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ON GENERALIZED JORDAN DERIVATIONS OF
LIE TRIPLE SYSTEMS

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Abstract. Under some conditions we prove that every generalized Jordan triple derivation on a Lie triple system is a generalized derivation. Specially, we conclude that every Jordan triple θ -derivation on a Lie triple system is a θ -derivation.

Keywords: Lie triple system, (φ, ψ) -derivation, Jordan triple (φ, ψ) -derivation, θ -derivation, Jordan triple θ -derivation

MSC 2010: 16W25, 17A40

1. INTRODUCTION

Throughout this paper \mathcal{R} will represent an associative ring. Given an integer $n > 1$, a ring \mathcal{R} is said to be *n-torsion free*, if for $x \in \mathcal{R}$, $nx = 0$ implies that $x = 0$. Recall that a ring \mathcal{R} is *prime* if for $a, b \in \mathcal{R}$, $aRb = (0)$ implies that either $a = 0$ or $b = 0$, and is *semiprime* in case $a \in \mathcal{R}$, $aRa = (0)$ implies that $a = 0$. An additive mapping $D: \mathcal{R} \rightarrow \mathcal{R}$ is called a *derivation* if $D(ab) = D(a)b + aD(b)$ holds for all pairs $a, b \in \mathcal{R}$. An additive mapping $D: \mathcal{R} \rightarrow \mathcal{R}$ is called a *Jordan derivation* if $D(a^2) = D(a)a + aD(a)$ holds for all $a \in \mathcal{R}$. Obviously, every derivation is a Jordan derivation. The converse is, in general, not true. I. N. Herstein [7] proved that every Jordan derivation on a 2-torsion free prime ring is a derivation. M. Brešar [3] extended this result to 2-torsion free semiprime rings. An additive mapping $D: \mathcal{R} \rightarrow \mathcal{R}$ is called a *Jordan triple derivation* if $D(aba) = D(a)ba + aD(b)a + abD(a)$ holds for all pairs $a, b \in \mathcal{R}$. M. Brešar [4] has proved that any Jordan triple derivation on a 2-torsion free semiprime ring is a derivation. Let θ, φ be automorphisms of \mathcal{R} . An additive mapping $\delta: \mathcal{R} \rightarrow \mathcal{R}$ is called a (θ, φ) -*derivation* if $\delta(xy) = \delta(x)\theta(y) + \varphi(x)\delta(y)$ holds for all $x, y \in \mathcal{R}$. An additive mapping $\delta: \mathcal{R} \rightarrow \mathcal{R}$ is called a *Jordan (θ, φ) -derivation* if $\delta(x^2) = \delta(x)\theta(x) + \varphi(x)\delta(x)$ holds for all $x \in \mathcal{R}$. An additive mapping $\delta: \mathcal{R} \rightarrow \mathcal{R}$

is called a *Jordan triple* (θ, φ) -*derivation* if $\delta(xy) = \delta(x)\theta(y)\theta(x) + \varphi(x)\delta(y)\theta(x) + \varphi(x)\varphi(y)\delta(x)$ holds for all $x, y \in \mathcal{R}$. It is easy to see that every (θ, φ) -derivation is a Jordan (triple) (θ, φ) -derivation. A result of [5] states that every Jordan (θ, φ) -derivation is a Jordan triple (θ, φ) -derivation. An additive mapping $\xi: \mathcal{R} \rightarrow \mathcal{R}$ is called a *generalized* (θ, φ) -*derivation* if there exists a (θ, φ) -derivation $\delta: \mathcal{R} \rightarrow \mathcal{R}$ such that $\xi(xy) = \xi(x)\theta(y) + \varphi(x)\delta(y)$ holds for all $x, y \in \mathcal{R}$ (see [6], [11]). In [13], Liu and Shiue proved that every Jordan (triple) (θ, φ) -derivation on a 2-torsion free semiprime ring is a (θ, φ) -derivation. Also, they introduced a concept of generalized Jordan (θ, φ) -derivation and generalized Jordan triple (θ, φ) -derivation (see also [10]). An additive mapping $\xi: \mathcal{R} \rightarrow \mathcal{R}$ is called a *generalized Jordan* (θ, φ) -*derivation* if $\xi(x^2) = \xi(x)\theta(x) + \varphi(x)\delta(x)$ holds for all $x \in \mathcal{R}$ where $\delta: \mathcal{R} \rightarrow \mathcal{R}$ is a Jordan (θ, φ) -derivation. An additive mapping $\xi: \mathcal{R} \rightarrow \mathcal{R}$ is called a *generalized Jordan triple* (θ, φ) -*derivation* if $\xi(xy) = \xi(x)\theta(y)\theta(x) + \varphi(x)\delta(y)\theta(x) + \varphi(x)\varphi(y)\delta(x)$ holds for all pairs $x, y \in \mathcal{R}$ where $\delta: \mathcal{R} \rightarrow \mathcal{R}$ is a Jordan triple (θ, φ) -derivation. A result of [1] states that every generalized Jordan (θ, φ) -derivation is a generalized Jordan triple (θ, φ) -derivation. Liu and Shiue [13] proved that every generalized Jordan (triple) (θ, φ) -derivation on a 2-torsion free semiprime ring is a generalized (θ, φ) -derivation (see also [14]).

The concept of Lie triple system was first introduced by N. Jacobson [8], [9] (see also [12]). A Lie triple system is a \mathcal{R} -module \mathcal{L} with a \mathcal{R} -trilinear mapping $\mathcal{L} \times \mathcal{L} \times \mathcal{L} \ni (x, y, z) \mapsto [x, y, z] \in \mathcal{L}$ satisfying the following axioms

- (a) $[x, x, y] = 0$,
- (b) $[x, y, z] + [y, z, x] + [z, x, y] = 0$,
- (c) $[u, v, [x, y, z]] = [[u, v, x], y, z] + [x, [u, v, y], z] + [x, y, [u, v, z]]$,

for all $u, v, x, y, z \in \mathcal{L}$. It follows from (a) that $[x, y, z] = -[y, x, z]$ for all $x, y, z \in \mathcal{L}$.

It is clear that every Lie algebra with product $[\cdot, \cdot]$ is a Lie triple system with respect to $[x, y, z] := [[x, y], z]$. Conversely, any Lie triple system \mathcal{L} can be considered as a subspace of a Lie algebra (Bertram [2], Jacobson [9]).

In the rest of this paper, we let \mathcal{L} be an \mathcal{R} -module which is a Lie triple system and $\theta, \varphi: \mathcal{L} \rightarrow \mathcal{L}$ be \mathcal{R} -linear mappings.

Definition 1.1. An \mathcal{R} -linear mapping $D: \mathcal{L} \rightarrow \mathcal{L}$ is called a (θ, φ) -*derivation* if

$$D([x, y, z]) = [D(x), \theta(y), \theta(z)] + [\varphi(x), D(y), \theta(z)] + [\varphi(x), \varphi(y), D(z)]$$

for all $x, y, z \in \mathcal{L}$. If $\varphi = \theta$, a (θ, φ) -derivation is called a θ -*derivation*. If $\varphi = \theta = I_{\mathcal{L}}$, where $I_{\mathcal{L}}$ is the identity map on \mathcal{L} , a (θ, φ) -derivation is called a *derivation* (see [12]).

Let $u, v \in \mathcal{L}$ and $D_{u,v}: \mathcal{L} \rightarrow \mathcal{L}$ be the mapping defined by

$$D_{u,v}(x) := [u, v, x]$$

for all $x \in \mathcal{L}$. It is clear that $D_{u,v}$ is \mathcal{R} -linear and we get from (c) that the mapping $D_{u,v}$ is a derivation on \mathcal{L} .

Following Jing and Lu [10], we introduce a concept of Jordan triple (θ, φ) -derivation.

Definition 1.2. An \mathcal{R} -linear mapping $D: \mathcal{L} \rightarrow \mathcal{L}$ is called a *Jordan triple (θ, φ) -derivation* if

$$D([x, y, x]) = [D(x), \theta(y), \theta(x)] + [\varphi(x), D(y), \theta(x)] + [\varphi(x), \varphi(y), D(x)]$$

for all $x, y \in \mathcal{L}$. If $\varphi = \theta$, a Jordan triple (θ, φ) -derivation is called a *Jordan triple θ -derivation*. If $\varphi = \theta = I_{\mathcal{L}}$, a Jordan triple (θ, φ) -derivation is called a *Jordan triple derivation*.

Following Liu and Shiue [13], we introduce a concept of generalized (θ, φ) -derivations and generalized Jordan triple (θ, φ) -derivations on Lie triple systems.

Definition 1.3. Let $\delta: \mathcal{L} \rightarrow \mathcal{L}$ be a (θ, φ) -derivation. An \mathcal{R} -linear mapping $D: \mathcal{L} \rightarrow \mathcal{L}$ is called a *generalized (θ, φ) -derivation with respect to δ* if

$$D([x, y, z]) = [\delta(x), \theta(y), \theta(z)] + [\varphi(x), \delta(y), \theta(z)] + [\varphi(x), \varphi(y), D(z)]$$

for all $x, y, z \in \mathcal{L}$.

Definition 1.4. Let $\delta: \mathcal{L} \rightarrow \mathcal{L}$ be a Jordan triple (θ, φ) -derivation. An \mathcal{R} -linear mapping $D: \mathcal{L} \rightarrow \mathcal{L}$ is called a *generalized Jordan triple (θ, φ) -derivation with respect to δ* if

$$D([x, y, x]) = [\delta(x), \theta(y), \theta(x)] + [\varphi(x), \delta(y), \theta(x)] + [\varphi(x), \varphi(y), D(x)]$$

for all $x, y \in \mathcal{L}$.

2. MAIN RESULTS

It is clear that every (θ, φ) -derivation on a Lie triple system is a Jordan triple (θ, φ) -derivation. In this section under some conditions we prove that every generalized Jordan triple (θ, φ) -derivation on a Lie triple system \mathcal{L} is a generalized (θ, φ) -derivation. So we conclude that every Jordan triple θ -derivation on \mathcal{L} is a θ -derivation.

Throughout this section $\theta, \varphi, D, \delta: \mathcal{L} \rightarrow \mathcal{L}$ are \mathcal{R} -linear mappings and $A_{\theta, \varphi}^{\delta, D}: \mathcal{L} \times \mathcal{L} \times \mathcal{L} \rightarrow \mathcal{L}$ is a mapping defined by

$$A_{\theta, \varphi}^{\delta, D}(x, y, z) := [\delta(x), \theta(y), \theta(z)] + [\varphi(x), \delta(y), \theta(z)] + [\varphi(x), \varphi(y), D(z)]$$

for all $x, y, z \in \mathcal{L}$. It is clear that the mapping $A_{\theta, \varphi}^{\delta, D}$ is \mathcal{R} -trilinear and $A_{\theta, \varphi}^{\delta, D}(x, x, y) = [\delta(x), (\theta - \varphi)(x), \theta(y)]$ for all $x, y \in \mathcal{L}$. We denote $A_{\theta, \varphi}^{\delta, \delta}$ by $A_{\theta, \varphi}^{\delta}$.

Lemma 2.1. *Let \mathcal{R} be a 3-torsion free ring and let $D: \mathcal{L} \rightarrow \mathcal{L}$ be a generalized Jordan triple (θ, φ) -derivation with respect to the Jordan triple (θ, φ) -derivation δ . If*

$$(2.1) \quad [\varphi(x), \varphi(y), (D - \delta)(z)] + [\varphi(y), \varphi(z), (D - \delta)(x)] + [\varphi(z), \varphi(x), (D - \delta)(y)] = 0$$

for all $x, y, z \in \mathcal{L}$, then

$$(2.2) \quad (D - \delta)([x, y, z]) = [\varphi(x), \varphi(y), (D - \delta)(z)],$$

$$(2.3) \quad B(x, y, z) + B(y, z, x) + B(z, x, y) = 0$$

for all $x, y, z \in \mathcal{L}$, where $B := A_{\theta, \varphi}^{\delta, D} - A_{\theta, \varphi}^{\delta}$.

Proof. It is clear that $(D - \delta)([x, y, x]) = [\varphi(x), \varphi(y), (D - \delta)(x)]$ for all $x, y, z \in \mathcal{L}$. We must show that the \mathcal{R} -trilinear mapping

$$F: \mathcal{L} \times \mathcal{L} \times \mathcal{L} \rightarrow \mathcal{L}, \quad (x, y, z) \mapsto (D - \delta)([x, y, z]) - [\varphi(x), \varphi(y), (D - \delta)(z)]$$

vanishes identically. We have $F(x, y, x) = 0$ and by (a), $F(x, x, y) = 0$ for all $x, y \in \mathcal{L}$. Since F is \mathcal{R} -trilinear and $F(x + y, x, x + y) = 0$, we get that $F(y, x, x) = 0$ for all $x, y \in \mathcal{L}$. Therefore $F(x, y, z) = F(y, z, x) = F(z, x, y)$ and it follows from (b) and (2.1) that

$$3F(x, y, z) = F(x, y, z) + F(y, z, x) + F(z, x, y) = 0,$$

for all $x, y, z \in \mathcal{L}$. So F is identically 0 since there is no 3-torsion. To prove (2.3), we have from (2.2) that

$$B(x, y, z) = (D - \delta)[x, y, z]$$

for all $x, y, z \in \mathcal{L}$. Since $D - \delta$ is additive, we get (2.3) from (b). \square

Remark 2.2. If $D: \mathcal{L} \rightarrow \mathcal{L}$ is a generalized Jordan triple (θ, φ) -derivation with respect to the Jordan triple (θ, φ) -derivation δ satisfying (2.2), then one checks that (2.1) holds.

Lemma 2.3. Let \mathcal{R} be a 3-torsion free ring and let $F: \mathcal{L} \times \mathcal{L} \times \mathcal{L} \rightarrow \mathcal{L}$ be an \mathcal{R} -trilinear mapping satisfying $F(x, y, x) = F(x, x, y) = 0$ and

$$(2.4) \quad F(x, y, z) + F(y, z, x) + F(z, x, y) = 0$$

for all $x, y, z \in \mathcal{L}$. Then F is identically 0.

Proof. Since F is \mathcal{R} -trilinear and $F(x+y, x, x+y) = 0$, we get that $F(y, x, x) = 0$ for all $x, y \in \mathcal{L}$. Therefore $F(x, y, z) = F(y, z, x) = F(z, x, y)$ for all $x, y, z \in \mathcal{L}$. Hence it follows from (2.4) that $3F(x, y, z) = 0$ for all $x, y, z \in \mathcal{L}$. So F is identically 0 since there is no 3-torsion. \square

Theorem 2.4. Let \mathcal{R} be a 3-torsion free ring and let $D: \mathcal{L} \rightarrow \mathcal{L}$ be a Jordan triple (θ, φ) -derivation. Then D is a (θ, φ) -derivation if and only if

- (i) $[D(x), (\theta - \varphi)(x), \theta(y)] = 0$;
- (ii) $A_{\theta, \varphi}^D(x, y, z) + A_{\theta, \varphi}^D(y, z, x) + A_{\theta, \varphi}^D(z, x, y) = 0$

for all $x, y, z \in \mathcal{L}$.

Proof. Let D be a (θ, φ) -derivation, then $A_{\theta, \varphi}^D(x, y, z) = D([x, y, z])$ for all $x, y, z \in \mathcal{L}$. So $A_{\theta, \varphi}^D(x, x, y) = 0$ for all $x, y \in \mathcal{L}$. This proves (i). Since $[x, y, z] + [y, z, x] + [z, x, y] = 0$, (ii) is valid for all $x, y, z \in \mathcal{L}$.

Conversely, we prove that D is a (θ, φ) -derivation if (i) and (ii) hold. We show that the \mathcal{R} -trilinear mapping

$$F: \mathcal{L} \times \mathcal{L} \times \mathcal{L} \rightarrow \mathcal{L}, \quad (x, y, z) \longmapsto D([x, y, z]) - A_{\theta, \varphi}^D(x, y, z)$$

vanishes identically. It follows from (a) and (i) that $F(x, x, y) = 0$ and since $D: \mathcal{L} \rightarrow \mathcal{L}$ is a Jordan triple (θ, φ) -derivation, we have $F(x, y, x) = 0$ for all $x, y \in \mathcal{L}$. By (b) and (ii), F satisfies (2.4). Hence by Lemma 2.3, F vanishes identically. \square

Theorem 2.5. Let \mathcal{R} be a 3-torsion free ring and let $D: \mathcal{L} \rightarrow \mathcal{L}$ be a generalized Jordan triple (θ, φ) -derivation with respect to the Jordan triple (θ, φ) -derivation δ satisfying (2.1). If

- (i) $[\delta(x), (\theta - \varphi)(x), \theta(y)] = 0$;
- (ii) $A_{\theta, \varphi}^\delta(x, y, z) + A_{\theta, \varphi}^\delta(y, z, x) + A_{\theta, \varphi}^\delta(z, x, y) = 0$

for all $x, y, z \in \mathcal{L}$, then δ is a (θ, φ) -derivation and D is a generalized (θ, φ) -derivation with respect to the (θ, φ) -derivation δ .

Proof. It follows from Theorem 2.4 that δ is a (θ, φ) -derivation. Applying Lemma 2.1, we get from (2.3) and (ii) that

$$(2.5) \quad A_{\theta, \varphi}^{\delta, D}(x, y, z) + A_{\theta, \varphi}^{\delta, D}(y, z, x) + A_{\theta, \varphi}^{\delta, D}(z, x, y) = 0$$

for all $x, y, z \in \mathcal{L}$. The rest of the proof is similar to the proof of Theorem 2.4. \square

Crollary 2.6. Let \mathcal{R} be a 3-torsion free ring and let $D: \mathcal{L} \rightarrow \mathcal{L}$ be a generalized Jordan triple θ -derivation with respect to the Jordan triple θ -derivation δ satisfying

$$[\theta(x), \theta(y), (D - \delta)(z)] + [\theta(y), \theta(z), (D - \delta)(x)] + [\theta(z), \theta(x), (D - \delta)(y)] = 0$$

for all $x, y, z \in \mathcal{L}$. Then δ is a θ -derivation and D is a generalized θ -derivation with respect to the θ -derivation δ .

Proof. It is clear that condition (i) of Theorem 2.5 is valid when $\theta = \varphi$. For condition (ii) of Theorem 2.5, we have from (b) that

$$\begin{aligned} & A_{\theta, \theta}^{\delta}(x, y, z) + A_{\theta, \theta}^{\delta}(y, z, x) + A_{\theta, \theta}^{\delta}(z, x, y) \\ &= ([\delta(x), \theta(y), \theta(z)] + [\theta(y), \theta(z), \delta(x)] + [\theta(z), \delta(x), \theta(y)]) \\ &\quad + ([\theta(x), \delta(y), \theta(z)] + [\delta(y), \theta(z), \theta(x)] + [\theta(z), \theta(x), \delta(y)]) \\ &\quad + ([\theta(x), \theta(y), \delta(z)] + [\theta(y), \delta(z), \theta(x)] + [\delta(z), \theta(x), \theta(y)]) = 0 \end{aligned}$$

for all $x, y, z \in \mathcal{L}$. So condition (ii) of Theorem 2.4 is valid if $\varphi = \theta$. Hence δ is a θ -derivation and D is a generalized θ -derivation with respect to δ . \square

Corollary 2.7. Let \mathcal{R} be a 3-torsion free ring. Then $D: \mathcal{L} \rightarrow \mathcal{L}$ is a Jordan triple θ -derivation if and only if D is a θ -derivation.

Corollary 2.8. Let \mathcal{R} be a 3-torsion free ring. Then $D: \mathcal{L} \rightarrow \mathcal{L}$ is a Jordan triple derivation if and only if D is a derivation.

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