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A DIRECT APPROACH TO THE WEISS CONJECTURE FOR BOUNDED ANALYTIC SEMIGROUPS

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Abstract. We give a new proof of the Weiss conjecture for analytic semigroups. Our approach does not make any recourse to the bounded H^{∞}-calculus and is based on elementary analysis.

Keywords: infinite dimensional systems, analytic semigroups, unbounded observation operator, admissibility, fractional power

MSC 2010: 35XX, 34K35, 35Q93, 47XX

1. Introduction

In this paper we consider the abstract differential system

$$\begin{cases} \dot{x}(t) = Ax(t), \ t \geqslant 0, \\ x(0) = x_0, \\ y(t) = Cx(t), \end{cases} \tag{1.1}$$

where the operator A generates a C_0-semigroup $\mathbb{T} := (\mathbb{T}(t))_{t\geqslant 0}$ on a Banach (state) space X. We denote by $\omega(A)$ the growth bound of $\mathbb{T}$. The operator C is a bounded Y-valued operator from the domain $D(A)$ of A, with respect to the graph norm to the second (output) Banach space Y. That is, there exists a constant $M_0 > 0$ such that

$$\|Cx\|_Y \leqslant M_0\big(\|x\|_X + \|Ax\|_X\big), \ x \in D(A).$$

For $x \in D(A)$ and $t > 0$, $\mathbb{T}(t)x \in D(A)$, the resulting function $t \mapsto C\mathbb{T}(t)x$ is continuous from $(0, \infty)$ into Y. An important question is whether the system (1.1) is well-posed. Of course, since A generates a C_0-semigroup on X, the state equation has a unique mild solution $x(t) = \mathbb{T}(t)x_0$. However, since C is not a bounded operator on X, it is not clear whether the output equation is well-posed. The output equation makes sense if C is bounded on X. However, one could relax this to the question whether the output trajectory is locally square integrable. Therefore the following definition of an admissible observation operator has been introduced. According to [21], we say that C is finite-time p-admissible if for some (or hence for all) $\tau \in (0, \infty)$ there exists $\kappa_p(\tau) > 0$ such that

$$\int_0^\tau \|C\mathbb{T}(t)x\|_Y^p \, \mathrm{d}t \leqslant \kappa_p^p(\tau)\|x\|_X^p. \tag{1.2}$$

A variant of admissibility called the infinite-time admissibility when the integral on $(0, \tau)$ in (1.2) is replaced by the whole time axis $(0, \infty)$ has also been extensively studied (see e.g., [7], [8], [23], [15], [6], [16], [24]). The notion of finite time p-admissibility is invariant under scalings $\mathrm{e}^{-\alpha \cdot}\mathbb{T}(\cdot)$. Hence, if we want to investigate finite-time admissibility of observation operators, then we may assume that the semigroup is exponentially stable. We refer to ([8], [22], [20]) and the references therein for historical background and applications of admissibility. Since the resolvent is the Laplace transform of the semigroup, the finite-time p-admissibility always implies

$$\sup_{z \in \mathbb{C}_\alpha} (\mathrm{Re}(z))^{1-1/p}\|CR(z, A)\| < \infty \tag{1.3}$$

for some $\alpha > \omega(A)$, where $\mathbb{C}_\alpha = \{z \in \mathbb{C} \text{ s.t. } \mathrm{Re}(z) > \alpha\}$.

For $p = 2$, it has been proved in [21] that the converse does not hold in the general Banach space context. However, in [21] it was conjectured that if X and Y are Hilbert spaces, then C is finite-time L^2-admissible observation operator if and only if (1.3) holds. Since then, this problem, which is known as the Weiss conjecture, has received much attention. Zwart in [24] presents some sufficient conditions for finite (or infinite) time L^2-admissibility. It was first shown in [17] that the isometric right shift semigroup in $H^2(\mathbb{C}^+, X)$ satisfies the Weiss conjecture for scalar observation operators (in the case $X = \mathbb{C}$). By using the Sz. Nagy Foias functional model for contraction semigroups on a Hilbert space and applying the same proof as in [17] for X a general separable Hilbert space, it was shown in [11] that the Weiss conjecture holds for general contraction semigroups on separable Hilbert spaces with a scalar observation operator. The proof in [11] was later simplified in [19], Section 10.7 by using an isometric extension of the semigroup. For $Y = \mathbb{C}$ and $\mathbb{T}(t)$ being a contraction semigroup, it was shown in [11] that the Weiss conjecture holds. But in [12] the authors showed that if $\dim(Y) = \infty$ then the Weiss conjecture can fail even for a semigroup of isometries. The papers [23] and [13] constructed bounded, analytic semigroups for which the Weiss conjecture fails with Y of finite or infinite dimension respectively. The paper [15] gives other examples of bounded analytic semigroups which are not similar to contraction semigroups for which the Weiss conjecture holds for all Banach spaces Y. Papers [10] and [20] contain the special case when the semigroup is normal and analytic. In the Banach space context and concerning the infinite-time admissibility, LeMerdy in his paper [15] showed that the infinite-time Weiss conjecture holds for a bounded analytic semigroup if and only if the fractional power $(-A)^{1/2}$ is admissible for A. For a contractive analytic semigroup on a Hilbert space X, it is shown in [15] that the Weiss conjecture holds. In particular, the author extends the result by Hansen and Weiss [10] and by Weiss [20] concerning the case when the semigroup is bounded analytic and normal (and hence contractive). In [15], essential use is made of the bounded H^∞-functional calculus.

For $p \in [1, \infty]$, there are a few results on the p-admissibility and its associated Weiss conjecture. The author in [4] characterized the finite time p-admissibility of control and observation operators. In [5] this result was extended to the infinite-time p-admissibility. Recently, the authors in [9] have extended the result in [15] on the Weiss conjecture for 2-admissibility to the case of p-admissibility for bounded analytic semigroups.

The aim of this paper is to present new and much shorter proofs of the results of the Weiss-conjecture for analytic semigroups proved in [15] for $p = 2$ and in [9] for $p \in (1, \infty]$, eventually even generalizing them. We will also prove that the analyticity assumption on the semigroup cannot be omitted. Our approach does not make any recourse to the H^∞-functional calculus and is based on elementary analysis. Similar results can be obtained for the weighted admissibility of observation operators studied in [9] and this will be the subject of a forthcoming paper.

2. Definition and results

The following definition explains what does it mean exactly that A *satisfies the finite time p-Weiss property.*

Definition 2.1. Let A generate a bounded C_0-semigroup $\mathcal{L}(D(A), Y)$ on a Banach space X. We say that A satisfies the finite-time p-Weiss property if for any Banach space Y and $C \in \mathcal{L}(D(A), Y)$, the following statements are equivalent:
(i) C is finite-time p-admissible for A.
(ii) C satisfies the estimate (1.3).

Definition 2.1 has an analogue version for the infinite p-Weiss property, henceforth called the p-Weiss property for short.

Our main result is based on the following ergodic result.

Proposition 2.2. *Let $p \in [1,\infty)$ and $T > 0$. Let X be a Banach space and let us define operators L_0 and L as*

$$(L_0 f)(t) := \frac{1}{t}\int_0^t f(s)\,\mathrm{d}s \text{ and } (Lf)(t) := \int_t^T \frac{f(s)}{s}\,\mathrm{d}s \text{ for } f \in L^p([0,T],X)$$

and $0 < t < T$. Then:

(i) *L_0 is bounded on $L^p([0,T],X)$ and $\|L_0\|_p \leqslant p/(p-1)$ for $p > 1$.*

(ii) *L is bounded on $L^p([0,T],X)$ and $\|L\|_p \leqslant p$ for $p \geqslant 1$.*

Proof. By virtue of the denseness of $\mathcal{C}([0,T],X)$ (the set of continuous functions on $[0,T]$) in $L^p([0,T],X)$ it suffices only to prove the result for $f \in \mathcal{C}([0,T],X)$.

(i) Let $p > 1$ and $0 < t \leqslant T$. It is easy to see that

$$\|L_0 f\|_p^p = \int_0^T \frac{1}{t^p}\left\|\int_0^t f(s)\,\mathrm{d}s\right\|^p \mathrm{d}t \leqslant \int_0^T \frac{1}{t^p}\left(\int_0^t \|f(s)\|\,\mathrm{d}s\right)^p \mathrm{d}t = \|L_0(\|f\|)\|_p^p.$$

Then it suffices to prove the statement for $X = \mathbb{C}$ and $f \geqslant 0$. Let p' be the conjugate of p $(1/p + 1/p' = 1)$. Note that $(p-1)p' = p$ and $p - p/p' = 1$.

We perform integration by parts to obtain

$$\begin{aligned}
\|L_0(f)\|_p &= \frac{1}{1-p}\|f\|_1^p + \frac{p}{p-1}\int_0^T f(t)\left(\frac{1}{t}\int_0^t f(s)\,\mathrm{d}s\right)^{p-1}\mathrm{d}t \\
&= \frac{1}{1-p}\|f\|_1^p + \frac{p}{p-1}\int_0^T f(t)(L_0(f)(t))^{p-1}\,\mathrm{d}t \\
&\leqslant \frac{1}{1-p}\|f\|_1^p + \frac{p}{p-1}\|f\|_p\left(\int_0^T (L_0(f)(t))^{(p-1)p'}\,\mathrm{d}t\right)^{1/p'} \quad \text{(by Hölder inequality)} \\
&= \frac{1}{1-p}\|f\|_1^p + \frac{p}{p-1}\|f\|_p\|L_0(f)\|_p^{p/p'}.
\end{aligned}$$

This implies that $\|L_0 f\|_p \leqslant \frac{p}{p-1}\|f\|_p$ for all $f \in \mathcal{C}([0,T],\mathbb{R}^+)$.

Now, we prove (ii). As above,

$$\|Lf\|_p^p = \int_0^T \left\|\int_t^T \frac{f(s)}{s}\,\mathrm{d}s\right\|^p \mathrm{d}t \leqslant \int_0^T \left(\int_t^T \frac{\|f(s)\|}{s}\,\mathrm{d}s\right)^p \mathrm{d}t = \big\|L(\|f\|)\big\|_p^p,$$

and it suffices again to prove the statement for $X = \mathbb{C}$ and $f \geqslant 0$. We have by integration by parts

$$\begin{aligned}
\|Lf\|_p^p &= \left[t\left(\int_t^T \frac{f(s)}{s}\,\mathrm{d}s\right)^p\right]_0^T + p\int_0^T f(t)\left(\int_t^T \frac{f(s)\|}{s}\,\mathrm{d}s\right)^{p-1}\mathrm{d}t \\
&\leqslant p\|f\|_p\big\|Lf\big\|_p^{p/p'} \text{ (by Hölder inequality)}
\end{aligned}$$

whence $\|Lf\|_p \leqslant p\|f\|_p$ for all $f \in \mathcal{C}\big([0,T],\mathbb{R}^+\big)$. □

Before giving the main result on the p-Weiss property for $p \in [1, \infty]$, we need the following lemmas.

Lemma 2.3. *Let A generate a bounded analytic C_0-semigroup on a Banach space X and let $C \in \mathcal{L}(D(A), Y)$ where Y is an another Banach space. Let $p \in [1, \infty]$. Consider the following statements:*

(i) $\sup_{z \in \mathbb{C}_0} (\operatorname{Re}(z))^{1/p'} \|CR(z, A)\| < \infty$.

(ii) $\sup_{t \geqslant 0} t^{1/p} \|C\mathbb{T}(t)x\| \leqslant M_p \|x\|$ *for* $x \in X$ *and some* $M_p > 0$.

Then (ii) $\Rightarrow$ (i) *for* $p \in (1, \infty]$ *and* (i) $\Rightarrow$ (ii) *for* $p \in [1, \infty]$.

P r o o f. (ii) $\Rightarrow$ (i). Let $p \in (1, \infty]$. By continuity of C on $D(A)$, for all $x \in X$, $z \in \mathbb{C}_0$, we have

$$CR(z, A)x = \int_0^\infty \mathrm{e}^{-zt} C\mathbb{T}(t)x \, \mathrm{d}t.$$

It follows that

$$\begin{aligned}
\|CR(z, A)x\| &\leqslant \int_0^\infty \mathrm{e}^{-\operatorname{Re}(z)t} \|C\mathbb{T}(t)x\| \, \mathrm{d}t \\
&\leqslant M_p \|x\| \int_0^\infty \frac{\mathrm{e}^{-\operatorname{Re}(z)t}}{t^{1/p}} \, \mathrm{d}t \\
&= \frac{M_p \|x\|}{\operatorname{Re}(z)^{1-1/p}} \int_0^\infty \frac{\mathrm{e}^{-s}}{s^{1/p}} \, \mathrm{d}s \quad (s := \operatorname{Re}(z)t) \\
&= \frac{\Gamma(1/p') M_p \|x\|}{\operatorname{Re}(z)^{1/p'}}
\end{aligned}$$

where Γ is the usual Gamma function. This shows the assertion.

(i) $\Rightarrow$ (ii). Let $p \in [1, \infty]$. For $\theta \in (0, \pi]$, we denote by S_θ the open sector of all $z \in \mathbb{C} \setminus \{0\}$ such that $\operatorname{Arg}(z) \in (-\theta, \theta)$, $\overline{S_\theta}$ its closure and by Γ_θ its boundary oriented counterclockwise. As $\mathbb{T}(t)$ is a bounded and analytic semigroup, it is well known that the spectrum of its generator A is contained in some $\mathbb{C} \setminus \overline{S_\omega}$ with $\omega \in (\pi/2, \pi)$. By the Cauchy integral formula, we have

$$\mathbb{T}(t) = \frac{1}{2\pi \mathrm{i}} \int_\Gamma \mathrm{e}^{tz} R(z, A) \, \mathrm{d}z,$$

where $\Gamma = \Gamma_\gamma$ with $\gamma \in (\pi/2, \omega)$.

Then for all $x \in D(A)$

$$\begin{aligned}
\|C\mathbb{T}(t)x\| &= \left\| \frac{1}{2\pi \mathrm{i}} \int_\Gamma \mathrm{e}^{tz} CR(z, A)x \, \mathrm{d}z \right\| \\
&\leqslant \frac{1}{2\pi} \int_\Gamma \mathrm{e}^{t \operatorname{Re}(z)} \|CR(z, A)x\| \, |\mathrm{d}z|.
\end{aligned}$$

Note that $|\operatorname{Re}(z)|/|z| = \sin(\gamma)$ for all $z \in \Gamma$. By virtue of the resolvent equation and using the analyticity of the semigroup $\mathbb{T}(t)$, the statement (i) implies that

$$|z|^{1/p'}\|CR(z,A)\| \leqslant M_\Gamma \sup_{s\in\mathbb{C}_0}(\operatorname{Re}(s)^{1/p'})\|CR(s,A)\| \quad \text{for all } z \in \Gamma,$$

for some constant $M_\Gamma > 0$.

On the other hand, it is straightforward to see that

$$\begin{aligned}\|C\mathbb{T}(t)x\| &\leqslant \frac{1}{t^{1/p}}\int_\Gamma |\lambda|^{1/p}\mathrm{e}^{\operatorname{Re}(\lambda)}\frac{|\,\mathrm{d}\lambda|}{|\lambda|}M_\Gamma \sup_{z\in\mathbb{C}_0}(\operatorname{Re}(z)^{1/p'})\|CR(z,A)\|\|x\| \quad (\lambda := tz)\\ &= \frac{M_p}{t^{1/p}}\|x\|,\end{aligned}$$

where

$$M_p = M_\Gamma \sup_{z\in\mathbb{C}_0}(\operatorname{Re}(z)^{1/p'})\|CR(z,A)\| \int_\Gamma |\lambda|^{-1/p'}\mathrm{e}^{\operatorname{Re}(\lambda)}|\mathrm{d}\lambda|. \tag{2.1}$$

This completes the proof. □

Before stating the next lemma, we recall some basic facts on fractional powers. The study of fractional powers of sectorial operators which are classical objects in semigroup theory has a long history. Many results can be found in the book of Amann [1], and in the original papers of Balakrishnan [3], Komatsu [14] and others.

Let A be a generator of a bounded semigroup. Then $-A$ is sectorial. In particular, the resolvent set $\varrho(-A)$ contains $(-\infty, 0)$ and the resolvent satisfies $\sup_{\lambda>0}\|\lambda(\lambda - A)^{-1}\| < \infty$, and then for all $0 < \theta \leqslant 1$ the fractional power $(-A)^\theta$ is well-defined. We refer to [14] for more details and references on fractional powers. In the case $\theta = 1/2$, we know from Arendt [[2], page 168], that $(-A)^{1/2}$ is the unique closed operator satisfying for $x \in D(A)$, $((-A)^{1/2})^2x = -Ax$ and

$$(-A)^{1/2}x = \lim_{\varepsilon\to 0}(\varepsilon - A)^{1/2}x = -\frac{1}{\pi}\int_0^\infty \frac{1}{s^{1/2}}R(s,A)Ax\,\mathrm{d}s.$$

Moreover, $D(A)$ is a core for $(-A)^{1/2}$, and $D((-A)^{1/2}) = D((\varepsilon - A)^{1/2})$ $(\varepsilon > 0)$. If, in addition, $\mathbb{T}(t)$ is a bounded analytic C_0-semigroup and $\theta \in (0,1]$ then $t^\theta(-A)^\theta\mathbb{T}(t)$ is uniformly bounded on $\mathbb{R}^+$ (see Theorem 12.1 [14]). In the rest of this paper we set

$$a_\theta = \sup_{t\in\mathbb{R}^+}\|t^\theta(-A)^\theta\mathbb{T}(t)\| \ (0 < \theta \leqslant 1). \tag{2.2}$$

In this case the fractional power $(-A)^{\theta}$ is always finite-time p-admissible for all $\theta < 1/p$.

For $\theta = 1/p$ we have the following first characterization of the finite-time p-admissibility of $(-A)^{1/p}$ for $p \in [1,\infty]$.

Lemma 2.4. *Let A generate a bounded analytic C_0-semigroup on a Banach space X and $p \in [1,\infty]$. Then the following statements are equivalent:*

(i) *$(-A)^{1/p}$ is finite-time p-admissible for A.*

(ii) *For all $T > 0$ there exists $C_p(T) > 0$ such that $\|t^{1-1/p}\mathbb{T}(t)Ax\|_{L^p([0,T],X)} \leqslant C_p(T)\|x\|$ for all $x \in D(A)$. (i.e. A is finite-time p-admissible of type $1 - 1/p = 1/p'$ see [9])*

Proof. For $p = 1$, the assertions (i) and (ii) with $C_1(T) = \kappa_1(T)$ are the same. For $p = \infty$, both the assertions (i) and (ii) are always true with $C_\infty(T) = a_1 T$.

(i) $\Rightarrow$ (ii) Let $p \in [1,\infty)$. For $x \in D(A)$ we have $t^{1-1/p}\mathbb{T}(t)Ax = -t^{1-1/p} \times (-A)^{1-1/p}\mathbb{T}(t/2)(-A)^{1/p}\mathbb{T}(t/2)x$.

Since $(-A)^{1/p}$ is finite-time p-admissible for A, for all $T > 0$ and $x \in D(A)$ we have

$$\int_0^T \|(-A)^{1/p}\mathbb{T}(t)x\|^p \,\mathrm{d}t \leqslant \kappa_p^p(T)\|x\|^p$$

for some constant $\kappa_p(T)$ depending only on T.

Thus, the assertion follows according to (2.2) with $C_p(T) = 2^{1+1/p'}a_{1/p'}\kappa_p(T)$ for $1 \leqslant p < \infty$.

(ii) $\Rightarrow$ (i). Let $p \in (1,\infty)$. For $x \in D(A)$ and $t > 0$ we can write

$$t(-A)^{1/p}\mathbb{T}(t)Ax = t^{1/p}(-A)^{1/p}\mathbb{T}(t/2)t^{1-1/p}\mathbb{T}(t/2)Ax.$$

Again by uniform boundedness of the operator $t^{1/p}(-A)^{1/p}\mathbb{T}(t/2)$ on $[0,T]$ and the fact that the function $t \mapsto t^{1-1/p}\mathbb{T}(t/2)Ax$ lies in $L^p([0,T],X)$, we deduce that the function $t \mapsto t(-A)^{1/p}\mathbb{T}(t)Ax$ is so. By applying the semigroup identity we get

$$\begin{aligned}(-A)^{1/p}\mathbb{T}(t)x = \mathbb{T}(t)(-A)^{1/p}x &= (-A)^{1/p}\mathbb{T}(T)x - \int_t^T (-A)^{1/p}\mathbb{T}(s)Ax\,\mathrm{d}s \\ &= (-A)^{1/p}\mathbb{T}(T)x - \int_t^T \frac{s(-A)^{1/p}\mathbb{T}(s)Ax}{s}\,\mathrm{d}s\end{aligned}$$

for all $0 \leqslant t \leqslant T$.

Since $T^{1/p}(-A)^{1/p}\mathbb{T}(T)$ is bounded and the function $t \mapsto t(-A)^{1/p}\mathbb{T}(t)Ax$ lies in $L^p([0,T],X)$, Proposition 2.2 yields the finite-time p-admissibility of $(-A)^{1/p}$ and we have precisely

$$\|(-A)^{1/p}\mathbb{T}(t)x\|_{L^p([0,T],X)} \leqslant \kappa_p(T)\|x\|$$

for all $x \in D(A)$ with

$$\kappa_p(T) = a_{1/p}\big(1 + 2^{2p+1} p C_p(T)\big).$$

$\square$

Corollary 2.5. *Let A generate a bounded analytic C_0-semigroup and let $C \in \mathcal{L}(D(A), Y)$. For $p \in [1, \infty]$, if $(-A)^{1/p}$ is finite-time p-admissible for A and $\sup_{z \in \mathbb{C}_0} (\operatorname{Re}(z)^{1/p'}) \|CR(z, A)\| < \infty$, then for all $T > 0$ there exists $C'_p(T) > 0$ such that $\|tC\mathbb{T}(t)Ax\|_{L^p([0,T],X)} \leqslant C'_p(T)\|x\|$ for all $x \in D(A)$.*

Proof. For $p = \infty, p' = 1$, $(-A)^{1/p} = I$ is always finite time ∞-admissible. Let $x \in D(A)$ and $t \in (0, T]$. Since $\sup_{z \in \mathbb{C}_0} (\operatorname{Re}(z)^{1/p'}) \|CR(z, A)\| < \infty$, Lemma 2.3 implies that $\sup_{t \in [0,T]} \|C\mathbb{T}(t)x\| \leqslant M_\infty \|x\|$ and we have $tC\mathbb{T}(t)Ax = 2C\mathbb{T}(t/2)t/2A\mathbb{T}(t/2)x$. The result follows with $C'_\infty(T) = 2a_1 M_\infty$.

Now, we prove the corollary for $p \in [1, \infty)$. Let $t \geqslant 0$ and $x \in D(A)$. We have again $tC\mathbb{T}(t)Ax = t^{1/p} C\mathbb{T}(t/2) t^{1-1/p}\mathbb{T}(t/2)Ax$. According to Lemmas 2.3 and 2.4 we get the claim. More precisely, if $(-A)^{1/p}$ is finite-time p-admissible for A, then there exists $\kappa_p^p(T) > 0$ such that for all $x \in D(A)$

$$\int_0^T \|(-A)^{1/p}\mathbb{T}(t)x\|^p \,\mathrm{d}t \leqslant \kappa_p^p(T)\|x\|^p.$$

Thus we obtain

$$\|tC\mathbb{T}(t)Ax\|_{L^p([0,T],X)} \leqslant C'_p(T)\|x\|$$

with

$$C'_p(T) = 2^{3+1/p} a_{1/p'} \kappa_p(T) M_p \quad (M_p \text{ is given by (2.1)}).$$

$\square$

The following theorem is the main result of the paper. It yields a characterization of the finite-time Weiss-property in terms of the admissibility of $(-A)^{1/p}$.

Theorem 2.1. *Let A generate a bounded analytic C_0-semigroup on a Banach space X and $p \in [1, \infty]$. Then the following assertions are equivalent:*

(i) *$(-A)^{1/p}$ is finite-time p-admissible for A.*

(ii) *A possesses the finite-time p-Weiss property on $\mathbb{C}_0$.*

As mentioned above this result was first obtained by Le Merdy [15] for $p = 2$ and by Haak-Kunstmann [9] for $p \in (1, \infty]$. The proof below includes the case $p = 1$.

P r o o f. (ii) $\Rightarrow$ (i) For $p \in [1, \infty]$ it follows directly from (2.2) since the resolvent of A is given by the Laplace transform of the semigroup.

(i) $\Rightarrow$ (ii) For $p = +\infty$ the required result is given by Lemma 2.3.

For $p = 1$, assume that A is finite-time 1-admissible for itself. Now, let $\varepsilon > 0$ and let $C\colon D(A) \to Y$ be a continuous operator. Since $C\mathbb{T}(t)x = C(\varepsilon - A)^{-1}(\varepsilon - A)\mathbb{T}(t)x$ for all $x \in D(A)$ and the fact that $M := \sup\limits_{0<s<1} \|C(s-A)^{-1}\|$ is finite, by letting $\varepsilon \downarrow 0$, we obtain for all $x \in D(A)$

$$\|C\mathbb{T}(t)x\| \leqslant M\|(-A)\mathbb{T}(t)x\|,$$

which implies that C is finite-time 1-admissible for A.

For $p \in (1, \infty)$, assume that $(-A)^{1/p}$ is a finite-time p-admissible observation for A. Consider a continuous operator $C\colon D(A) \to Y$ such that $\sup\limits_{z\in\mathbb{C}_0} \operatorname{Re}(z)^{1/p'}\|CR(z,A)\| < \infty$. Thanks to Corollary 2.5, for all $x \in D(A)$ and $T > 0$ the function $f(t) := tC\mathbb{T}(t)Ax$ lies in $L^p([0,T], X)$. Moreover, for any $x \in D(A)$ and $t \in (0, T]$ we have

$$C\mathbb{T}(t)x = C\mathbb{T}(T)x - \int_t^T C\mathbb{T}(s)Ax\,\mathrm{d}s = C\mathbb{T}(T)x - (Lf)(t).$$

By Lemma 2.3 $\sup\limits_{T>0} T^{1/p}\|C\mathbb{T}(T)\| < \infty$ and by applying Proposition 2.2 to the function f we obtain

$$\int_0^T \|C\mathbb{T}(s)x\|^p\,\mathrm{d}s \leqslant 2^p\bigl(M_p^p + 2^{2p+1}p^pM_p^p\kappa_p^p(T)a_{1/p'}^p\bigr)\|x\|^p,$$

where the constant M_p is given by (2.1). This gives the claim. □

Corollary 2.6. *Let A generate a bounded analytic C_0-semigroup on a Banach space X and let $p \in [1, \infty]$. Then the following assertions are equivalent:*

(i) *$(-A)^{1/p}$ is p-admissible for A.*

(ii) *A possesses the p-Weiss property on $\mathbb{C}_0$.*

P r o o f. (i) $\Rightarrow$ (ii) for $p = \infty$ and (ii) $\Rightarrow$ (i) for $p \in [1, \infty]$ are obtained as in Proposition 2.1.

So, we prove (i) $\Rightarrow$ (ii) for $p \in [1, \infty)$. Assume that $(-A)^{1/p}$ is p-admissible for A with

$$\int_0^\infty \|(-A)^{1/p}\mathbb{T}(t)x\|^p\,\mathrm{d}t \leqslant \kappa_p^p\|x\|^p$$

for all $x \in D(A)$ and for some $\kappa_p > 0$.

The fact that the semigroup $\mathbb{T}$ is bounded and analytic, implies that the function $t \mapsto t^{1-1/p}(-A)^{1-1/p}\mathbb{T}(t)$ is uniformly bounded on $\mathbb{R}^+$ with the bound $a_{1/p'}$ given by (2.2). Again, Lemma 2.4 implies that $t \mapsto t^{1-1/p}\mathbb{T}(t)Ax$ is in $L^p(\mathbb{R}^+, X)$ with

$$\int_0^\infty \|t^{1-1/p}\mathbb{T}(t)Ax\|^p \,\mathrm{d}t \leqslant 2^{2p-1} a_{1/p'}^p \kappa_p^p \|x\|^p.$$

Now, consider a continuous operator $C\colon D(A) \to Y$ satisfying the estimate $\sup_{z\in\mathbb{C}_0} \operatorname{Re}(z)^{1/p'} \|CR(z,A)\|$ is finite. Thanks to Corollary 2.5, it is easy to see that $f(t) = tC\mathbb{T}(t)Ax$ is also in $L^p(\mathbb{R}^+, X)$ and

$$\int_0^\infty \|f(t)\|^p \,\mathrm{d}t \leqslant 2^{p+1} a_{1/p'}^p \kappa_p^p a_p^p M_p^p \|x\|^p. \tag{2.3}$$

As above, for $x \in D(A)$ and $t \in (0, T]$ we have

$$C\mathbb{T}(t)x = C\mathbb{T}(T)x - \int_t^T C\mathbb{T}(s)Ax \,\mathrm{d}s = C\mathbb{T}(T)x - (Lf)(t).$$

Hence,

$$\|C\mathbb{T}(t)x\|^p \leqslant 2^p\bigl(\|C\mathbb{T}(T)x\|^p + \|(Lf)(t)\|^p\bigr).$$

According to Lemma 2.3 and Proposition 2.2 we obtain

$$\begin{aligned}\int_0^T \|C\mathbb{T}(t)x\|^p \,\mathrm{d}t &\leqslant 2^p\bigl(T\|C\mathbb{T}(T)x\|^p + \|(Lf)\|_p^p\bigr)\\ &\leqslant 2^p(M_p^p\|x\|^p + p^p\|f\|_p^p)\\ &\leqslant 2^p M_p^p\bigl(1 + 2^{p+1} a_{1/p'}^p \kappa_p^p a_p^p\bigr)\|x\|^p,\end{aligned}$$

which implies that the operator C is finite-time p-admissible for A and

$$\sup_{T>0} \int_0^T \|C\mathbb{T}(t)x\|^p \,\mathrm{d}t < \infty.$$

Since $\int_0^\infty \|C\mathbb{T}(t)x\|^p \,\mathrm{d}t = \sup_{T>0} \int_0^T \|C\mathbb{T}(t)x\|^p \,\mathrm{d}t$, the proof is complete. □

Now we will show that the assumption that A generates an analytic semigroup in Theorem 2.1 cannot be omitted.

Proposition 2.7. *Let A be a generator of a bounded C_0-semigroup on a Banach space X. If $(-A)^{1/p}$ is finite-time p-admissible for A with $p \in [1,\infty)$, then A generates an analytic semigroup on X.*

Proof. Assume that $(-A)^{1/p}$ is a finite-time p-admissible observation operator for A. It suffices to show that there exists $K > 0$ such that $\|A\mathbb{T}(t)\| \leqslant K/t$, $0 < t \leqslant 1$ (see, e.g. [2]). Indeed, for $x \in D(A)$, Hahn-Banach's Theorem implies that there exists $\varphi_{t,x} \in X^*$ with $\|\varphi_{t,x}\| = 1$ such that

$$\begin{aligned} t\|(-A)^{1/p}\mathbb{T}(t)x\| &= t|\langle(-A)^{1/p}\mathbb{T}(t)x, \varphi_{t,x}\rangle| \\ &= t|\langle(-A)^{1/p}\mathbb{T}(t-s)\mathbb{T}(s)x, \varphi_{t,x}\rangle| \ (0 \leqslant s \leqslant t) \\ &= t|\langle(-A)^{1/p}\mathbb{T}(t-s)x, \mathbb{T}^*(s)\varphi_{t,x}\rangle|. \end{aligned}$$

Hence, using the Cauchy-Schwartz inequality we find

(2.4)
$$\begin{aligned} t\|(-A)^{1/p}\mathbb{T}(t)x\| &\leqslant \int_0^t \|(-A)^{1/p}\mathbb{T}(t-s)x\|\|\mathbb{T}^*(s)\varphi_{t,x}\| \,\mathrm{d}s \\ &\leqslant \Big(\int_0^t \|(-A)^{1/p}\mathbb{T}(s)x\|^p \,\mathrm{d}s\Big)^{1/p}\Big(\int_0^t \|\mathbb{T}^*(s)\varphi_{t,x}\|^{p'} \,\mathrm{d}s\Big)^{1/p'}. \end{aligned}$$

Since $(-A)^{1/p}$ is finite-time p-admissible for A we have

$$\int_0^t \|(-A)^{1/p}\mathbb{T}(s)x\|^p \,\mathrm{d}s \leqslant \kappa_p^p\|x\|^p \tag{2.5}$$

for all $t \in (0,1]$ and for some constant $M_p > 0$ not depending on $x \in D(A)$. Since $\mathbb{T}(t)$ is bounded on X, $\mathbb{T}^*(t)$ is also bounded on X^*. Combining (2.5) and (2.4) we deduce that

$$\|t^{1/p}(-A)^{1/p}\mathbb{T}(t)x\| \leqslant \alpha_p\|x\|, \quad x \in D(A) \tag{2.6}$$

for some constant $\alpha_p > 0$. By density we deduce that (2.6) is true for any $x \in X$. For n integer greater than p we obtain

$$\|t^{n/p}(-A)^{n/p}\mathbb{T}(t)x\| \leqslant \alpha_p^n n^{n/p}\|x\|, \quad x \in X, \tag{2.7}$$

and by the fact that $1 \in [\frac{1}{p}, \frac{n}{p}[$, the real interpolation completes the proof. □

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