setminus \mathcal{C}$ has exactly two components $G' = \langle S \cup N(S)\rangle$ and G'' which are black bank and white bank of $\mathcal{C}$ respectively. Similarly to the proof of Lemma 2.3, we can show that $\mathcal{C}$ is a minimal edge-cut of G. Hence, $\mathcal{C}$ is an elementary edge-cut of G. However, $|B(G_b(\mathcal{C}))| = |N(S)| < |S| = |W(G_b(\mathcal{C}))|$. □

Remark. The elementary edge-cut $\mathcal{C}$ in Theorem 3.1 need not have two edges on the boundary of G. For example, the plane bipartite graph G in Fig. 2 has

$|B(G)| = |W(G)|$, and $|B(G_b(\mathcal{C}))| \geqslant |W(G_b(\mathcal{C}))|$ for any elementary edge-cut $\mathcal{C}$ of G with two edges on the boundary of G. Nonetheles, $|B(G_b(\mathcal{C}))| < |W(G_b(\mathcal{C}))|$ for the elementary edge-cut $\mathcal{C}$ of G shown in the figure. Hence, G does not have a perfect matching by Theorem 3.1.

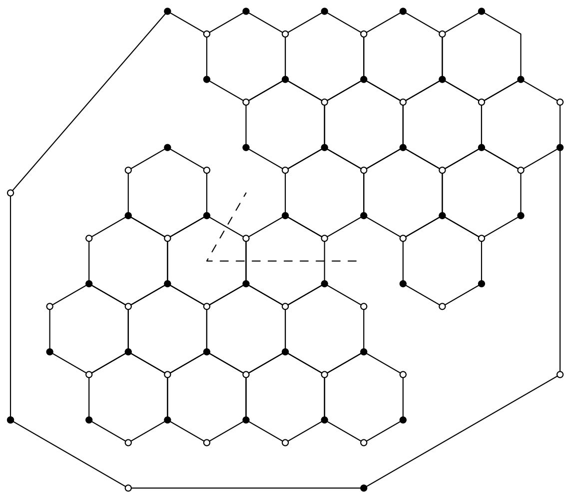

Figure 2. An elementary edge-cut of a plane bipartite graph

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