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Czechoslovak Mathematical Journal, Vol. 60 (2010), No. 2, 513–516

Persistent URL: <http://dml.cz/dmlcz/140585>

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INTERPOLATION OF BOUNDED SEQUENCES

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(Received December 18, 2008)

Abstract. This paper deals with an interpolation problem in the open unit disc \mathbb{D} of the complex plane. We characterize the sequences in a Stolz angle of \mathbb{D} , verifying that the bounded sequences are interpolated on them by a certain class of not bounded holomorphic functions on \mathbb{D} , but very close to the bounded ones. We prove that these interpolating sequences are also uniformly separated, as in the case of the interpolation by bounded holomorphic functions.

Keywords: interpolating sequence, Carleson's theorem, uniformly separated, Blaschke product, Lipschitz class

MSC 2010: 30D50, 30E05

1. INTRODUCTION

Let l^∞ be the space of all bounded sequences of complex numbers. We write (z_n) for any sequence in \mathbb{D} . As usual, we will put c for all positive constants. $\text{Lip}_{(z_n)}$ denotes the space of all complex functions Φ on (z_n) such that

$$|\Phi(z_k) - \Phi(z_m)| \leq c|z_k - z_m|, \quad \forall k, m \in \mathbb{N}.$$

H^∞ is the space of all bounded holomorphic functions on \mathbb{D} . We denote by A the disc algebra and by Lip the Lipschitz class on \mathbb{D} , that is,

$$\text{Lip} = \{f \in A: |f(z) - f(w)| \leq c|z - w|, \forall z, w \in \mathbb{D}\}.$$

Recall that $f \in \text{Lip}$ is equivalent to $f' \in H^\infty$.

Let $\psi(z, w) = |z - w|/|1 - \bar{z}w|$ be the pseudo-hyperbolic distance between $z, w \in \mathbb{D}$. The Blaschke product in \mathbb{D} with zeros at (z_n) is the function in H^∞ defined by

$$B(z) = \prod_{n \in \mathbb{N}} \frac{|z_n|}{z_n} \frac{z - z_n}{1 - \bar{z}_n z}.$$

We denote by B_m and $B_{m,k}$ the Blaschke products in \mathbb{D} with zeros at $(z_n) \setminus \{z_m\}$ and $(z_n) \setminus \{z_m, z_k\}$, respectively. The Stolz angle with vertex at the point $\eta \in \partial\mathbb{D}$ and aperture $\beta > 0$ is the set

$$S_\beta(\eta) = \{z \in \mathbb{D}: |z - \eta| < (1 + \beta)(1 - |z|)\}.$$

Finally, (z_n) is called uniformly separated if

$$|B_m(z_m)| \geq c, \quad \forall m \in \mathbb{N}.$$

The well known Carleson's theorem ([2]) asserts for a sequence (z_n) that given any $(w_n) \in l^\infty$, there exists $f \in H^\infty$ such that $f(z_n) = w_n, \forall n \in \mathbb{N}$, if and only if (z_n) is uniformly separated. If $h \in H^\infty$ and (z_n) is the zero set of the function h , then $h = Bg$, where B is the Blaschke product in \mathbb{D} with zeros at (z_n) and g is a function in H^∞ not vanishing on (z_n) . Now, if we differentiate B and 'integrate' g to 'compensate', rather, take $f \in \text{Lip}$, then it is possible to pose a new interpolation problem for a bounded sequence, that consists of finding an interpolating function in the form $B'f$, instead of a function in H^∞ . More precisely,

Definition 1. (z_n) is called a balanced interpolating sequence, if given any $(w_n) \in l^\infty$, there exists $f \in \text{Lip}$ such that $(B'f)(z_n) = w_n, \forall n \in \mathbb{N}$.

Our result is the following one:

Theorem 1. (z_n) in a Stolz angle is a balanced interpolating sequence if and only if it is uniformly separated.

In view of this theorem, the uniformly separated sequences continue being the interpolating sequences for l^∞ , though we consider a weaker space of interpolating functions.

2. PROOF OF THE THEOREM

Proof. Let (z_n) be a balanced interpolating sequence in a Stolz angle $S_\beta(\eta)$. For $m \in \mathbb{N}$ fixed, let (w_n) be defined by: $w_m = 1; w_k = 0$, if $k \neq m$. As $(w_n) \in l^\infty$, we take $f \in \text{Lip}$ such that $(B'f)(z_n) = w_n, \forall n \in \mathbb{N}$.

For a given $f \in \text{Lip}$ vanishing on $(z_n) \setminus \{z_m\}$, it is proved in [4] that

$$|f(z)| \leq c|z - z_k| |B_{m,k}(z)|, \quad \forall k \in \mathbb{N}, k \neq m.$$

Writing this inequality for $z = z_m$ and taking z_k as the point of (z_n) nearest to z_m in the Euclidean distance, say $z_{m'}$, we have

$$(1) \quad \frac{1}{|B'(z_m)|} \leq c|z_m - z_{m'}| |B_{m,m'}(z_m)|, \quad \forall m \in \mathbb{N}.$$

It is straightforward to obtain

$$(2) \quad |B'(z_m)| = \frac{|B_m(z_m)|}{1 - |z_m|^2}, \quad \forall m \in \mathbb{N}.$$

Using (2), the expression (1) becomes

$$\frac{|B_m(z_m)|}{1 - |z_m|^2} |z_m - z_{m'}| |B_{m,m'}(z_m)| \geq c, \quad \forall m \in \mathbb{N}.$$

Since $1 - |z_m|^2 > 1 - |z_m| > c|z_m - \eta|$ and $|B_{m,m'}(z_m)| < 1$, then we have

$$|B_m(z_m)| \geq c \frac{|z_m - \eta|}{|z_m - z_{m'}|}, \quad \forall m \in \mathbb{N}.$$

From this inequality, it follows immediately that $|B_m(z_m)| \geq c, \forall m \in \mathbb{N}$, that is, (z_n) is uniformly separated.

Now, let (z_n) be uniformly separated in a Stolz angle and $(w_n) \in l^\infty$. We take $h \in H^\infty$ such that $h(z_n) = w_n, \forall n \in \mathbb{N}$ and see that $h/B' \in \text{Lip}_{(z_n)}$.

We will use that any $g \in H^\infty$ satisfies the following inequalities (see [1] for (5)):

$$(3) \quad |g(z) - g(w)| \leq c\psi(z, w), \quad \forall z, w \in \mathbb{D}.$$

$$(4) \quad |g'(z)| \leq \frac{c}{1 - |z|^2}, \quad \forall z \in \mathbb{D}.$$

$$(5) \quad |g'(z)(1 - |z|^2) - g'(w)(1 - |w|^2)| \leq c\psi(z, w), \quad \forall z, w \in \mathbb{D}.$$

For $z_k, z_m \in (z_n)$, the triangle inequality gives

$$(6) \quad \left| \frac{h(z_k)}{B'(z_k)} - \frac{h(z_m)}{B'(z_m)} \right| \leq \frac{|h(z_k) - h(z_m)|}{|B'(z_m)|} + \frac{|h(z_k)| |B'(z_m) - B'(z_k)|}{|B'(z_k)| |B'(z_m)|}.$$

Taking into account (2), for a uniformly separated sequence it holds

$$|B'(z_i)| \geq \frac{c}{1 - |z_i|^2}, \quad \forall i \in \mathbb{N},$$

and then, the sum in (6) is bounded by

$$(7) \quad c|h(z_k) - h(z_m)|(1 - |z_m|^2) + c|h(z_k)| |B'(z_m) - B'(z_k)|(1 - |z_k|^2)(1 - |z_m|^2).$$

Using (3), the first summand in (7) is bounded by

$$c\psi(z_k, z_m)(1 - |z_m|^2) \leq c\psi(z_k, z_m)(1 - |z_m|) \leq c|z_k - z_m|.$$

By the triangle inequality, the second summand in (7) is bounded by

$$c\{|B'(z_k)|(1 - |z_k|^2)|(1 - |z_k|^2) - (1 - |z_m|^2)| \\ + |B'(z_m)(1 - |z_m|^2) - B'(z_k)(1 - |z_k|^2)|(1 - |z_k|^2)\},$$

and using (4) and (5), by

$$c\{|z_m| - |z_k|(|z_m| + |z_k|) + \psi(z_k, z_m)(1 - |z_k|^2)\} \leq c|z_k - z_m|.$$

Thus $h/B' \in \text{Lip}_{(z_n)}$. For a sequence (z_n) in a Stolz angle, it is proved in [3] that given any $\Phi \in \text{Lip}_{(z_n)}$, there exists $f \in \text{Lip}$ such that $f(z_n) = \Phi(z_n)$, $\forall n \in \mathbb{N}$, if and only if (z_n) is the union of two uniformly separated sequences. Hence, there exists $f \in \text{Lip}$, such that $f(z_n) = h(z_n)/B'(z_n)$, $\forall n \in \mathbb{N}$, that is, $(fB')(z_n) = w_n$, $\forall n \in \mathbb{N}$, and consequently, (z_n) is a balanced interpolating sequence. \square

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